CHAPTER 5 - ENTROPY AND APPLICATIONS

This chapter is an introduction to entropy (or Lyapunov) methods for general (possibly nonlinear) dynamical system and some applications to some evolution PDEs (mostly linear, positivity preserving and mass preserving) including a general Fokker-Planck model, a scattering (or linear Boltzmann) equation and a growth-fragmentation equation.

Contents

1. Dynamic system, equilibrium and entropy methods

1.1. Existence of steady states.

Definition 1.1. *We say that $(S_t)_{t\ge 0}$ is a dynamical system (or a continuous (possibly nonlinear) semigroup) on a metric space $(\mathcal{Z}, d)$ if*

(S1) $\forall t \ge 0$, $S_t \in C(\mathcal{Z},\mathcal{Z})$ (continuously defined on $\mathcal{Z}$);

(S2) $\forall x \in \mathcal{Z}$, $t \mapsto S_t\, x \in C([0,\infty),\mathcal{Z})$ (trajectories are continuous);

(S3) $S_0 = I$; $\forall\, s,\, t \ge 0$, $S_{t+s} = S_t\, S_s$ (semigroup property).

We say that $\bar z \in \mathcal{Z}$ is invariant (or is a steady state, a stationary point) if $S_t \bar z = \bar z$ for any $t \ge 0$. We denote by $\mathcal{E}$ the set of all steady states,

$$\mathcal{E} := \{y \in \mathcal{Z};\ S_t y = y\ \forall\, t \ge 0\}.$$

We remark that $\mathcal{E}$ is closed by definition ($\mathcal{E} = \cap_{t\ge 0}(S_t - I)^{-1}(\{0\})$).

Theorem 1.2. (Dynamic system and steady state). *Consider a bounded and convex subset $\mathcal{Z}$ of a Banach space X which is sequentially compact when it is endowed with the metric associated to the norm $\|\cdot\|_X$ (strong topology), to the weak topology $\sigma(X,X')$ or to the weak-$\star$ topology $\sigma(X,Y)$, $Y'=X$ (see Section 7.1 for the precise requirement we need on X and $\mathcal{Z}$). Then any dynamical system $(S_t)_{t\ge 0}$ on $\mathcal{Z}$ admits at least one steady state, that is $\mathcal{E}\neq\emptyset$.*

Proof of Theorem 1.2. For any $t>0$, there exists $z_t\in\mathcal{Z}$ such that $S_t z_t = z_t$ thanks to the Schauder or the Tychonoff point fixed Theorem (see Section 7.1). On the one hand, from the semigroup property *(S3)*

$$S_{i\,2^{-m}} z_{2^{-n}} = z_{2^{-n}} \quad \text{for any} \quad i,n,m\in\mathbb{N}, \quad m\le n. \tag{1.1}$$

On the other hand, by compactness of $\mathcal{Z}$, we may extract a subsequence $(z_{2^{-n_k}})_k$ which converges weakly to a limit $\bar z\in\mathcal{Z}$. By the continuity assumption *(S1)* on S_t, we may pass to the limit $n_k\to\infty$ in (1.1) and we obtain $S_t\,\bar z = \bar z$ for any dyadic time $t\ge 0$. We conclude that $\bar z$ is a stationary point by the trajectorial continuity assumption *(S2)* on S_t and the density of the dyadic real numbers in the real line. □

1.2. **ω-limit set of trajectories compact dynamical system.** Consider a dynamical system $(S_t)_{t\ge0}$ on a metric space $(\mathcal{Z},d)$. For any given $z\in\mathcal{Z}$, we define the associated *omega-limit set* as

$$\omega(z) = \{y\in\mathcal{Z};\ \exists t_n \nearrow \infty \ \text{ et } \ S_{t_n} z \to y\},$$

or equivalently

$$\omega(z) := \bigcap_{T>0} \omega_T(z), \quad \omega_T(z) := \overline{\{S_t z;\ t\ge T\}}. \tag{1.2}$$

We obviously have

$$\mathcal{E}_z := \{y\in\omega_0(z);\ S_t y = y\ \forall\, t\ge 0\} \subset \omega(z)$$

and $\mathcal{E}_z=\{\bar z\}$ if $S_t z\to\bar z$ when $t\to\infty$.

Theorem 1.3. (Dynamic system and ω-limit set). *Consider a dynamical system $(S_t)_{t\ge0}$ on a metric space $(\mathcal{Z},d)$ which trajectories are relatively compact. More precisely, we assume*

(S4) $\omega_0(z)$ is compact for some fixed $z\in\mathcal{Z}$.

Then there hold

(i) $S_t(\omega(z)) = \omega(z)$ $\forall t\ge0$;

(ii) $\omega(z)$ is a nonempty connected and compact subset of $\mathcal{Z}$. More precisely, any $y\in\omega(z)$ belongs to an eternal trajectory in the sense that there exists $(y_t)_{t\in\mathbb{R}}\subset\omega(z)$ such that $y_0=y$ and $y_{s+t}=S(s)y_t$ for any $t\in\mathbb{R}$ and any $s\ge0$;

(iii) $d(S_t\,z,\omega(z))\to0$ as $t\to\infty$;

(iv) If furthermore $\omega(z)$ is a discrete set, then $\omega(z)$ is a singleton and $\omega(z)\subset\mathcal{E}_z$. More explicitly, there exists $\bar z\in\mathcal{Z}$ such that $\omega(z)=\{\bar z\}\subset\mathcal{E}_z$ or equivalently such that $S_t z\to\bar z$ as $t\to\infty$.

Proof of Theorem 1.3. (i) On the one hand, for any $y\in\omega(z)$, there exists (t_n) such that $S_{t_n}z\to y$, so that $S_{t_n+t}z\to S_t y$ and $S_t y\in\omega(z)$. That proves $S_t(\omega(z))\subset\omega(z)$. On the other hand, given $y\in\omega(z)$ and $t_n\to\infty$ such that $S_{t_n}z\to y$, there exists $w\in\mathcal{Z}$ and a subsequence $(t_{n'})$ such that $S_{t_{n'}-t}z\to w$ because of assumption *(S4)*, and then $w\in\omega(z)$. We deduce

$$S_t w = S_t(\lim S_{t_{n'}-t} z) = \lim S_{t_{n'}} z = y.$$

That proves the reverse inclusion $\omega(z)\subset S_t(\omega(z))$.

(ii) For any $n\ge0$, the set $\omega_n(z)$ is a nonempty connected and compact subset of $\mathcal{Z}$ by assumption *(S4)*. The sequence $(\omega_n(z))$ being decreasing, we have $\omega(z)=\lim\omega_n(z)$ which is nothing but (1.2) and thus (ii). More precisely, consider $y\in\omega(z)$ and (t_n) such that $S(t_n)z\to y$. For any $t\in\mathbb{R}_-$, we may extract a subsequence of $S(t_n+t)z$ which converges to a limit y_t. Better, thanks to Cantor's diagonal process, there exists one subsequence (t_{n_k}) such that for any $t\in\mathbb{Z}_-$ there holds $S(t_{n_k}+t)z\to y_t$ and next, for any $t\in\mathbb{R}_-$,

$$S(t_{n_k}+t)z = S(-[t]+1+t)S(t_{n_k}+[t]-1)z \to S(-[t]+1-t)y_{[t]-1} =: y_t.$$

As a consequence, $y_t\in\omega(z)$, $y_0=y$ and $y_{t+s}=\lim S(t_{n_k}+s+t)z=\lim S(s)S(t_{n_k}+t)z=S(s)y_t$ for any $t\in\mathbb{R}$ and $s\in\mathbb{R}_-$.

(iii) We argue by contradiction. Assume that there exist a sequence $t_n \to \infty$ and a real number $\epsilon > 0$ such that $d(S_{t_n} z, \omega(z)) \geq \epsilon$. From assumption *(S4)*, there exists a subsequence $(t_{n'})$ such that $S_{t_{n'}} z \to w \in \omega(z)$ and then $d(S_{t_{n'}} z, \omega(z)) \to 0$, which is absurd.

(iv) First, $\omega(z)$ is a singleton as a discrete and connected nonempty set, we then have $\omega = \{\bar{z}\}$. Next, by uniqueness of the possible limits, we deduce $S_t z \to \bar{z}$ as $t \to \infty$. □

1.3. **Dissipation of entropy method.** Consider a dynamical system $(S_t)_{t\geq 0}$ on a metric space $(\mathcal{Z}, d)$. We say that a functional $\mathcal{H} : \mathcal{Z} \to \mathbb{R}$ is an entropy if there exists a *dissipation of entropy* functional $\mathcal{D} : \mathcal{Z} \to \mathbb{R}_+$ such that for any $z \in \mathcal{Z}$ there holds

$$\frac{d}{dt}\mathcal{H}(S_t z) = -\mathcal{D}(S_t z) \leq 0 \quad \forall t > 0,$$

or equivalently

$$\mathcal{H}(S_t z) + \int_0^t \mathcal{D}(S_s z)\, ds = \mathcal{H}(z). \tag{1.3}$$

As a consequence $t \mapsto \mathcal{H}(S_t z)$ is a decreasing function, and more importantly here, under the additional lower bound assumption

$$\mathcal{H}_z > -\infty, \quad \mathcal{H}_z := \inf_{y\in\omega_0(z)} \mathcal{H}(y), \tag{1.4}$$

there holds

$$\int_0^\infty \mathcal{D}(S_s z)\, ds \leq \mathcal{H}(z) - \mathcal{H}_z < \infty. \tag{1.5}$$

We define

$$\omega_{\mathcal{D}}(z) := \{y \in \omega_0(z);\ \mathcal{D}(S_t y) = 0\ \forall t \geq 0\},$$

and we observe that $\mathcal{E}_z \subset \omega_{\mathcal{D}}(z)$ at least when (1.3) holds.

Theorem 1.4. (Dissipation of entropy method - weak version). *Consider a dynamical system $(S_t)_{t\geq 0}$ on a metric space $(\mathcal{Z}, d)$ and $z \in \mathcal{Z}$. We assume*

(S4′) $(S_t z)_{t\geq 0}$ is "locally uniformly compact" in the sense that $(\mathcal{S}^{z,T}_t)_{t\geq 0}$ is relatively compact in $C([0,T]; \mathcal{Z})$ for any fixed time $T \in \mathbb{R}_+$, where we have defined $s \mapsto \mathcal{S}^{z,T}_t(s) := S_{t+s} z$;

(H1) there exists a lsc dissipation of entropy functional $\mathcal{D}$ on $\mathcal{Z}$ such that $t \mapsto \mathcal{D}(S_t z) \in L^1$.

Then, we have $\omega(z) \subset \omega_{\mathcal{D}}(z)$, and therefore $d(S_t z, \omega_{\mathcal{D}}(z)) \to 0$ as $t \to \infty$.

Proof of Theorem 1.4. We define $z^t := \mathcal{S}^{z,T}_t \in C([0,T]; \mathcal{Z})$, $T > 0$, and we observe that

$$\int_0^T \mathcal{D}(z^t(s))\, ds = \int_t^{t+T} \mathcal{D}(S_s z)\, ds \leq \int_t^\infty \mathcal{D}(S_s z)\, ds.$$

Consider $y \in \omega(z)$ and a sequence $t_n \to \infty$ such that $S_{t_n} z \to y$ as $n \to \infty$. From the compactness assumption *(S4′)* and a diagonal Cantor procedure, there exist a subsequence $(t_{n'})$ and a function $z^* \in C([0,\infty); \mathcal{Z})$ such that $z^{t_{n'}} \to z^*$ in $C([0,T]; \mathcal{Z})$ for any $T > 0$ and obviously $z^*(s) = S_s y$ for any $s \geq 0$. From the assumptions *(H1)* made on the dissipation of entropy and the above inequality, we then deduce

$$\int_0^T \mathcal{D}(z^*(s))\, ds \leq \liminf_{n'\to\infty} \int_{t_{n'}}^\infty \mathcal{D}(S_s z)\, ds = 0.$$

As a consequence $\mathcal{D}(z^*(s)) = 0$ for any $s \geq 0$ and then $y \in \omega_{\mathcal{D}}(z)$. We conclude thanks to (iii) in Theorem 1.3. □

Exercise 1.5. *Assume furthermore that $\omega_{\mathcal{D}}(z) = \{\bar{z}\}$. Taking up again the proof of Theorem 1.4, prove directly (without using Theorem 1.3 nor Theorem 1.6) that $\omega(z) = \{\bar{z}\} \subset \mathcal{E}$.*

Theorem 1.6. (Dissipation of entropy method - strong version). *We assume furthermore that*

$$\omega_{\mathcal{D}}(z) \quad \text{is discrete.} \tag{1.6}$$

Then, $\omega(z)$ is a singleton and $\omega(z) \subset \mathcal{E}_z$. More explicitly, we have $\omega(z) = \{z^*\} \subset \mathcal{E}_z \cap \omega_{\mathcal{D}}(z)$ for some $z^* \in \mathcal{Z}$ or equivalently $S_t z \to z^*$ as $t \to \infty$.

Proof of Theorem 1.6. From Theorem 1.4 we have $\omega(z) \subset \omega_{\mathcal{D}}(z)$ which is assumed to be discrete. We conclude thanks to (iv) in Theorem 1.3. $\square$

1.4. **Lyapunov functional and La Salle invariance principle.**

Definition 1.7. *Consider a dynamical system $(S_t)_{t\ge 0}$ on a metric space $(\mathcal{Z}, d)$.*
- We say that $\mathcal{H}$ is a Lyapunov functional if $\mathcal{H} \in C(\mathcal{Z}, \mathbf{R})$ and $t \mapsto \mathcal{H}(S_t z)$ is decreasing.
- For a given $z \in \mathcal{Z}$ we recall that $\mathcal{H}_z$ is defined in (1.4) and we define

$$\omega_{\mathcal{H}}(z) := \{y \in \omega_0(z);\ \mathcal{H}(S_t y) = \mathcal{H}_z \ \forall\, t \ge 0\}.$$

Theorem 1.8. (La Salle invariance principle). *Consider a dynamical system $(S_t)_{t\ge 0}$ on a metric space $(\mathcal{Z}, d)$ and $z \in \mathcal{Z}$. Assuming that*

(S4) $(S_t z)_{t\ge 0}$ is relatively compact;

(H2) $\mathcal{H}$ is a Lyapunov functional;

there holds $\omega(z) \subset \omega_{\mathcal{H}}(z)$, and more precisely

$$\mathcal{H}_z \in \mathbb{R}, \quad \mathcal{H}(S_t z) \searrow \mathcal{H}_z \ \text{as } t \to \infty \quad \text{and} \quad d(S_t z, \omega_{\mathcal{H}}(z)) \to 0 \ \text{as } t \to \infty.$$

Proof of Theorem 1.8. On the one hand, $\mathcal{H}(S_t z)$ is decreasing so that $\lim \mathcal{H}(S_t z) = \mathcal{H}_z$ and bounded (because the trajectories are relatively compact) so that $\mathcal{H}_z \in \mathbb{R}$. On the other hand, for any $y \in \omega(z)$ there exists $t_n \to \infty$ such that $S_{t_n} z \to y$ which in turns implies $\mathcal{H}_z = \lim \mathcal{H}(S_{t_n+s} z) = \mathcal{H}(\lim S_{t_n+s} z) = \mathcal{H}(S_s y)$ for any $s \ge 0$. In other words, we have $\omega(z) \subset \omega_{\mathcal{H}}(z)$ and the second convergence result is a consequence of (iii) in Theorem 1.3. $\square$

We immediately deduce

Theorem 1.9. (Lyapunov method). *Assuming furthermore that*

$$\omega_{\mathcal{H}}(z) \quad \text{is discrete},$$

there holds $\omega(z) = \{z^*\}$ for some $z^* \in \mathcal{E}_z$, or equivalently $S_t\, z \to z^*$ as $t \to \infty$.

Proof of Theorem 1.9. Since then $\omega(z) \subset \omega_{\mathcal{H}}(z)$ is discrete, we may use (iv) in Theorem 1.3 and conclude. $\square$

1.5. **Discussions on the entropy methods.** For the sake of simplicity, consider here the situation when the semigroup (S_t) is (formally) associated to an (abstract) evolution equation

$$\frac{d}{dt} z_t = \mathcal{Q}(z_t) \text{ on } (0,\infty), \quad z_0 \in \mathcal{Z}. \tag{1.7}$$

More precisely, we assume that for any $z_0 \in \mathcal{Z}$ there exists a unique solution $z_t \in C([0,\infty); \mathcal{Z})$ to the equation (1.7), and for any $z \in \mathcal{Z}$ we set $S_t z = z_t$ where z_t is the solution to (1.7) associated to the initial datum $z_0 = z$. We may observe that

$$\mathcal{E}_z = \{y \in \omega(z);\ Q(y) = 0\}.$$

For any function $\mathcal{H} : \mathcal{Z} \to \mathbb{R}$, we have (formally)

$$\frac{d}{dt}\mathcal{H}(S_t z) = \mathcal{H}'(z_t) \cdot \frac{d}{dt} z_t = \mathcal{H}'(z_t) \cdot \mathcal{Q}(z_t).$$

The condition

$$\forall\, z \in \mathcal{Z} \quad \mathcal{D}(z) := -\mathcal{H}'(z) \cdot \mathcal{Q}(z) \ge 0$$

then (formally) guaranties that the functional $\mathcal{H}$ is an entropy (decreases along trajectories) and $\mathcal{D}$ is a *dissipation of entropy functional.*

In the two entropy methods and for a given metric space $(\mathcal{Z}, d)$, the compactness condition $(S4')$ is clearly stronger than the condition $(S4)$. It is however not difficult to deduce $(S4')$ from $(S4)$ for an evolution equation in the applications we have in mind.

The first main difference between the two entropy methods lies on the fact that we assume that
- $\mathcal{D}$ is lower semicontinuous in the first method;
- $\mathcal{H}$ is continuous in the second method.

In many applications, the lower semicontinuity condition on $\mathcal{D}$ is easier to prove than the continuity condition on $\mathcal{H}$.

More importantly, the decreasing condition on $\mathcal{H}$ is obtained by writing the identity (1.3) while the integrability condition (1.5) is a consequence of the mere inequality

$$\mathcal{H}(S_t z) + \int_0^t \mathcal{D}(S_s z)\, ds \le \mathcal{H}(z) \quad \forall\, t \ge 0. \tag{1.8}$$

Again, that last inequality is easier to obtain than the identity (1.3): in many cases it can be proved by an approximation procedure and using the fact that both $\mathcal{H}$ and $\mathcal{D}$ are lower semicontinuous.

Let us then discuss the accuracy of the two methods. For that purpose we introduce the subsets

$$\mathcal{E}_{\mathcal{H}}(z) := \{y \in \omega_0(z);\ \mathcal{H}(y) = \mathcal{H}_z\}, \quad \mathcal{E}_{\mathcal{D}}(z) := \{y \in \omega_0(z);\ \mathcal{D}(y) = 0\}$$

which are defined through "a stationary formulation" (they are not related to the semigroup or the evolutionary problem). We easily check the following inclusions

$$\mathcal{E}_z \subset \omega(z) \subset \omega_{\mathcal{H}}(z) = \mathcal{E}_{\mathcal{H}}(z) \subset \omega_{\mathcal{D}}(z) \subset \mathcal{E}_{\mathcal{D}}(z)$$

from the convergence theorems proved before and thanks to the inequality (1.8). We deduce that the conclusion in Theorem 1.8 is a bit stronger than in Theorem 1.4 because the target set is in general smaller and because of the additional convergence of the entropy functional. However, in the case the identity (1.3) holds, we have $\omega_{\mathcal{H}}(z) = \omega_{\mathcal{D}}(z)$ and the target sets are the same in both theorems. In practice, in order to identify the possible limit set, we try to characterize the set $\omega_{\mathcal{D}}(z)$ or the set

$$\{y \in \omega_0(z);\ \mathcal{H}'(y) \perp \omega_0(z)\}$$

which clearly contains the set $\mathcal{E}_{\mathcal{H}}(z)$.

The above entropy methods are quite general and efficient. The shortcoming of the method is that it does not give any rate of convergence to the stationary state. In order to overcome that lack of convergence, the usual strategy is to try to prove some functional inequality of the kind

$$\mathcal{D}(y) \ge \Theta(\mathcal{H}(y|\bar z)), \quad \mathcal{H}(y|\bar z) := \mathcal{H}(y) - \mathcal{H}(\bar z), \quad \mathcal{H}(\bar z) = \mathcal{H}_z,$$

for some function $\Theta \in C^1(\mathbb{R};\mathbb{R})$, $\Theta(s) > 0$ for $s \ne 0$, $\Theta(0) = 0$. But that is another story ... .

2. Elements of spectral analysis in an Hilbert space

For linear (or linearized) equation (and its associated continuous semigroup of linear operators) there exists an efficient way to understand the long time asymptotic behaviour of the solutions. That consists in making the spectral analysis of the corresponding operator and then deduce the spectral analysis of the associated semigroup (when the spectral mapping theorem applies). We present now two situations where an accurate spectral analysis of unbounded operators can be done without too much difficulty (but we will not present full detail for the sake of conciseness). Because we deal with spectrum which is a complex plane subset, we assume that X is a complex Banach space. Otherwise, starting from a real Banach $X_{\mathbb{R}}$, we may associate in a natural way the complex Banach space $X_{\mathbb{C}}$ defined by

$$X_{\mathbb{C}} = X_{\mathbb{R}} + i\, X_{\mathbb{R}} = \{f = g + ih;\ g, h \in X_{\mathbb{R}}\}.$$

Both (real and complex) Banach spaces are just denoted by X.

2.1. **Generator with compact resolvent.** We denote by $\mathscr{B}(X)$ the space of linear and bounded operators. For $T \in \mathscr{B}(X)$, we denote $\mathrm{Im}T = R(T) = T(X)$ the range of T and $\ker T = N(T) = T^{-1}(0)$ the null space of T. For $Y \subset X$ we denote

$$Y^\perp := \{y \in X';\ \langle y, x\rangle = 0\ \forall\, x \in X\} \subset X'.$$

For $T \in \mathscr{B}(X)$, we denote $T^* \in \mathscr{B}(X')$ the adjoint operator defined through the relation

$$\langle T^* y, x\rangle = \langle y, Tx\rangle \quad \forall\, x \in X,\, y \in X'.$$

Definition 2.1. *We say that a bounded operator $T \in \mathscr{B}(X)$ is compact, we note $T \in \mathscr{K}(X)$, if $T(B_X)$ is relatively compact in X, or equivalently, if for any bounded sequence (u_n) of X we may extract a subsequence of (Tu_n) such that this one converges.*

Exercise 2.2. *Show that for an operator $T \in \mathscr{B}(H)$ in an Hilbert space H, so that we make the identification $H' = H$, we have $T \in \mathscr{K}(H)$ iff $T^* \in \mathscr{K}(H)$.*

Theorem 2.3. (Fredholm alternative [1, Théorème VI.6]**).** *Consider $T \in \mathcal{K}(X)$ in a Banach space X. Then*

(a) $ker(I-T)$ is finite dimensional;

(b) $Im(I-T)$ is closed and $Im(I-T) = (ker(I-T^*))^\perp$;

(c) $ker(I-T) = \{0\}$ if, and only if, $Im(I-T) = X$;

(d) $\dim ker(I-T) = \dim ker(I-T^*)$.

The Fredholm alternative is essentially a consequence of the Riesz Theorem about the not compactness of the unit ball in an infinite dimension normed space.

For $T \in \mathscr{B}(X)$, we define the resolvent set

$$\rho(T) := \{\lambda \in \mathbb{C};\ T - \lambda \text{ is a bijection on } X\} \subset \mathbb{C}.$$

It is worth emphasising that because the good framework to look for eigenvalues and spectrum is the complex numbers framework (as in the finite dimensional case with matrix) we will consider X as a Banach space on the scalar field of complex numbers. When we start with a Banach space $X = X^{(\mathbb{R})}$ on the scalar field of real numbers, we define the new vectors space $X^{(\mathbb{C})} := X^{(\mathbb{R})} + iX^{(\mathbb{R})}$ and for any real operator $\mathcal{L} \in \mathscr{B}(X^{(\mathbb{R})})$ we immediately extend it (by linearity) to $\mathscr{B}(X^{(\mathbb{C})})$. We will note specify the scalar field on which the vectors filed is based in general.
As a direct consequence of the fact that $I - U$ is invertible if $U \in \mathscr{B}(X)$ and $\|U\| < 1$, just because then the inverse is given by the Neumann series

$$(I-U)^{-1} = \sum_{k=0}^{\infty} U^k,$$

we get that the resolvent set is open. We define the spectrum $\Sigma(T)$ as the complementary set

$$\Sigma(T) := \mathbb{C} \backslash \rho(T).$$

It is a closed, nonempty and bounded set, more precisely $\Sigma(T) \subset B(0, \|T\|)$. We say that ξ is an eigenvalue (or a punctual spectral value) and we note $\xi \in \Sigma_P(T)$ if $N(T-\xi) \neq 0$ (so that in particular $\xi \notin \rho(T)$, or in other words $\Sigma_P(T) \subset \Sigma(T)$). For $\xi \in \Sigma_P(T)$, we say that $N(T-\xi)$ is the associated eigenspace. The dimension of $N(T-\xi)$ is called the *geometric multiplicity* of ξ. If it is equal to one, we say that ξ is geometrically simple. For $\xi \in \Sigma_P(T)$, consider M_ξ the larger subspace of X such that M_ξ is invariant under the action of T and $\Sigma(T_{|M_\xi}) = \{\xi\}$. We call *algebraic multiplicity* of ξ the dimension of M_ξ. The algebraic multiplicity is then larger than the geometric multiplicity. We say that $\xi \in \Sigma_P(T)$ is semisimple if the algebraic multiplicity and the geometric multiplicity are the same, and it is simple if they are both equal to 1.

Theorem 2.4. (Spectrum of a compact operator [1, Théorème VI.8]**).** *Consider $T \in \mathcal{K}(X)$ and assume $\dim X = +\infty$. Then there hold*

(a) $0 \in \Sigma(T)$;

(b) $\Sigma(T)\backslash\{0\}$ is empty, finite or is a sequence which tends to 0;

(c) $\Sigma(T)\backslash\{0\} = \Sigma_P(T)\backslash\{0\}$ and the algebraic multiplicity of any eigenvalue is finite.

The proof of this theorem uses the Riesz Theorem (points (a) and (b)) and the Fredholm alternative (point (c)).

We consider now the generator $\mathcal{L}$ of a semigroup of linear and continuous operators $S_t = S_{\mathcal{L}}(t)$ on a Banach space X. Similarly as for bounded operators, we define the resolvent set $\rho(\mathcal{L})$ and the resolvent operators $R_{\mathcal{L}}(z)$, $z \in \rho(\mathcal{L})$, by

$$\rho(\mathcal{L}) := \{z \in \mathbb{C};\ (\mathcal{L} - z) \text{ invertible and } R_{\mathcal{L}}(z) := (\mathcal{L} - z)^{-1} \in \mathscr{B}(X)\}.$$

We then define again the spectrum $\Sigma(\mathcal{L})$ as the complementary of the resolvent set.

The next "particular case of spectral mapping theorem" makes possible to reduce the spectral analysis of a generator $\mathcal{L}$ to the spectral analysis of one of its revolvent $R_{\mathcal{L}}(z_0)$, $z_0 \in \rho(\mathcal{L})$.

Exercise 2.5. *(a) Prove the resolvent identity*

$$R_{\mathcal{L}}(z) - R_{\mathcal{L}}(z_0) = (z - z_0) R_{\mathcal{L}}(z) R_{\mathcal{L}}(z_0) \quad \forall\, z, z_0 \in \rho(\mathcal{L}). \tag{2.1}$$

(Hint. Use the definition and nothing else).
(b) Prove that $R_{\mathcal{L}^*}(z) = R_{\mathcal{L}}(z)^*$ for any $z \in \rho(\mathcal{L}^*) = \rho(\mathcal{L})$.

Proposition 2.6. *For any (unbounded) operator $\mathcal{L}$, the following identity holds*

$$\forall\, z \in \rho(\mathcal{L}) \quad \Sigma(\mathcal{L}) = \{z + \xi^{-1};\ \xi \in \Sigma(\mathcal{R}_{\mathcal{L}}(z)) \backslash \{0\}\}.$$

Proof of Proposition 2.6. Consider an (unbounded) operator T such that $0 \in \rho(T)$, and then $T^{-1} \in \mathscr{B}(X)$. For any $\xi \in \rho(T)$, we have

$$T R_T(\xi) = I + \xi R_T(\xi) \in \mathscr{B}(X).$$

As a consequence, if furthermore $\xi \neq 0$, using the above identity and the fact that T and $R_T(\xi)$ commute, we get

$$(T^{-1} - \xi^{-1})\,(-\xi T R_T(\xi)) = (-\xi T R_T(\xi))\,(T^{-1} - \xi^{-1}) = I.$$

That means that $(T^{-1} - \xi^{-1})$ is invertible and then $\xi^{-1} \in \rho(T^{-1})$. On the other way round, if $\xi \neq 0$, $\xi^{-1} \in \rho(T^{-1})$, we may write

$$T - \xi = (\xi^{-1} - T^{-1})\xi T$$

where the right hand side term is invertible, so that $\xi \in \rho(T)$. As a conclusion, we have shown that for an invertible (unbounded) operator T, we have

$$\Sigma(T^{-1}) \backslash \{0\} = \{1/\xi;\ \xi \in \Sigma(T)\},$$

and then

$$\Sigma(T) = \{1/\xi;\ \xi \in \Sigma(T^{-1}) \backslash \{0\}\}.$$

Observing that for any $z \in \rho(\mathcal{L})$, we have $\Sigma(\mathcal{L}) = \Sigma(\mathcal{L} - z) + z$, we immediately conclude by applying the previous identity with $T := \mathcal{L} - z$. □

In order to simplify the discussion we will restrict the analysis to the generator $\mathcal{L}$ of a semigroup $S_{\mathcal{L}}$. We may then assume that the semigroup satisfies the growth bound

$$\|S_{\mathcal{L}}(t)\|_{\mathscr{B}(X)} \leq M\, e^{bt} \tag{2.2}$$

for some $M \geq 1$ and $b \in \mathbb{R}$. For any $a \in \mathbb{R}$, we define the half complex plane

$$\Delta_a := \{z \in \mathbb{C};\ \Re e\, z > a\}.$$

Lemma 2.7. *Consider a semigroup $S = S_\Lambda$ on a Banach space X which satisfies the growth bound* (2.2)*. Then for any $z \in \Delta_b$, the operator*

$$\mathcal{R}_\Lambda(z) := -\int_0^\infty S_\Lambda(t)\, e^{zt}\, dt$$

is well defined in $\mathscr{B}(X)$, $R(\mathcal{R}_\Lambda(z)) = D(\Lambda)$ and it is the resolvent operator of Λ, namely

$$\mathcal{R}_\Lambda(z)\,(\Lambda - z) = I_{D(\Lambda)}, \quad (\Lambda - z)\,\mathcal{R}_\Lambda(z) = I_X.$$

In other words, $\Lambda - z$ is invertible, with inverse $(\Lambda - z)^{-1} = \mathcal{R}_\Lambda(z)$. In particular, $\rho(\mathcal{L}) \supset \Delta_b \neq \emptyset$.

Proof of Lemma 2.7. Thanks to the growth estimate, we can define

$$\Delta_b \to \mathscr{B}(X),\ z \mapsto \mathcal{U}_z := \int_0^\infty e^{-z\,t} S(t)\, dt.$$

From the semigroup property of S, for any $h > 0$ and $f \in X$, we have

$$\frac{S(h)\mathcal{U}_z f - \mathcal{U}_z f}{h} = -\frac{e^{zh}}{h}\int_0^h e^{-zt}\, S(t)\, f\, dt + \left(\frac{e^{zh} - 1}{h}\right)\int_0^\infty e^{-zt}\, S(t) f\, dt.$$

Passing to the limit $h \to 0^+$, we get

$$\lim_{h \to 0^+} \frac{S(h)\mathcal{U}_z f - \mathcal{U}_z f}{h} = -f + z\mathcal{U}_z\, f,$$

which in turn implies $\mathcal{U}_z f \in D(\Lambda)$ and $\Lambda(\mathcal{U}_z f) = -f + z\mathcal{U}_z\, f$. In other words,

$$\forall\, f \in X, \qquad (\Lambda - z)\,(-\mathcal{U}_z f) = f. \tag{2.3}$$

On the other hand, if $f \in D(\Lambda)$, we have

$$\begin{aligned}\Lambda \mathcal{U}_z f &= \Lambda \int_0^\infty e^{-z\,t}\, S(t) f\, dt = \int_0^\infty e^{-z\,t}\, \Lambda S(t) f\, dt \\ &= \int_0^\infty e^{-z\,t}\, S(t)\Lambda f\, dt = \mathcal{U}_z(\Lambda f).\end{aligned}$$

Using that commutative relation in (2.3), we find

$$\forall\, f \in D(\Lambda), \qquad (-\mathcal{U}_z)(\Lambda - z) f = f.$$

Together, these identities imply that $\Lambda - z$ is invertible, with inverse $-\mathcal{U}_z \in \mathscr{B}(X)$. □

Lemma 2.8. (Generator with compact resolvent). *Consider a generator $\mathcal{L}$ such that $R_\mathcal{L}(z_0)$ is compact for some $z_0 \in \rho(\mathcal{L})$. Then $R_\mathcal{L}(z)$ is a compact operator for any $z \in \rho(\mathcal{L})$. We say that $\mathcal{L}$ is a generator with compact resolvent.*

Proof of Lemma 2.8. Use the resolvent identity (2.1). □

Theorem 2.9. (Spectrum for generator with compact resolvent). *Consider a Banach space X and a generator $\mathcal{L}$ with compact resolvent. Then*

$$\Sigma(\mathcal{L}) = \Sigma_P(\mathcal{L}) = \{\lambda_n;\, n \in I\},$$

where I is empty, finite or equals to $\mathbb{N}$, and in that last case (λ_n) is a sequence of decreasing (in the sense that $(\Re e\lambda_n)$ is decreasing) eigenvalues such that $\lambda_n \to -\infty$ (in the sense that $\Re e\lambda_n \to -\infty$).

Proof of Theorem 2.9. Take $x_0 > b$ given by (2.2) so that $x_0 \in \rho(\mathcal{L})$ thanks to lemma 2.7 and $R_\mathcal{L}(x_0)$ is compact thanks to Lemma 2.8. Theorem 2.4-(c) implies

$$\Sigma(R_\mathcal{L}(x_0))\backslash\{0\} = \Sigma_P(R_\mathcal{L}(x_0))\backslash\{0\},$$

there exists I empty, finite or $I = \mathbb{N}$ such that $(\mu_n)_{n\in I}$ is the set of eigenvalues of $\Sigma(R_\mathcal{L}(x_0))$ and any eigenvalue has finite algebraic multiplicity. More precisely,

$$\Sigma(R_\mathcal{L}(x_0))\backslash\{0\} = \Sigma_P(R_\mathcal{L}(x_0))\backslash\{0\} = \{\mu_n;\ n \in I\} \subset \mathbb{C},$$

with $\mu_n \to 0$ as $n \to \infty$ if $I = \mathbb{N}$. We only consider this case now. Denoting $\Sigma(\mathcal{L}) = \{\lambda_n;\, n \in J\}$, we have

$$\lambda_n = x_0 + \mu_n^{-1}, \quad \forall\, n, \tag{2.4}$$

thanks to Proposition 2.6. We deduce that $J = \mathbb{N}$. Thanks to Lemma 2.7, we also deduce

$$\Re e\mu_n^{-1} = \Re e\lambda - x_0 \le b - x_0 < 0, \quad \forall\, n \in \mathbb{N}.$$

We can then choose (relabeled) (μ_n) such that $(\Re e\mu_n)$ is an increasing and convergent to 0 sequence. We conclude using again (2.4). □

Example 2.10. *The case I is an empty set (or is finite set) cannot be excluded in the previous result. Here is an example. Take $H = L^2(0,1)$, $\mathcal{L}f := -f'$ with domain*

$$D(\mathcal{L}) := \{f \in H^1(0,1),\ f(0) = 0\}.$$

It is clear that $\mathcal{L}$ is dissipatif, since

$$(\mathcal{L}f, f) = -\int_0^1 f'\, f\, dx = \frac{1}{2}[f(0)^2 - f(1)^2] \le 0.$$

Moreover, if $\lambda \in \mathbb{C}$ and $g \in L^2(0,1)$, the unique solution to

$$-f' - \lambda f = \mathcal{L}f - \lambda f = g,$$

with $f \in D(\mathcal{L})$, is given by

$$f(x) = (R_\mathcal{L}(\lambda)g)(x) = -\int_0^x \exp(\lambda(y - x)) g(y)\, dy,$$

so that $\lambda \in \rho(\mathcal{L})$ for any $\lambda \in \mathbb{C}$. Therefore $\Sigma(\mathcal{L}) = \emptyset$ and $R_\mathcal{L}(\lambda)$ is a compact operator.

2.2. **Self-adjoint operator.** Consider an Hilbert space H and make the identification $H' = H$. In such a way, $T^* \in \mathscr{B}(H)$ for any $T \in \mathscr{B}(H)$. We say that T is self-adjoint if $T^* = T$.

Theorem 2.11. (Spectrum of a bounded self-adjoint operator). *Consider an Hilbert space H and a self-adjoint operator $T \in \mathscr{B}(H)$. We define*

$$m := \inf_{u \in H, \, |u|=1} (Tu, u), \qquad M := \sup_{u \in H, \, |u|=1} (Tu, u).$$

Then $\Sigma(T) \subset [m, M]$, $m, M \in \Sigma(T)$. In particular, $T = 0$ if furthermore $\Sigma(T) = \{0\}$. Moreover, any eigenvalue $\lambda \in \Sigma_P(T)$ is semisimple.

Elements of the proof of Theorem 2.11. We only deal with the supremum value M. In order to prove the same result for the infimum value m we just have to change T in $-T$. We split the proof into four steps.

Step 1. Consider $\lambda \in \mathbb{R}$, $\lambda > M$, and observe that the bilinear form on H defined by $a_\lambda(u, v) = (\lambda \, u - T \, u, v)$ satisfies

$$\forall \, u \in H \qquad a_\lambda(u, u) = \lambda \, |u|^2 - M \, |u|^2 \geq \alpha \, |u|^2, \quad \alpha := \lambda - M > 0.$$

Thanks to the Lax-Milgram theorem we deduce that for any $f \in H$ the equation $\lambda \, u - T \, u = f$ has an unique solution. In other words, $\lambda \in \rho(T)$, and then $]M, \infty[\, \subset \rho(T)$.

Step 2. In the case $\xi \in \mathbb{C} \backslash \mathbb{R}$, we use a complex version of the Lax-Milgram theorem together with the identity

$$\|(T - \xi)u\|^2 = \|(T - \Re e \xi)u\|^2 + (\Im m \xi)^2 \|u\|^2,$$

in order to deduce that $\xi \in \rho(T)$.

Step 3. Let us show that $M \in \Sigma(T)$. The bilinear form a_M being symmetric and linear, the Cauchy-Schwarz inequality writes

$$|(M \, u - T \, u, v)| \leq (M \, u - T \, u, u)^{1/2} \, (M \, v - T \, v, v)^{1/2} \leq C \, (M \, u - T \, u, u)^{1/2} \, |v| \quad \forall \, u, \, v \in H,$$

and then

$$|M \, u - T \, u| \leq C \, (M \, u - T \, u, u)^{1/2} \qquad \forall \, u \in H. \tag{2.5}$$

Consider a sequence (u_n) such that $|u_n| = 1$ and $(T \, u_n, u_n) \to M$. From (2.5), we deduce that $|M \, u_n - T \, u_n| \to 0$. Assuming by contradiction that $M \in \rho(T)$, we would get $u_n = (M - T)^{-1} \, (M \, u_n - T \, u_n) \to 0$ which is absurd.

Step 4. In the case $\Sigma(T) = \{0\}$, the already proved properties imply that $(Tu, u) = 0$ for any $u \in H$. We deduce

$$(Tu, v) = \frac{1}{2} \big((T(u + v), u + v) - (Tu, u) - (Tv, v) \big) = 0 \quad \forall \, u, v \in H,$$

and then $T = 0$.

Step 5. Consider $\lambda \in \Sigma_P(T)$ and prove that $N((T - \lambda)^k) = N(T - \lambda)$ for any $k \geq 1$. We may assume $\lambda = 0$ and we just have to prove that $N(T^2) = N(T)$ since then $N(T^{2^k}) = N(T)$ for any $\ell \geq 1$ and $N(T^k) \subset N(T^\ell)$ if $\ell \geq k$. Now, if $u \in N(T^2)$, we have $|Tu|^2 = (T^2 u, u) = 0$, so that $u \in N(T)$. In other words, $N(T^2) \subset N(T)$. □

Theorem 2.12. (Spectral decomposition of a compact self-adjoint operator). *Consider a separable Hilbert space H and a self-adjoint compact operator $T \in \mathcal{L}(H)$. Then there exists an Hilbert basis made up of a sequence of eigenvectors of T.*

Elements of the proof of Theorem 2.12. We introduce E_n, $n \in \mathbb{N}$, the family of eigenspaces (the eigenvalues are semisimple) associated to the family of eigenvalues (λ_n), $\lambda_0 = 0$ and $\lambda_n \to 0$ if that family is not finite. We observe that $E_i \perp E_j$ for any $i \neq j$. We define F the space generated by the (E_n) and we show that $T_0 := T_{|F^\perp}$ is self-adjoint compact and that $\Sigma(T_0) = \{0\}$. That implies that $T_0 = 0$ and then $\bar{F} = H$. □

Exercise 2.13. *(a) Prove in all details Theorem 2.12.*
(b) Prove that $R_\mathcal{L}$ is self-adjoint if $\mathcal{L}$ is self-adjoint. (Hint. Use the resolvent identity (2.1)*).*

Theorem 2.14. (Spectrum and semigroup spectral gap for self-adjoint generator). *Consider an infinite dimensional and separable Hilbert space H and a self-adjoint generator $\mathcal{L}$ with compact resolvent. Then*

$$\Sigma(\mathcal{L}) = \Sigma_P(\mathcal{L}) = \{\lambda_n;\, n \ge 1\},$$

where (λ_n) is a sequence of real, strictly decreasing and semisimple eigenvalues such that $\lambda_n \to -\infty$. Furthermore, the functional $\mathcal{H}(f) := \|(I - \Pi_1) f\|^2$ is a Lyapunov function, where Π_1 stands for the orthogonal projection on the eigenspace associated to the eigenvalue λ_1, and

$$\|e^{t\mathcal{L}}(I - \Pi_1)\|_{\mathscr{B}(H)} \le e^{t\lambda_2} \, \|I - \Pi_1\|_{\mathscr{B}(H)} \qquad \forall\, t \ge 0.$$

Proof of Theorem 2.14. We split the proof into two parts.

Step 1. Take $x_0 > b := \omega(\mathcal{L})$. Using that $R_{\mathcal{L}}(x_0)$ is self-adjoint (see exercise 2.13) and Theorem 2.12, we deduce that the family $(\mu_n)_{n \in I}$ built in Theorem 2.9 is made of real numbers. Moreover, $I = \mathbb{N}$ because each eigenvalue μ_n has finite multiplicity (Theorem 2.9), the family of associated eigenspace (E_n) generates a dense space in H (Theorem 2.12) and the space is infinite dimensional. As a consequence

$$\Sigma(\mathcal{L}) = \{\lambda_n;\, n \ge 1\}, \quad \lambda_n = x_0 + \mu_n^{-1} \in \mathbb{R},$$

with $\lambda_n \searrow -\infty$ because $\mu_n \nearrow 0$.

Step 2. As in Theorem 2.12, we introduce E_n, $n \in \mathbb{N}^*$, the sequence of (finite dimensional) eigenspaces associated to the sequence of eigenvalues (λ_n). We define Π_n as the orthogonal projection on E_n and the Bessel-Parseval equality writes

$$\forall\, f \in D(\mathcal{L}) \quad \|f\|^2 = \sum_{n \ge 1} \|\Pi_n f\|^2.$$

From the fact that $\Sigma(\mathcal{L}_{E_n}) = \{\lambda_n\}$ and λ_n is semisimple, we deduce for any $f \in D(\mathcal{L})$ and with the notation $f_t := S_{\mathcal{L}}(t) f$ that

$$\begin{aligned} \frac{d}{dt}\|(I - \Pi_1) f_t\|^2 &= \sum_{n \ge 2} \frac{d}{dt}\|\Pi_n f_t\|^2 = 2 \sum_{n \ge 2} (\Pi_n \mathcal{L} f_t, \Pi_n f_t) \\ &= 2 \sum_{n \ge 2} \lambda_n \|\Pi_n f_t\|^2 \le 2 \sum_{n \ge 2} \lambda_2 \|\Pi_n f_t\|^2 \\ &\le 2\lambda_2 \|(I - \Pi_1) f_t\|^2. \end{aligned}$$

We conclude thanks to the Gronwall lemma. □

2.3. **Krein-Rutman theorem for dissipative generator with compact resolvent.** In this section we present a version of the Krein-Rutman theorem for positive operator. We split the section into three parts. In a first part, we introduce the notion of Banach lattice, positive operator, Kato's inequality, weak and strong maximum and we state the Krein-Rutman theorem. In a second part, we prove the existence part of the theorem by exhibiting a positive eigenvector. In a third part, we establish the qualitative properties, and provide then a characterization, of the positive eigenvector.

2.3.1. *Statement of the Krein-Rutman theorem.* The Hilbert space $H = L^2(\mathscr{U}, d\mu)$ is endowed with a (partial) order, denoted by $\ge$ (or $\le$), defined by

$$\text{for } f, g \in H \qquad (f \ge g) \quad \text{iff} \quad (f(x) \ge g(x) \text{ for } \mu\text{-a.e. } x \in \mathcal{U}).$$

That order structure is compatible with the norm on H and for any $f \in H$, we may write $f = f_+ - f_-$ with $f_\pm(x) := \max(\pm f(x), 0)$, so that H is a *Banach lattice.*

We recall that a Banach lattice X is a Banach space endowed with an order such that:
- (i) The set $X_+ := \{f \in X;\ f \ge 0\}$ is a nonempty convex closed cone.
- (ii) For any $f \in X$, there exist some unique minimal $f_\pm \in X_+$ such that $f = f_+ - f_-$, we then denote $|f| := f_+ + f_- \in X_+$.
- (iii) For any $f, g \in X$, $0 \le f \le g$ implies $\|f\| \le \|g\|$.
- (iv) X contains at least one strictly positive element (definite below) and X has a sign operator. More precisely, for any $f \in X_{\mathbb{R}}$ there exists $(\operatorname{sign} f) \in \mathscr{B}(X)$ which fulfils

$$|(\operatorname{sign} f)\, g| \le |g|, \quad \forall\, g \in X, \qquad (\operatorname{sign} f)\, f = |f|.$$

We may define a dual order $\geq$ (or $\leq$) on X' by writing for $\psi \in X'$

$$\psi \geq 0 \ \text{(or } \psi \in X'_+) \quad \text{iff} \quad \forall f \in X_+ \ \langle \psi, f \rangle \geq 0.$$

One may show that X' is also a Banach lattice for that definition of order.

Exercise 2.15. *(a) In the Hilbert space $H = L^2$ show that the duality definition of order in H' is nothing but the pointwise definition of order in L^2*
(b) For $\psi \in X'$, we define $\psi_\pm$ by

$$\psi_+(f) := \sup\{\psi(g);\ 0 \leq g \leq f\} \quad \text{for} \quad f \geq 0,$$

$\psi_+(f) = \psi_+(f_+) - \psi_+(f_-)$ for $f \in X$ and $\psi_- = (-\psi)_+$. Show that $\psi = \psi_+ - \psi_-$ and that $\psi_\pm$ are minimal elements in X'_+ which fulfills such a splitting.

In the space of functions $H = L^2(\mathscr{U}, d\mu)$, we may define without difficulty the composition functions $\theta(f)$ and then the sign function as

$$(\text{sign} f)h := \theta'(s) \cdot h = \frac{1}{2|f|}(f\bar{h} + \bar{f}h).$$

Also observe that in any Hilbert space

$$f \geq 0 \text{ implies } f^* = f \geq 0.$$

Definition 2.16. *Let us consider a Banach lattice X and an bounded operator $T \in \mathcal{B}(X)$. We say that T is positive, we note $T \geq 0$, if $Tf \in X_+$ for any $f \in X_+$.*

Exercise 2.17. *1. Show that for a bounded operator $T \in \mathcal{B}(H)$ in a Hilbert space H, there holds $T \geq 0$ implies $T^* \geq 0$.*
2. Show that for an operator $T \in \mathcal{B}(X)$, $X = L^2$, such that $Tg > 0$ a.e. if $g \in X_+ \backslash \{0\}$, then if f is an eigenfunction associated to the eigenvalue $\lambda := \|T\|$, we also have $T|f| = \lambda |f|$. (Hint. Use that $(Tf, f) = \lambda \|f\|^2$ and that $(Tg, g) \leq \lambda \|g\|^2$ for $g = f_\pm$ in order to prove $f_+ = 0$ or $f_- = 0$).

Definition 2.18. *On a Hilbert space H we say that $\mathcal{L}$ is a dissipative generator if it generates a semigroup $S_\mathcal{L}$ and satisfies one of the two equivalent additional property*
(i) $\Re e\langle f, \mathcal{L}f \rangle \leq \gamma \|f\|^2$ for any $\gamma > b$;
(ii) $\|S_\mathcal{L}(t)\|_{\mathscr{B}(H)} \leq e^{\gamma t}$ for any $\gamma > b$.
We denote

$$\omega_d(\mathcal{L}) := \inf\{b \in \mathbb{R};\ (ii) \ holds\} = \inf\{b \in \mathbb{R};\ (i) \ holds\}. \tag{2.6}$$

Definition 2.19. *Let us consider a Banach lattice X and a generator $\mathcal{L}$ of a semigroup $S_\mathcal{L}$ on X.*
(a) - We say that the semigroup $S_\mathcal{L}$ is positive if $S_\mathcal{L}(t)$ is positive for any $t \geq 0$.
(b) - We say that a generator $\mathcal{L}$ on X satisfies Kato's inequality if

$$\forall f \in D(\mathcal{L}) \quad \mathcal{L}|f| \geq (sign f) \cdot (\mathcal{L}f). \tag{2.7}$$

(c) - We say that a generator $\mathcal{L}$ satisfies a "weak maximum principle" if for any $a > \omega(\mathcal{L})$ and $g \in X_-$ there holds

$$f \in D(\mathcal{L}) \ and \ (\mathcal{L} - a)f = g \quad imply \quad f \geq 0. \tag{2.8}$$

(d) - For a generator $\mathcal{L}$, we say that the opposite of the resolvent is a positive operator if for any $a > \omega(\mathcal{L})$ and $g \in X_+$ there holds $-R_\mathcal{L}(a)g \in X_+$.

Here the correct way to understand Kato's inequality (2.9) is

$$\forall f \in D(\mathcal{L}),\ \forall \psi \in D(\mathcal{L}^*) \cap X'_+ \quad \langle |f|, \mathcal{L}^*\psi \rangle \geq \langle (\text{sign} f) \cdot (\mathcal{L}f), \psi \rangle.$$

The definitions of dissipative operator $\mathcal{L}$ and "dissipative growth rate" $\omega_d(\mathcal{L})$ are presented in Section III.5.3. It is worth emphasize that in the case $X = L^2$, we clear have that Kato's inequality also implies

$$\forall f \in D(\mathcal{L}) \cap X_\mathbb{R} \quad \mathcal{L}f_+ \geq \theta'(f)(\mathcal{L}f), \tag{2.9}$$

with $\theta(s) := s_+$ and $\theta'(s) = \mathbf{1}_{s>0}$. Indeed, we just have to remark that $s_+ = (|s| + s)/2$.

In that simple framework of dissipative generators in the Hilbert space L^2 one can show that all the preceding properties are equivalent when we replace the growth bound $\omega(\mathcal{L})$ by the dissipative growth bound $\omega_d(\mathcal{L})$. We only show the following.

Lemma 2.20. *In the Hilbert space $H = L^2$ and for a dissipative generator $\mathcal{L}$, Kato's inequality (b) implies the positivity properties (a), (c) and (d).*

Proof. *Step 1. (b) implies (a).* Consider $f_0 \le 0$ and denotes $f := S_\mathcal{L}(t) f_0 \, e^{-at}$. We compute successively

$$\partial_t f = (\mathcal{L} - a) f,$$

next with $\theta(s) := s_+$ and thanks to Kato's inequality

$$\partial_t f_+ = \theta'(f) \, (\mathcal{L} - a) f \le (\mathcal{L} - a) f_+,$$

and finally with $f_+^* \in X_+$ such that $\langle f_+, f_+^* \rangle = \|f_+\|^2$,

$$\frac{1}{2} \frac{d}{dt} \|f_+\|^2 = \langle \partial_t f_+, f_+^* \rangle \le \langle (\mathcal{L} - a) f_+, f_+^* \rangle \le 0.$$

Since $f_0 \le 0$ we have $\|(f_0)_+\| = 0$, next $\|(f_t)_+\| = 0$ and finally $S_\mathcal{L}(t) f_0 = e^{at} f_t \le 0$ for any $t \ge 0$.
Step 2. (b) implies (c). Consider $a \in \mathbb{R}$, $f, g \in X$ as required in (2.8). From Kato's inequality, we have with $\theta(s) = s_+$

$$(\mathcal{L} - a) f_+ \ge (\mathcal{L} f) \, \theta'(f) - a f_+ = (a f - g) \, \theta'(f) - a f_+ = (-g) \, \theta'(f) \ge 0.$$

By definition of $\omega_d(\mathcal{L})$ and because $f_+^* = f_+ \ge 0$, we deduce that for any $b \in (\omega(\mathcal{L}), a)$

$$0 \le \langle (\mathcal{L} - a) f_+, f_+^* \rangle \le (b - a) \, \|f_+\|^2,$$

so that $f_+ = 0$ and then $f \le 0$.
Step 3. The implication (c) $\Rightarrow$ (d) is just straightforward. □

Last, we need some strict positivity notion on X and some strict positivity (or irreducibility) assumption on $\mathcal{L}$ that we will formulate in term of *"strong maximum principle"*. For that purpose, we define the strict order $>$ (or $<$) on X by writing for $f \in X$

$$f > 0 \quad \text{iff} \quad \forall \, \psi \in X_+ \backslash \{0\} \ \ \langle \psi, f \rangle > 0,$$

and similarly a strict order $>$ (or $<$) on X' by writing for $\psi \in X'$

$$\psi > 0 \quad \text{iff} \quad \forall \, g \in X'_+ \backslash \{0\} \ \ \langle \psi, g \rangle > 0.$$

It is worth emphasizing that from the Hahn-Banach Theorem, for any $f \in X_+$ there exists $\psi \in X'_+$ such that $\|\psi\|_{X'} = 1$ and $\langle \psi, f \rangle = \|f\|_X$ from which we easily deduce that

$$\forall \, f, g \in X, \quad 0 \le f < g \quad \text{implies} \quad \|f\|_X < \|g\|_X. \tag{2.10}$$

Also notice that in the case of the Hilbert space $H := L^2$ that notion corresponds to the usual pointwise definition

$$\text{for } f, g \in H \qquad (f > g) \quad \text{iff} \quad (f(x) > g(x) \text{ for } \mu\text{-a.e. } x \in \mathcal{U}).$$

Definition 2.21. *We say that $\mathcal{L}$ satisfies a "strong maximum principle" if for any given eigenvalue $\mu \in \mathbb{R}$ and any associated eigenvector $f \in \mathcal{D}(\mathcal{L})$ the case of equality in Kato's inequality*

$$(\mathcal{L} - \mu) \theta(f) = (\mathcal{L} f - \mu f) \cdot \theta'(f) = 0$$

for $\theta(s) = |s|$, $s \in \mathbb{C}$, implies

$$f = 0 \quad \textit{or} \quad |f| > 0 \ \textit{and} \ \exists \, u \in \mathbb{C}, \ f = u|f|.$$

It is worth emphasizing that we can see the strong maximum principle as a consequence of the weak maximum principle together with the existence of a family of strictly positive barrier functions. We give a typical result which can be applied (or modified in order to be apply) in many situations.

Lemma 2.22. *We assume that*
(i) $(\mathcal{L} - a)$ satisfies the weak maximum principle;
(ii) for any $f \ge 0$, $f \ne 0$, there exists $g > 0$ such that $(g - f)_+ \in X$ and $(\mathcal{L} - a) g \ge 0$.
Then $\mathcal{L} - a$ satisfies the following strong maximum principle (in its usual form): for any $f \in X_+ \backslash \{0\}$ such that $(\mathcal{L} - a) f \le 0$ there holds $f > 0$.

Proof. We define $h := (g-f)_+ \in X$ and we remark that from Kato's inequality

$$(\mathcal{L}-a)h \geq \mathbf{1}_{h>0}(\mathcal{L}-a)(g-f) \geq 0.$$

As a consequence of the weak maximum principle, we have $h \leq 0$. That implies $h = 0$, and then $g - f \leq 0$. □

Also observe the following: that last strong maximum principle together with the (quite natural) assumption $D(\mathcal{L}^n) \subset C(\mathcal{U})$ for n large enough in the Hilbert case $H = L^2(\mathcal{U})$ implies the following other version of "strong maximum principle": for any given $f \in X$ and $\mu \in \mathbb{R}$, there holds

$$|f| \in D(\mathcal{L})\backslash\{0\} \text{ and } (\mathcal{L}-\mu)|f| \leq 0 \quad \text{imply} \quad f > 0 \text{ or } f < 0$$

We can now state the following version of the Krein-Rutman Theorem in a general and abstract setting.

Theorem 2.23. *We consider an opeartor $\mathcal{L}$ on the Hilbert space $H = L^2$ and we assume that*
(1) *$\mathcal{L}$ is dissipative with compact resolvent and generates a semigroup;*
(2) *for some $a^* \in \mathbb{R}$, there exists $\psi \in D(\mathcal{L}^*)$, $\psi > 0$, such that $\mathcal{L}^*\psi \geq a^*\psi$ or there exists $g \in D(\mathcal{L})$, $g > 0$, such that $\mathcal{L}g \geq a^* g$;*
(3) *$\mathcal{L}$ satisfies Kato's inequality;*
(4) *$\mathcal{L}$ satisfies a strong maximum principle.*
Defining $\lambda_1 = s(\mathcal{L}) := \sup\{\Re e\xi,\ \xi \in \Sigma(\mathcal{L})\}$ and $\Delta_a := \{\xi \in \mathbb{C},\ \Re e\xi > a\}$, there holds

$$\exists\, a \in [a^*, \lambda_1), \quad \Sigma(\mathcal{L}) \cap \Delta_a = \{\lambda_1\} \qquad \text{and} \qquad \lambda_1 \text{ is simple},$$

and there exists $0 < f_1 \in D(\mathcal{L})$ and $0 < \phi \in D(\mathcal{L}^*)$ such that

$$\mathcal{L} f_1 = \lambda_1\, f_1, \quad \mathcal{L}^*\phi = \lambda_1\, \phi, \quad R\Pi_{\mathcal{L},\lambda} = Vect(f_1),$$

and then

$$\Pi_{\mathcal{L},\lambda} f = \langle f, \phi\rangle\, f_1 \quad \forall\, f \in X.$$

2.3.2. *Proof of the Krein-Rutman theorem - existence part.*

Proposition 2.24. *Consider a Banach lattice X and an operator $T \in \mathscr{B}(X)$. We assume*
(1) $T \geq 0$ and T is compact.
(2) $\exists\, \psi \in X'$, $\psi > 0$, $\exists\, \alpha > 0$ such that $T^*\psi \geq \alpha\, \psi$.
Then, there exists $\mu_1 \geq \alpha$, $u_1 \in X_+\backslash\{0\}$ such that $Tu_1 = \mu_1 u_1$.

Proof of Proposition 2.24. Step 1. A modified problem. We define $\mathcal{B}_+ := \{f \in X;\ f \geq 0,\ \|f\|_X \leq 1\}$ and we fix $g \in \mathcal{B}_+$ such that $Tg \neq 0$ (that is possible since $T^* \neq 0$ and then $T \neq 0$) and $\varepsilon \in (0,1)$. We observe that thanks to the positivity property of T, for $f \in \mathcal{B}_+$, there holds

$$+\infty > (1+\varepsilon)\, \|T\|_{\mathscr{B}(X)} \geq \|T(f+\varepsilon g)\|_X \geq \|T(\varepsilon g)\|_X > 0.$$

Thanks to that lower bound we may define for any $\varepsilon > 0$ the continuous mapping

$$\Phi_\varepsilon : \mathcal{B}_+ \to \mathcal{B}_+, \quad \Phi_\varepsilon(f) := T(f+\varepsilon\, g)/\|T(f+\varepsilon\, g)\|_X.$$

The Schauder's fixed point Theorem 7.1 implies that the function Φ_ε has at least one fixed point G_ε which then satisfies

$$G_\varepsilon \in \mathcal{B}_+, \quad T(G_\varepsilon + \varepsilon g) = \mu_\varepsilon\, G_\varepsilon, \ \ \mu_\varepsilon := \|T(G_\varepsilon + \varepsilon g)\|_X, \tag{2.11}$$

and in particular $\|G_\varepsilon\|_X = 1$.

Step 2. A stability argument. On the one hand, by definition, $\mu_\varepsilon \leq (1+\varepsilon)\|T\|_{\mathscr{B}(X)}$. On the other hand, by hypothesis (2) and because $g \geq 0$, we have

$$\alpha\, \langle G_\varepsilon, \psi\rangle \leq \langle G_\varepsilon, T^*\psi\rangle \leq \langle T\,(G_\varepsilon + \varepsilon g), \psi\rangle = \mu_\varepsilon \langle G_\varepsilon, \psi\rangle,$$

and then $\mu_\varepsilon \geq \alpha$. Up to the extraction of a subsequence, we get $\mu_\varepsilon \to \mu \in [\alpha, \|T\|_{\mathscr{B}(X)}]$, and therefore $\mu \neq 0$. By compactness of T and up to the extraction of a subsequence, we also get $\mu_\varepsilon\, G_\varepsilon \to h$ in X, and therefore $G_\varepsilon \to f := h/\mu$ in X. We conclude by passing to the limit in (2.11). □

Proposition 2.25. *Consider a Banach lattice X and an (possibly unbounded) operator $\mathcal{L} \in \mathscr{G}(X)$. We assume that for $a, b \in \mathbb{R}$, $a \le b$,*
(1) $-R_{\mathcal{L}}(b) \ge 0$ (resp. $-R_{\mathcal{L}^*}(b) \ge 0$);
(2) $R_{\mathcal{L}}(b)$ is compact (resp. $R_{\mathcal{L}^*}(b)$ is compact);
(3) $\exists\, \psi \in X'$, $\psi > 0$, such that $\mathcal{L}^* \psi \ge a\, \psi$ (resp. $\exists\, g \in X$, $g > 0$, such that $\mathcal{L} g \ge a\, g$).
Then there exist $\lambda_1 \in [a, b]$, $f_1 \in X_+ \backslash \{0\}$ such that $\mathcal{L} f_1 = \lambda_1 f_1$ (resp. there exist $\lambda_1^* \in [a, b]$, $\phi \in X'_+ \backslash \{0\}$ such that $\mathcal{L}^* \phi = \lambda_1^* \phi$).

Proof of Proposition 2.25. We only prove the result concerning $\mathcal{L}$ because the proof for the dual problem is similar. We may assume $b > a$. We define $T = -R_{\mathcal{L}}(b)$ which obviously satisfies (1) in Proposition 2.24. We define $\alpha := (b-a)^{-1} > 0$ and we observe that

$$\alpha (b - \mathcal{L}^*) \psi \le \psi$$

or equivalently T satisfies (2) in Proposition 2.24. Thanks to proposition 2.24, there exist $\mu_1 \ge \alpha$ and $u_1 \in X_+ \backslash \{0\}$ such that $T u_1 = \mu_1 u_1$. Defining $f_1 = T u_1 \in X_+ \backslash \{0\}$, we get $\mu_1^{-1} f_1 = (b - \mathcal{L}) f_1$, and the result holds with $\lambda_1 := b - \mu^{-1} \in [a, b)$. □

Exercise 2.26. *Consider a Banach space X such that $X = Y'$ for a separable Banach Y and consider a generator $\mathcal{L} \in \mathscr{G}(X)$. Assume that (1) $S_{\mathcal{L}} \ge 0$;*
(2) $\exists\, \phi \in Y \subset X'$, $\phi > 0$, such that $\mathcal{L}^* \phi = \lambda \phi$;
(3) there exists $\delta > 0$ such that

$$\forall\, f \ge 0 \quad \langle (\mathcal{L} - \lambda) f, f^* \rangle \le -\delta\, \|f\|_X^2 + \delta^{-1} \langle f, \phi \rangle^2$$

Prove that there exists $G \in X_+ \backslash \{0\}$ such that $\mathcal{L} G = \lambda G$. (Hint. Use Theorem 1.2).

2.3.3. *Qualitative properties.*

Proposition 2.27. *We assume that*
(2) there exist $\lambda, \lambda^* \in \mathbb{R}$, $G \in X_+ \backslash \{0\}$, $\phi \in X'_+ \backslash \{0\}$, such that $\mathcal{L} G = \lambda G$, $\mathcal{L}^* \phi = \lambda^* \phi$;
(1) $\mathcal{L}$ satisfies Kato's inequality and the strong maximum principle.
Then
(3) $\phi > 0$, $G > 0$, $\lambda = \lambda^*$,
(4) λ_1 is simple,
(5) $\Sigma_P(\mathcal{L}) \cap \bar{\Delta}_{\lambda_1} = \{\lambda_1\}$ and λ_1 is the only eigenvalue associated to a positive eigenvector, and more precisely

$$\lambda = \sup \{ r \in \mathbb{R};\ \exists f \in D(\mathcal{L}) \cap X_+,\ f \ne 0,\ \mathcal{L} f \ge r f \}.$$

Remark 2.28. *Observe that λ_1 is the only eigenvalue associated to a positive eigenvector for the dual problem.*

Proof of Proposition 2.27. Step 1. From the strong maximum principle we obviously have $G > 0$. As a consequence,

$$\lambda^* \langle \phi, G \rangle = \langle \mathcal{L}^* \phi, G \rangle = \langle \phi, \mathcal{L} G \rangle = \lambda \langle \phi, G \rangle$$

and $\lambda^* = \lambda$. Let us prove that $\phi > 0$. For $a > s(\mathcal{L})$ and $g \in X_+ \backslash \{0\}$, thanks to the weak and strong maximum principles, there exists $0 < f \in X$ such that

$$(-\mathcal{L} + a) f = g.$$

As a consequence, we have

$$\begin{aligned} \langle \phi, g \rangle &= \langle \phi, (-\mathcal{L} + a) f \rangle \\ &= \langle (a - \mathcal{L}^*) \phi, f \rangle = (a - \lambda) \langle \phi, f \rangle > 0. \end{aligned}$$

Since $g \in X_+$ is arbitrary, we deduce that $\phi > 0$.

Step 2. We prove that $N(\mathcal{L} - \lambda) = \mathrm{vect}(f_1)$. Consider a normalized eigenfunction $f \in X^{\mathbb{R}} \backslash \{0\}$ associated to the eigenvalue λ. First we observe that from Kato's inequality

$$\lambda |f| = \lambda f \,\mathrm{sign}(f) = \mathcal{L} f \,\mathrm{sign}(f) \le \mathcal{L} |f|.$$

That inequality is in fact an equality, otherwise we should have

$$\lambda \langle |f|, \phi \rangle \ne \langle \mathcal{L} |f|, \phi \rangle = \langle |f|, \mathcal{L}^* \phi \rangle = \lambda \langle |f|, \phi \rangle,$$

and a contradiction. As a consequence, $|f|$ is a solution to the eigenvalue problem $\lambda|f| = \mathcal{L}|f|$ so that the strong maximum principle assumption implies $f > 0$ or $f < 0$, and without lost of generality we may assume $f > 0$. Now, we write thanks to Kato's inequality again

$$\lambda(f - f_1)_+ = \mathcal{L}(f - f_1)\,\mathrm{sign}_+(f - f_1) \le \mathcal{L}(f - f_1)_+,$$

and for the same reason as above that last inequality is in fact an equality. Since $(f - f_1)_+ = |(f - f_1)_+|$, the strong maximum principle implies that either $(f - f_1)_+ = 0$, or in other words $f \le f_1$, either $(f - f_1)_+ > 0$ or in other words $f > f_1$. Thanks to (2.10) and to the normalization hypothesis $\|f\| = \|f_1\| = 1$ the second case in the above alternative is not possible. Repeating the same argument with $(f_1 - f)_+$ we get that $f_1 \le f$ and we conclude with $f = f_1$. For a general eigenfunction $f \in X^{\mathbb{C}}$ associated to the eigenvalue λ we may introduce the decomposition $f = f_r + i f_i$ and we immediately get that $f_\alpha \in X^{\mathbb{R}}$ is an eigenfunction associated to λ for $\alpha = r, i$. As a consequence of what we have just established, we have $f_\alpha = \theta_\alpha f_1$ for some $\theta_\alpha \in \mathbb{R}$ and we conclude that $f = (\theta_r + i\theta_i)\, f_1 \in \mathrm{vect}(f_1)$ again.

We finally claim that λ is algebraically simple. Indeed, if it is not the case, there would exist $f \in X^{\mathbb{R}}$ such that $\mathcal{L}f = \lambda f + f_1$ and then

$$\lambda\langle f, \phi\rangle = \langle f, \mathcal{L}^*\phi\rangle = \langle \mathcal{L}f, \phi\rangle = \langle \lambda f + f_1, \phi\rangle,$$

which in turns implies $\langle f_1, \phi\rangle = 0$ and a contradiction.

Step 3. Clearly if $\mathcal{L}g \ge rg$, with $g \in X_+\backslash\{0\}$, we have

$$r\langle g, \phi\rangle \le \langle \mathcal{L}g, \phi\rangle = \langle g, \mathcal{L}^*\phi\rangle = \lambda\langle g, \phi\rangle$$

and therefore $r \le \lambda$.

We finally prove that there is no other eigenvalue but λ with real part equal and larger than λ. We consider a couple (f, μ) of eigenfunction and eigenvalue with $\Re e\mu \ge \lambda$. By Kato's inequality for the modulus function θ, we have

$$\begin{aligned} (\Re e\,\mu)|f| &= (\mu f)\cdot\theta'(f) \\ &= (\mathcal{L}f)\cdot\theta'(f) \le \mathcal{L}|f|. \end{aligned}$$

By the preceding characterization of the first eigenvalue λ, there holds $\Re e\mu \le \lambda$. As a consequence, $\Re e\mu = \lambda$ and then

$$\lambda|f| = (\mathcal{L}f)\cdot\theta'(f) = \mathcal{L}|f|.$$

On the contrary, we multiply the equation by ϕ and we get a contradiction. The strong maximum principle says that $f = u\,|f|$, $u \in \mathbb{S}^1$, $|f| > 0$, so that f is an eigenfunction associated to λ, or in other words, $\mu = \lambda$. □

We come to the proof of the Krein-Rutman theorem.

Proof of Theorem 2.23. Step 1. First from Proposition 2.25 applied to $\mathcal{L}$ we have the existence of a positive eigenfunction, namely there exist $\lambda_1 \ge a$, $f_1 \in X_+\backslash\{0\}$ such that $\mathcal{L}f_1 = \lambda_1 f_1$. Thanks to the strong maximum we have $f_1 > 0$ and Proposition 2.25 applied to $\mathcal{L}^*$ gives the existence of a positive dual eigenfunction, namely there exist $\lambda_1^* \ge \lambda_1$, $\phi \in X'_+\backslash\{0\}$ such that $\mathcal{L}^*\phi = \lambda_1^*\phi$. As a consequence, we may apply Proposition 2.27. Next, thanks to Theorem 2.14, we know that $\Sigma(\mathcal{L}) = \Sigma_P(\mathcal{L})$ is at most a countable set $\{\lambda_n,\ n \in I\}$ and $\Re e\lambda_n \to -\infty$ if $I = \mathbb{N}$. We then deduce $\Re e\lambda_n \le \Re e\lambda_2 < \lambda_1$ for any $n \ge 2$. □

3. Relative entropy for linear and positive PDE

We consider the general evolution PDE

$$\partial_t f = \Delta f - a\cdot\nabla f + cf + \int b\, f_*, \quad \int b\, f_* := \int b(x, x_*) f(x_*)\, dx_*, \quad b \ge 0. \tag{3.1}$$

If $g > 0$ is a solution

$$\partial_t g = \Delta g - a\cdot\nabla g + cg + \int b\, g_*$$

and if $\phi \ge 0$ is a solution to the dual evolution problem

$$-\partial_t \phi = \Delta\phi + \mathrm{div}(a\,\phi) + c\,\phi + \int b_*\,\phi_*, \quad \int b_*\,\phi_* := \int b(x_*, x)\,\phi(x_*)\, dx_*,$$

we can exhibit a family of entropies associated to the evolution PDE (3.1). More precisely, we establish the following result (and in fact a bit more accurate formulation of it).

Theorem 3.1. *For any real values convex function H, the functional*

$$f \mapsto \mathcal{H}(f) := \int_{\mathbb{R}^d} H(f/g)\, g\, \phi,$$

is an entropy for the evolution PDE (3.1).

Step 1. First order PDE. We assume that

$$\begin{aligned} \partial_t f &= -a \cdot \nabla f + cf \\ \partial_t g &= -a \cdot \nabla g + cg \\ -\partial_t \phi &= \operatorname{div}(a\, \phi) + c\, \phi, \end{aligned}$$

and we show that

$$\partial_t(H(X) g\phi) + \operatorname{div}(aH(X)g\phi) = 0, \quad X = f/g.$$

We compute

$$\begin{aligned} &\partial_t(H(X) g\phi) + \operatorname{div}(aH(X)g\phi) \\ &= H'(X) g\phi \left[\partial_t X + a\nabla X\right] + H(x) \left[\partial_t(g\phi) + \operatorname{div}(ag\phi)\right] \end{aligned}$$

The first term vanishes because

$$\partial_t X + a\nabla X = \frac{1}{g}\left(\partial_t f + a\nabla f\right) - \frac{f}{g^2}\left(\partial_t g + a\nabla g\right) = \frac{1}{g}\left(cf\right) - \frac{f}{g^2}\left(cg\right) = 0.$$

The second term also vanishes because

$$\partial_t(g\phi) + \operatorname{div}(ag\phi) = \phi\left[\partial_t g + a\nabla g\right] + g\left[\partial_t \phi + \operatorname{div}(a\phi)\right] = \phi\left[\, - cg\right] + g\left[\, + c\phi\right] = 0.$$

Step 2. Second order PDE. We assume that

$$\begin{aligned} \partial_t f &= \Delta f + cf \\ \partial_t g &= \Delta g + cg \\ -\partial_t \phi &= \Delta\phi + c\, \phi, \end{aligned}$$

and we show

$$\partial_t(H(X) g\phi) - \operatorname{div}(\phi \nabla(H(X) g)) + \operatorname{div}(gH(X)\nabla\phi) = -H''(X) g\phi |\nabla X|^2.$$

We first observe that

$$\begin{aligned} \Delta X &= \operatorname{div}\Big(\frac{\nabla f}{g} - f\,\frac{1}{g^2}\,\nabla g\Big) \\ &= \frac{\Delta f}{g} - 2\nabla f\,\frac{\nabla g}{g^2} + 2\, f\,\frac{|\nabla g|^2}{g^3} - \frac{f}{g^2}\,\Delta g \\ &= \frac{\Delta f}{g} - \frac{f\,\Delta g}{g^2} - 2\,\frac{\nabla g}{g}\cdot \nabla X, \end{aligned}$$

which in turn implies

$$\partial_t X - \Delta X = 2\,\frac{\nabla g}{g}\cdot \nabla X.$$

We then compute

$$\begin{aligned} &\partial_t(H(X) g\phi) - \operatorname{div}(\phi\nabla(H(X)g)) + \operatorname{div}(gH(X)\nabla\phi) = \\ &= (\partial_t H(X))\, g\phi + H(X)\, \partial_t(g\phi) - \phi \operatorname{div}[gH'(X)\nabla X + H(X)\nabla g] + gH(X)\Delta\phi \\ &= H'(X) g\phi \left\{\partial_t X - \Delta X - 2\frac{\nabla g}{g}\cdot \nabla X\right\} - g\phi\, H''(X)\, |\nabla X|^2 + H(X)\left[\partial_t(g\phi) - \phi\Delta g + g\Delta\phi\right] \\ &= -g\phi\, H''(X)\, |\nabla X|^2, \end{aligned}$$

since the first term and the last term independently vanish.

Step 3. Integral equation. We assume that

$$\begin{aligned} \partial_t f &= cf + \int b f_* \\ \partial_t g &= cg + \int b g_* \\ -\partial_t \phi &= c\,\phi + \int b_* \phi_*, \end{aligned}$$

with the notations

$$\int b\psi_* := \int b(x,x_*)\,\psi(x_*)\,dx_*, \quad \int b_*\psi_* := \int b(x_*,x)\,\psi(x_*)\,dx_*,$$

and we show

$$\partial_t(H(X)g\phi) + \int H(X)g b_*\phi_* - \int bH(X_*)g_*\phi = -\int bg_*\phi\Big\{H(X_*) - H(X) - H'(X)(X_* - X)\Big\}$$

We compute indeed

$$\begin{aligned} \partial_t(g\phi H(X)) &= H(X)g\partial_t\phi + H(X)\phi\partial_t g + H'(X)\phi(\partial_t f - X\partial_t g) \\ &= -\int H(X)gb_*\phi_* + \int bH(X_*)g_*\phi \\ &\quad + \int bg_*\phi\Big\{-H(X_*) + H(X) + H'(X)X_* - H'(X)X\Big\} \end{aligned}$$

Step 4. Conclusion. For any solutions (f, g, ϕ) to the system of (full) equations, we have summing up the three computations

$$\begin{aligned} &\partial_t(g\phi H(X)) + \\ &+\mathrm{div}(aH(X)g\phi) - \mathrm{div}(\phi\nabla(H(X)g)) + \mathrm{div}(gH(X)\nabla\phi) + \int bH(X_*)g_*\phi - \int H(X)gb_*\phi_* \\ &= -g\phi\, H''(X)\,|\nabla X|^2 - \int bg_*\phi\Big\{H(X_*) - H(X) - H'(X)(X_* - X)\Big\}. \end{aligned}$$

Since when we integrate in the x variable the term on the second line vanishes, we find out

$$\frac{d}{dt}\mathcal{H}(f) = -D_{\mathcal{H}}(f),$$

with

$$D_{\mathcal{H}}(f) := \int g\phi\, H''(X)\,|\nabla X|^2 + \iint bg_*\phi\Big\{H(X_*) - H(X) - H'(X)(X_* - X)\Big\} \ge 0.$$

Exercise 3.2. *We consider a semigroup $S_t = e^{tL}$ of linear and bounded operators on L^1 and we assume that*
(i) $S_t \ge 0$;
(ii) $\exists\, g > 0$ such that $Lg = 0$, or equivalently $S_t g = g$ for any $t \ge 0$;
(iii) $\exists\, \phi$ such that $L^*\phi = 0$, or equivalently $\langle S_t h, \phi\rangle = \langle h, \phi\rangle$ for any $h \in L^1$ and $t \ge 0$.

Our aim is to generalize to that a bit more general (and abstract) framework the general relative entropy principle we have presented for the evolution PDE (3.1).

(a) Prove that for any real affine function ℓ, there holds $\ell[(S_t f)/g]g = S_t[\ell(f/g)g]$.

(b) Prove that for any convex function H and any $f \ge 0$, there holds $H[(S_t f)/g]g \le S_t[H(f/g)g]$. (Hint. Use the fact that $H = \sup_{\ell \le H} \ell$).

(c) Deduce that

$$\int H[(S_t f)/g]g\phi \le \int H[f/g]g\phi, \quad \forall\, t \ge 0.$$

4. First example: a general Fokker-Planck equation

In this section we consider the Fokker-Planck equation

$$\partial_t f = \mathcal{L} f = \Delta f + \mathrm{div}(E f), \tag{4.2}$$

on the density $f = f(t,x)$, $t \ge 0$, $x \in \mathbb{R}^d$, where the force field $E \in \mathbb{R}^d$ is a given fixed (exterior) vectors field or is a function of the density.

4.1. **Conservation, explicit steady states and self-adjointness property.**

Any solution f to the Fokker-Planck equation (4.2) is mass conservative in the sense that

$$\frac{d}{dt}\int f\,dx = \int \mathrm{div}(\nabla f + E f)\,dx = 0,$$

because of the divergence structure of the Fokker-Planck operator $\mathcal{L}$ and the Stokes formula.
In the case when

$$E = \nabla U + E_0, \quad \mathrm{div}(E_0\, e^{-U}) = 0, \tag{4.3}$$

for a confinement potential $U : \mathbb{R}^d \to \mathbb{R}$ and a non gradient force field perturbation $E_0 : \mathbb{R}^d \to \mathbb{R}^d$, we may observe that the positive function $e^{-U(x)+U_0}$ is a stationary state for any $U_0 \in \mathbb{R}$. When furthermore $e^{-U(x)} \in L^1(\mathbb{R}^d)$, we may fix $U_0 \in \mathbb{R}$ such that

$$G(x) := e^{-U(x)+U_0} \text{ is a stationary state and a probability.}$$

On the other way round, in the most general case we just assume there exists a steady state

$$G \in L^1(\mathbb{R}^d) \cap \mathbb{P}(\mathbb{R}^d), \quad \mathrm{div}(\nabla G + E\,G) = 0,$$

where $\mathbb{P}(\mathbb{R}^d)$ stands for the set of probability measures, we may observe that $G \in C^1(\mathbb{R}^d)$ thanks to a bootstrap regularization argument and $G > 0$ thanks to the strong maximum principle. Then we define $U := -\log G$ and $E_0 := E - \nabla U$, so that (4.3) holds again.
Consider a weight function $m : \mathbb{R}^d \to \mathbb{R}_+$ and the associated Lebesgue space $L^2(m)$ with $\|f\|_{L^2(m)} := \|fm\|_{L^2}$. For $f, g \in \mathcal{D}(\mathbb{R}^d)$, we compute

$$\begin{aligned} I &:= (\mathcal{L}f, g)_{L^2(m)} - (f, \mathcal{L}g)_{L^2(m)} \\ &= \int (\Delta f + \mathrm{div}(E\,f))\,g\,m^2 - \int (\Delta g + \mathrm{div}(E\,g))\,f\,m^2 \\ &= \int g\,m^2\,E\cdot\nabla f - g\,\nabla f\cdot\nabla m^2 + \int f\,\nabla g\cdot\nabla m^2 - f\,m^2\,E\cdot\nabla g \\ &= 2\int g\,(m^2\,E - \nabla m^2)\cdot\nabla f + \int g f(\mathrm{div}(m^2 E) - \Delta m^2). \end{aligned} \tag{4.4}$$

In the one hand, if $I(f,g) = 0$ for any f, g, by choosing f as a constant function, we get

$$0 = \int g(\Delta m^2 - \mathrm{div}(m^2 E))$$

for any g, and then

$$\Delta m^2 - \mathrm{div}(m^2 E) = 0.$$

Plugging that information into (4.4), we get

$$I \;=\; 2\int g\,(m^2\,E - \nabla m^2)\cdot\nabla f,$$

and the equation $I(f,g) = 0$ for any f, g, by choosing $f = x_i$, implies

$$\int g(\partial_i m^2 - m^2 E_i) = 0,$$

for any g. We deduce

$$\partial_i m^2 - m^2 E_i = 0$$

and then $E = \nabla U$ with $U := \log(m^2)$ or equivalently $m = e^{U/2}$. In other words, we just have proved that $\mathcal{L}$ is a self-adjoint operator in the Hilbert space $L^2(m)$ if and only if $E = \nabla U$ and $m = e^{U/2}$ for some confinement potential $U : \mathbb{R}^d \to \mathbb{R}$. In that case, $G = \exp(-U - U_0)$, $U_0 \in \mathbb{R}$, is the family of steady states.

4.2. **General a priori estimates and well-posedness issue.**

Lemma 4.3. *For any $f \in \mathcal{D}(\mathbb{R}^d)$ and any weight function $m : \mathbb{R}^d \to \mathbb{R}_+$, we have*

$$\int (\mathcal{L}f) f^{p-1} m^p = -(p-1) \int |\nabla f|^2 f^{p-2} m^p + \int f^p m^p \psi_1$$

with

$$\psi_1 := (p-1) \frac{|\nabla m|^2}{m^2} + \frac{\Delta m}{m} + \left(1 - \frac{1}{p}\right) div\, E - E \cdot \frac{\nabla m}{m}.$$

Proof of Lemma 4.3. It is a good exercise! Just perform two integrations by part: one on the term which involves the Laplacien, another on the term which involves the $E \cdot \nabla f$ function. □

Observe that (at least formally):

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^d} |f|^p m^p &= \frac{p}{2} \int_{\mathbb{R}^d} (|f|^2)^{p/2-1} \partial_t (f\, \bar f) m^p \\ &= \frac{p}{2} \int_{\mathbb{R}^d} |f|^{p-2} (\mathcal{L}f\, \bar f + f\, \bar{\mathcal{L}} f) m^p, \end{aligned}$$

so that defining $f^* := \|f\|^{2-p}_{L^p(m)}\, \bar f\, |f|^{p-2}$, we get

$$\begin{aligned} \frac{d}{dt} \|f\|^2_{L^p(m)} &= \frac{2}{p} (\|f\|^p_{L^p(m)})^{2/p-1} \frac{d}{dt} \|f\|^p_{L^p(m)} = \int_{\mathbb{R}^d} (\mathcal{L}f\, f^* + \bar f^*\, \mathcal{L}f) m^p \\ &= 2\, \Re e \langle \mathcal{L}f, f^* \rangle. \end{aligned} \tag{4.5}$$

As a consequence, (4.5) together with Lemma 4.3 lead to some differential inequality on the L^p-norm which provides an a priori estimate on a solution of (4.2) when the function ψ_1 in Lemma 4.3 is uniformly bounded above.

Exercise 4.4. *(a) Generalize Lemma 4.3 to the case of a complex valued function $f \in \mathcal{D}(\mathbb{R}^d; \mathbb{C})$.*
(b) For any $f > 0$, prove that (at least formally)

$$\int (\mathcal{L}f) \log f = -4 \int |\nabla \sqrt{f}|^2 + \int (div\, E)\, f.$$

As a consequence of the previous identity we obtain several existence results. In the sequel we assume either

$$E = E(t,x) \in L^\infty(0,T; W^{1,\infty}(\mathbb{R}^d)), \tag{4.6}$$

or that $E = E(x) \in W^{1,\infty}_{loc}$ and, for some $\gamma \geq 2$,

$$|E(x)| \leq K_1 \langle x \rangle^{\gamma-1}, \quad |\mathrm{div} E(x)| \leq K_2 \langle x \rangle^{\gamma-2}, \quad E(x) \cdot x \geq |x|^\gamma \quad \forall x \in \mathbb{R}^d. \tag{4.7}$$

We define

$$H := L^2(m), \quad V := H^1(m) \cap L^2(m_1) \tag{4.8}$$

with $m = m_1 = \langle x \rangle^k$, $k \geq 0$, in the first case, and with $m := e^{\kappa \langle x \rangle^\gamma}$, $m_1 := \langle x \rangle^{\gamma-1} e^{\kappa \langle x \rangle^\gamma}$, $\kappa := \gamma/4$, in the second case. We next define

$$X_T := C([0,T]; H) \cap L^2(0,T; V).$$

Proposition 4.5. *For any $f_0 \in H$, there exists a unique variational solution $f \in X_T$ to the Fokker-Planck equation (4.2). Moreover, if $f_0 \geq 0$ then $f(t) \geq 0$ for any $t \geq 0$; if $f_0 \in L^1$ then $f(t) \in L^1$ and $\langle f(t) \rangle = \langle f_0 \rangle$ for any $t \geq 0$.*

Proof of Proposition 4.5. We observe that the (possibly time dependent) bilinear form

$$\begin{aligned} a(t,f,g) &:= -\int \mathcal{L}(t) f\, g\, m^2 \\ &= \int \{ \nabla f \cdot \nabla g\, m^2 - \nabla f \cdot (\nabla m^2 + E m^2)\, g - \mathrm{div} E\, f\, g\, m^2 \}\, dx \end{aligned}$$

is continuous in V. Moreover, thanks to Lemma 4.3, it satisfies the following coercivity lower bound

$$a(t,f,f) = \int |\nabla f|^2\, m^2 + \int |f|^2\, m^2\, \psi_1$$

with

$$\psi_1 = -\frac{k^2}{\langle x\rangle^2} - \frac{k(k-1)}{\langle x\rangle^2} - \frac{1}{2}\mathrm{div}E + k\,E\cdot\frac{x}{\langle x\rangle^2} \ge C,$$

$C\in\mathbb{R}$, in the first case, and

$$\begin{aligned}\psi_1 &= -\frac{1}{16}|x|^2\langle x\rangle^{2\gamma-4} - \frac{d}{4}\langle x\rangle^{\gamma-2} - \frac{\gamma-2}{4}|x|^2\langle x\rangle^{\gamma-4} - \frac{1}{16}|x|^2\langle x\rangle^{2\gamma-4}\\ &\quad -\frac{1}{2}\mathrm{div}E + \frac{1}{4}E\cdot x\,\langle x\rangle^{\gamma-2} \ge \frac{1}{8}\langle x\rangle^{2\gamma-2} + C,\end{aligned}$$

$C\in\mathbb{R}$, in the second case. We conclude to the existence and the uniqueness of a variational solution $f\in X_T$ by applying Lions' Theorem 3.2 in Chapter 1. □

Proposition 4.6. *Assume $a\in W^{1,\infty}\cap L^2$. For any $f_0\in L^2_k$, $k>d/2$, there exists a unique solution*

$$f\in C([0,T);L^2_k)\cap L^2(0,T;H^1_k), \quad \forall T>0,$$

to the nonlinear Fokker-Planck equation

$$\partial_t f = \Delta f + div((a*f)f). \tag{4.9}$$

Proof of Proposition 4.6. Step 1. A priori bounds. On the one hand, we clearly have

$$\int_{\mathbb{R}^d}|f|\,dx \le \int_{\mathbb{R}^d}|f_0|\,dx,$$

and then

$$\begin{aligned}\frac{d}{dt}\int f^2\frac{\langle x\rangle^{2k}}{2} &= -\int|\nabla f|^2\langle x\rangle^{2k}\\ &\quad+\int f^2\langle x\rangle^{2k}\{\frac{k^2}{\langle x\rangle^2}+\frac{k(k-1)}{\langle x\rangle^2}+\frac{1}{2}(\mathrm{div}a)*f+k\,(a*f)\cdot\frac{x}{\langle x\rangle^2}\}\\ &\le -\int|\nabla f|^2\langle x\rangle^{2k}\\ &\quad+\{2k^2+\frac{1}{2}\|\nabla a\|_{L^\infty}\|f_0\|_{L^1}+k\,\|a\|_{L^\infty}\|f_0\|_{L^1}\}\int f^2\langle x\rangle^{2k}.\end{aligned}$$

Step 2. Existence. To prove the existence we consider the mapping $g\mapsto f$ defined for $g\in C([0,T];L^2_k)$, $k>d/2$, so that $L^2_k\subset L^1$, by solving the linear evolution PDE

$$\partial_t f = \Delta f + \mathrm{div}((a*g)f).$$

For the linear (and g dependent) problem, by repeating the same computations as in step 1, we also have

$$\sup_{[0,T]}\|f\|_{L^1}\le\|f_0\|_{L^1},\quad \sup_{[0,T]}\|f\|_{L^2_k}\le\mathcal{A}_T,$$

where $\mathcal{A}_T$ only depends on $\|f_0\|_{L^2_k}$, k, a and T. We then define

$$\mathcal{C}_T := \{f\in C([0,T];L^2_k),\ \|f(t)\|_{L^1}\le\|f_0\|_{L^1},\ \|f(t)\|_{L^2_k}\le\mathcal{A}_T\}$$

and we have $\Phi:\mathcal{C}_T\to\mathcal{C}_T$. We consider two solutions

$$\partial_t f_i = \Delta f_i + \mathrm{div}((a*g_i)f_i)$$

so that the differences $f=f_2-f_1$ and $g:=g_2-g_1$ satisfy

$$\partial_t f = \Delta f + \mathrm{div}((a*g_1)f) + \mathrm{div}((a*g)f_2).$$

As a consequence, using the Young inequality, we have

$$\begin{aligned}\frac{d}{dt}\|f\|^2_{L^2} &= -2\int|\nabla f|^2+\int(\nabla a*g_1)\,f^2-2\int(a*g)f_2\nabla f\\ &\le -\int|\nabla f|^2+\|\nabla a*g_1\|_{L^\infty}\int f^2+\|a*g\|^2_{L^\infty}\int f_2^2\\ &\le \|\nabla a\|_{L^\infty}\,\|f_0\|_{L^1}\|f\|^2_{L^2}+2\,\|a\|_{L^2}\,\mathcal{A}_T^2\,\|g\|^2_{L^2},\end{aligned}$$

from which we deduce

$$\sup_{[0,T]}\|f\|^2_{L^2}\le\varepsilon_T\,\sup_{[0,T]}\|g\|_{L^2}$$

with $\varepsilon_T \to 0$ as $T \to 0$. We conclude to the existence by a Banach fixed point theorem. □

Exercise 4.7. *Prove that the assumption $a \in L^2$ can be removed by making the contraction argument with the L^2_k norm.*

4.3. **Long-time behaviour.**
We briefly discuss the long-time asymptotics for the linear and nonlinear Fokker-Planck equations (4.2) and (4.9).

• In the case $E = \nabla U$, $U = \langle x\rangle^\gamma/\gamma$, $\gamma \ge 2$, $\mathcal{L}$ is self-adjoint and dissipative in $L^2(G^{-1/2})$ and the resolvent $R_{\mathcal{L}}(b)$ is compact, because for $b > 0$ large enough, the bilinear form $a'(f,g) := a(f,g) + b(f,g)$, with a defined in the proof of Proposition 4.5, is coercive in the space $V := H^1(G^{-1/2}) \cap L^2(\langle x\rangle^{\gamma-1}G^{-1/2})$ and V is compactly embedded in $L^2(G^{1/2})$. More precisely, performing just one integration by part on the first (Laplacian) term, we have

$$\begin{aligned} a'(f,f) &= \int [-(\mathcal{L}f)f + b\,f^2]\,G^{-1} \\ &= \int [-\Delta f - \Delta U\,f - \nabla U\cdot\nabla f + bf)\,fe^U \\ &= \int \{|\nabla f|^2 + (b-\Delta U)\,f^2\}\,e^U. \end{aligned}$$

Introducing the notation $g := f\,G^{-1/2}$ and observing that

$$\begin{aligned} \int [-\Delta f]\,f\,e^U &= \int \nabla(g\,G^{1/2})\cdot\nabla(g\,G^{-1/2}) \\ &= \int |\nabla g|^2 + \int g^2\,\nabla G^{1/2}\cdot\nabla G^{-1/2} \\ &= \int |\nabla g|^2 - \frac{1}{4}\int f^2\,G^{-1}\,|\nabla U|^2, \end{aligned}$$

as well as

$$-\int (\nabla U\cdot\nabla f)\,fe^U = \frac{1}{2}\int (\Delta U + |\nabla U|^2)\,f^2\,G^{-1},$$

we also have

$$a'(f,f) = \int |\nabla g|^2 + \int f^2\,[b - \frac{1}{2}\Delta U + \frac{1}{4}|\nabla U|^2]\,e^U.$$

Gathering these two identities, we deduce that for any $f \in \mathcal{D}(\mathbb{R}^d)$

$$a'(f,f) \ge \frac{1}{2}\int |\nabla f|^2\,e^U + \int \{b - \frac{1}{2}\Delta U + \frac{1}{8}\,|\nabla U|^2)\,f^2\}\,e^U.$$

Observing now that $|\nabla U|^2 \ge |x|^{2(\gamma-1)}$ and $|\nabla U| \le (d+\gamma-2)\,\langle x\rangle^{\gamma-2}$, we obtain by taking $b > 0$ large enough the following lower (coercivity) bound: for any $f \in \mathcal{D}(\mathbb{R}^d)$

$$a'(f,f) \ge \frac{1}{2}\int |\nabla f|^2\,e^U + \int \{\frac{b}{2} + \frac{1}{16}\,|x|^{2(\gamma-1)})\,f^2\}\,e^U =: \|f\|_V^2.$$

On the other hand, thanks to the first identity in Proposition 4.5, we also have

$$a'(f,g) = \int \{\nabla f\cdot\nabla g - 2\nabla f\cdot\nabla U\,g + (b - \Delta U f\,g\}\,e^U\,dx,$$

and then $|a'(f,g)| \le C\,\|f\|_V\,\|g\|_V$ for any $f, g \in V$. Thanks to the Lax-Milgram Theorem, we deduce that $b \in \rho(\mathcal{L})$. Moreover, for any $g \in H := L^2(G^{-1/2})$, the (variational) solution $f := R_{\mathcal{L}}(b)f \in V$ to the equation

$$(\mathcal{L} - b)f = g$$

satisfies

$$\|f\|_V^2 \le ((b-\mathcal{L})f, f)_H = (-g, f)_H \le C\,\|g\|_H\,\|f\|_V.$$

As a consequence, $\|R_{\mathcal{L}}(b)g\|_V \le C\,\|g\|_H$. Because $V \subset H$ is compactly embedded, we get that $R_{\mathcal{L}}(b)$ is a compact operator. We may apply Theorem 2.14: the $L^2(G^{-1/2})$-norm is a Lyapunov functional and any solution converges with exponential rate to the associated equilibrium (uniquely defined thanks to the mass conservation).

• In the case $E = a * f + x$ with $a = \nabla U$, U a convex function, we write

$$\partial_t f = \mathcal{L} f = \mathrm{div}[f \nabla(\log f + |x|^2/2 + U * f)].$$

We define

$$\mathcal{H}(f) := \int_{\mathbb{R}^d} f\{\log f + \frac{1}{2}|x|^2 + \frac{1}{2} U * f\}, \quad \mathcal{D}(f) := \int_{\mathbb{R}^d} f\,|\nabla(\log f + |x|^2/2 + U * f)|^2.$$

We may compute

$$\frac{d}{dt}\mathcal{H}(f) = \int_{\mathbb{R}^d} (\partial_t f)(1 + \log f + |x|^2/2 + U * f) = -\mathcal{D}(f).$$

The functional $\mathcal{H}$ is then an entropy. It is moreover a Lyapunov functional under some additional assumption on U. We accept the following result.

Lemma 4.8. *If U is a convex function then $\mathcal{H}$ is a convex functional and there exists a unique minimizer f_∞ to the minimizing problem*

$$\mathcal{H}(f_\infty) = \min\Big\{\mathcal{H}(f);\ 0 \le f \in L^1(\mathbb{R}^d),\ \int_{\mathbb{R}^d} f\,dx = 1\Big\}.$$

Moreover, f_∞ is smooth and positive, and we have $\mathcal{D}(f) = 0$ implies $f = f_\infty$.

Exercise 4.9. *Show the convergence of the solutions to the unique equilibrium f_∞ by applying Theorem 1.6 or Theorem 1.9.*

• In the case $E = \nabla U + E_0$, $U = \langle x\rangle^\gamma/\gamma$, $\gamma \ge 2$, $\mathrm{div}(E_0\, e^{-U}) = 0$, $E_0 \ne 0$, the operator $\mathcal{L}$ satisfies the same properties as the first case (when $E_0 = 0$) except that $\mathcal{L}$ is not self-adjoint anymore. Because $\mathcal{L}G = 0$ and $\mathcal{L}^* 1 = 0$, we may apply the GRE method which readily implies

$$\frac{d}{dt}\mathcal{H}(f) = -\mathcal{D}(f),$$

with

$$\mathcal{H}(f) := \int (f - \langle f_0\rangle G)^2 G^{-1} \quad \text{and} \quad \mathcal{D}(f) = \int |\nabla(f/G)|^2 G.$$

The equation $D(f) = 0$ is equivalent to $f = \langle f\rangle G$ and by conservation of mass $f = \langle f_0\rangle G$. As a consequence, $\mathcal{H}$ is a entropy, $\mathcal{D}$ is a dissipation of entropy functional (it is lsc for the weak L^2 convergence), and then Theorem 1.6 says that $f(t) \rightharpoonup \langle f_0\rangle G$ weakly in L^2 as $t \to \infty$ for any $f_0 \in H := L^2(G^{-1/2})$ (take $\mathcal{Z} := \{g \in H;\ \|g\|_H \le \|f_0\|_H,\ \langle g\rangle = \langle f_0\rangle\}$). In order to enlarge of the class of initial data and to strengthen the sense of convergence we may argue as follows (we present the argument in dimension $d = 1$ for the sake of simplicity of the notation). By developing the term $\mathcal{D}(f)$ or just using Lemma 4.3, we have for any $K > 0$ and for some $K_0 = K_0(\mathcal{H}(f_0), K)$

$$\begin{aligned}\frac{d}{dt}\int f^2 G^{-1} &= -\int (\partial f)^2 G^{-1} + \int f^2 G^{-1}\psi_1 \\ &\le -\int (\partial f)^2 G^{-1} - K\Big(\int f^2 G^{-1}\Big)^2 + K_0,\end{aligned}$$

because $\psi_1 \le C$ and $\mathcal{H}(f) \le \mathcal{H}(f_0)$.
The equation satisfied by ∂f is

$$\partial_t \partial f = \Delta \partial f + \partial(\mathrm{div}E) f + (\mathrm{div}E)\partial f + \partial E \cdot \nabla f + E \cdot \nabla \partial f,$$

from which we deduce for some $\theta \in (0,1)$

$$\begin{aligned}\frac{d}{dt}\int (\partial f)^2 &\le -\int (\partial^2 f)^2 + \int \big\{|D^2E|\,|f|\,|\nabla f| + \frac{3}{2}\,|\mathrm{div}E|\,|\nabla f|^2\big\} \\ &\le -\Big(\int f^2\Big)^{-1}\Big(\int (\partial f)^2\Big)^2 + \int (C\,f^2 + \theta^{-1}(\partial f)^2) G^{-1}.\end{aligned}$$

We define

$$u := \int f^2 G^{-1} + \theta \int (\partial f)^2,$$

which satisfies the differential ODE

$$\begin{aligned}\frac{du}{dt} &\le -K\Big(\int f^2 G^{-1}\Big)^2 + K_0 - \theta \|f_0\|_H^{-2} \Big(\int (\partial f)^2\Big)^2 + \theta C \, \|f_0\|_H^2 \\ &\le -\theta' u^2 + K_0',\end{aligned}$$

for some constants $\theta', K_0' > 0$, which only depend on $\|f_0\|_H$. Defining $K_1 = K_1(\|f_0\|_H) := K_0'/(2\theta')$ and the set

$$\mathcal{Z}_1 := \big\{ g \in H;\ \|g\|_H \le \|f_0\|_H,\ u[g] \le K_1 \big\},$$

we deduce that if $f_0 \in \mathcal{Z}_1$ then $f(t) \in \mathcal{Z}_1$ for any $t \ge 0$, and on the contrary, defining $\tau := \sup\{t > 0;\ u(t') > K_1\ \forall t' \in [0,t)\}$, we have

$$\frac{du}{dt} \le -\frac{\theta'}{2} u^2 \quad \text{on} \quad (0,\tau).$$

As a consequence, we get $u(t) \le 2/(\theta' t)$ on $(0,\tau)$, so that necessarily $u(t) \le K_1$ for some $t \le 2/(\theta' K) := T$. We have then proved that $\mathcal{Z}_1$ is invariant and attractive in the sense that $f(t) \in \mathcal{Z}_1$ for any $t \ge T$. Because $\mathcal{Z}_1 \subset L^1$ with compact embedding, we deduce from the previous (weak) convergence that $f(t) \to \langle f_0 \rangle G$ strongly in L^1. It is worth emphasizing that we get the same conclusion by using the La Salle invariance principle (Theorem 1.8) by observing that $\mathcal{H}$ is a Lyapunov functional in the set $\mathcal{Z}_1$.
For a general initial datum $f_0 \in L^1$, we use the splitting $f_0 = f_{0,n} + g_{0,n}$ with $f_{0,n} = \mathbf{1}_{B(0,n)}\, f_0 * \rho_n \in H$ for a mollifier sequence (ρ_n). We then have

$$\|f_n(t) - \langle f_{0,n}\rangle G\|_{L^1} \to 0 \text{ as } t \to \infty,\ \forall\, n \ge 0,$$

by the previous analysis, and

$$\sup_{t \ge 0} \|g_n(t)\|_{L^1} \le \|g_{0,n}\| \to 0 \text{ as } n \to \infty.$$

Putting together the two above estimates and the fact that $\langle f_{0,n}\rangle \to \langle f_0 \rangle$ as $n \to \infty$, we conclude to $f(t) \to \langle f_0 \rangle G$ in L^1 as $t \to \infty$.

• In the general case when E satisfies (4.7), we verify that $\mathcal{L} - b$ is dissipative in the space H defined in (4.8) for $b > 0$ large enough and again that $R_{\mathcal{L}}(b)$ is compact. Moreover $\mathcal{L}$ satisfies Kato's inequality and the strong maximum principle as stated and proved bellow. We may then apply the existence result Theorem 1.2 (see Exercise 2.26) and we obtain that there exists of nonnegative and mass normalized steady state $f_1 \in H$. Alternatively, we obtain the same existence result by applying the existence part Proposition 2.25 of the Krein-Rutman Theorem 2.23 and observing that $\mathcal{L}^* 1 = 0$ so that the first eigenvalue is 0 and the first eigenfunction f_1 given by the Krein-Rutman theorem is a steady state. We conclude by applying the GRE method as in the previous case, and we get again $f(t) \rightharpoonup \langle f_0 \rangle f_1$ weakly in L^2 as $t \to \infty$ for any $f_0 \in H$. We can improve (enlarge and strengthen) the above convergence by following the same argument as in the previous case.

Proposition 4.10. *The operator $\mathcal{L}$ satisfies "Kato's inequalities" and the "strong maximum principle" in H*

Proof of Proposition 4.10. Step 1. Kato's inequalities. For a convex function $\beta : \mathbb{R} \to \mathbb{R}$ such that $\beta(s) = s\beta'(s)$, we clearly have

$$\mathcal{L}\beta(f) = \beta''(f)|\nabla f|^2 + \beta'(f)\mathcal{L}f \ge \beta'(f)\mathcal{L}f.$$

For the square of the modulus function $s \in \mathbb{C} \mapsto \theta(s) = |s| = \sqrt{s\bar{s}}$, we have on the one hand

$$\begin{aligned}\mathcal{L}f \cdot \theta'(f) &:= [(\mathcal{L}f)\, \bar{f} + (\mathcal{L}\bar{f})\, f]/(2|f|) \\ &= [(\Delta f)\, \bar{f} + (\Delta \bar{f})\, f]/(2|f|) + \operatorname{div}(E|f|).\end{aligned}$$

On the other hand, introducing the real part R and the imaginary part I in such a way that $f = R + iI$, $R, I \in \mathbb{R}$, we easily compute

$$\begin{aligned} \Delta|f| &= \mathrm{div}\big(\frac{\nabla|f|^2}{2|f|}\big) = \mathrm{div}\big(\frac{\nabla f\,\bar f + \nabla \bar f\, f}{2|f|}\big) \\ &= \frac{\Delta f\,\bar f + \Delta \bar f\, f}{2|f|} + \frac{|\nabla f|^2}{|f|} - \frac{1}{4}\frac{\big|\nabla |f|^2\big|^2}{|f|^3} \\ &= \frac{\Delta f\,\bar f + \Delta \bar f\, f}{2|f|} + \frac{(I\nabla R - R\nabla I)^2}{|f|}. \end{aligned} \tag{4.10}$$

The two identities together imply

$$\begin{aligned} \mathcal{L}|f| &= \Delta|f| + \mathrm{div}(E|f|) \\ &\geq [(\Delta f)\bar f + f\,(\Delta \bar f)]/(2|f|) + \mathrm{div}(E|f|) = \mathcal{L}f \cdot \theta'(f). \end{aligned} \tag{4.11}$$

It is worth emphasizing that (4.10) is clearly true for a $W^{2,d}_{loc}(\mathbb{R}^d)$ and not vanishing function f. For a function $f \in W^{2,d}_{loc}(\mathbb{R}^d)$ which may vanish, we introduce the quantity $|f|_\varepsilon := (\varepsilon^2 + |f|^2)^{1/2}$, and we similarly have

$$\Delta|f|_\varepsilon = \frac{\Delta f\,\bar f + \Delta \bar f\, f}{2|f|_\varepsilon} + \frac{|\nabla f|^2}{|f|_\varepsilon} - \frac{1}{4}\frac{\big|\nabla |f|^2\big|^2}{|f|^3_\varepsilon} \geq \frac{\Delta f\,\bar f + \Delta \bar f\, f}{2|f|_\varepsilon}.$$

By passing to the limit $\varepsilon \to 0$, we recover (4.11).

Step 2. Strong maximum principle for a real values function. Consider $f \in H\backslash\{0\}$ such that $\mathcal{L}f = 0$. By a bootstrap regularization argument, we classically have $f \in W^{2,d}_{loc}(\mathbb{R}^d) \subset C(\mathbb{R}^d)$. By assumption there exist then $x_0 \in \mathbb{R}^d$, $c, r > 0$, such that $|f(x)| \geq c$ on $B(x_0, r)$. From Lemma 4.3, we also have that $\mathcal{L} - a$ is -1-dissipative for $a \geq 0$ large enough, in the sense that

$$\forall\, h \in D(\mathcal{L}) \quad ((\mathcal{L} - a)h, h)_H \leq -\|h\|^2_H. \tag{4.12}$$

We next observe that for $\sigma > 0$ large enough, the function $g(x) := c\exp(\sigma r^\gamma - \sigma|x - x_0|^\gamma)$ satisfies $g = c$ on $\partial B(x_0, r)$ and

$$\begin{aligned} (-\mathcal{L} + a)g &= \big[-\sigma^2\gamma^2|x - x_0|^{2(\gamma-1)} + \sigma\gamma(d + \gamma - 2)|x - x_0|^{\gamma-2} \\ &\quad -\mathrm{div}E + E\cdot(x - x_0)\gamma\sigma|x - x_0|^{\gamma-2} - a\big]\, g \leq 0 \quad \text{on} \quad B(x_0, r)^c. \end{aligned}$$

We define $h := (g - |f|)_+$ and $\Omega := \mathbb{R}^d\backslash B(x_0, r)$. We have $h \in H^1_0(\Omega, m dx)$ and

$$\begin{aligned} (\mathcal{L} - a)h &\geq \theta'(g - |f|)\,\mathcal{L}(g - |f|) - a\,h \\ &= \theta'(g - |f|)\,[(\mathcal{L} - a)g + a|f|] \geq 0, \end{aligned}$$

where we have used the notation $\theta(s) = s_+$. Thanks to a straightforward generalization of (4.12) to $H^1_0(\Omega, m)$, we deduce

$$0 \leq ((\mathcal{L} - a)h, h)_{L^2(\Omega,m)} \leq -\|h\|^2_{L^2(\Omega,m)},$$

and then $h = 0$. That implies $|f| \geq g$ on Ω, next $|f| > 0$ on $\mathbb{R}^d$ and then $f > 0$ or $f < 0$ because $f \in C(\mathbb{R}^d)$.

Step 3. Strong maximum principle for a complex values function. Consider a complex values function $f \in D(\mathcal{L})\backslash\{0\}$ such that

$$\mathcal{L}\theta(f) = \mathcal{L}f \cdot \theta'(f) = 0$$

for $\theta(s) = |s|$. The strong maximum principle for a real values function implies that $|f| > 0$ and we may assume that both R and I do not vanish in some open set $\mathcal{O}$. Using the case of equality in Kato's inequality, we deduce with the notation of Step 2 that

$$\frac{(I\nabla R - R\nabla I)^2}{|f|} = 0$$

which in turns implies

$$|\nabla \log R - \nabla \log I|^2 = 0.$$

We have then proved $R = CI$ for some constant $C \in \mathbb{R}$ in $\mathcal{O}$ and then in $\mathbb{R}^d$, which exactly means $f = u\,|f|$ for some $u \in \mathbb{C}^*$. ☐

5. Second example: the scattering equation

The linear Boltzmann (or scattering) equation of the density function $f = f(t,v) \ge 0$, $t \ge 0$, $v \in \mathcal{V} \subset \mathbb{R}^d$, writes

$$\partial_t f = \mathcal{L}\, f := \int_{\mathcal{V}} (b_* f_* - b\, f)\, dv_*, \tag{5.1}$$

where $f = f(v)$, $f_* = f(v_*)$, $b = b(v,v_*)$ and $b_* = b(v_*,v)$, $b \ge 0$ is a given function (the rate of collisions), or more generally

$$\partial_t f = \mathcal{L}\, f := \int_{\mathcal{V}} b_* f_*\, dv_* - B(v)\, f, \tag{5.2}$$

and we assume that there exists a function $\phi > 0$ such that

$$\mathcal{L}^*\phi := \int_{\mathcal{V}} b\, \phi_*\, dv_* - B\,\phi = 0, \quad \text{in other words} \quad B(v) := \int_{\mathcal{V}} \frac{\phi_*}{\phi}\, b\, dv_*,$$

with again $\phi = \phi(v)$ and $\phi_* = \phi(v_*)$. The first equation (5.1) corresponds to the choice

$$B(v) = \int_{\mathcal{V}} b\, dv_*, \quad \phi \equiv 1,$$

in the second equation (5.2).

Example 1. We assume $\mathcal{V} \subset \mathbb{R}^d$, $b_* = k(v,v_*)\, F(v)$, for a symmetric function $k(v,v_*) = k(v_*,v) > 0$ and a given function $0 < F \in L^1(\mathcal{V}) \cap \mathbf{P}(\mathcal{V})$. The equation (5.1) becomes

$$\partial_t f = \mathcal{L}\, f := \int_{\mathcal{V}} k\, (F\, f_* - F_*\, f)\, dv_*. \tag{5.3}$$

It is worth noticing that $F = F(v)$ is a stationary solution to the equation (5.5) since

$$\partial_t F = 0 = \mathcal{L}\, F. \tag{5.4}$$

Example 2. We assume $\mathcal{V} = (0,\infty)$, $b_* = b_*\, \mathbf{1}_{v_* > v}$, $\phi(v) = v$, and then the equation (5.2) becomes the *fragmentation equation*

$$\partial_t f = \mathcal{L}\, f := \int_0^\infty b_*\, f_*\, dv_* - B(v)\, f(v), \quad B(v) := \int_0^v \frac{v_*}{v}\, b\, dv_*. \tag{5.5}$$

Conservation law. Without any additional assumption, we immediately deduce that the equation (5.2) has one law of conservation: any solution satisfies (at least formally)

$$\int_{\mathcal{V}} f(t,v)\, \phi(v)\, dv = \int_{\mathcal{V}} f(0,v)\, \phi(v)\, dv,$$

because

$$\frac{d}{dt} \int_{\mathcal{V}} f\, \phi\, dv = \int_{\mathcal{V}} (\mathcal{L} f)\, \phi\, dv = \int_{\mathcal{V}} f\, (\mathcal{L}^*\phi)\, dv = 0.$$

Lyapunov/entropy functional. We assume that there exists a function $0 < F \in L^1(\mathcal{V}) \cap \mathbf{P}(\mathcal{V})$ which is a stationary solution

$$\mathcal{L} F = \int_{\mathcal{V}} b_* F_*\, dv_* - \int_{\mathcal{V}} \frac{\phi_*}{\phi}\, b\, dv_*\, F = 0,$$

what it is the situation in Example 1. Then any solution f to the equation (5.2) satisfies (at least formally)

$$\frac{d}{dt} \int_{\mathcal{V}} f^2\, \frac{\phi}{F}\, dv = 2 \int_{\mathcal{V}} (\mathcal{L}\, f)\, \frac{f\,\phi}{F}\, dv = -D_2(f) \tag{5.6}$$

with

$$D_2(f) := \int_{\mathcal{V}}\int_{\mathcal{V}} b_*\, F\, \phi \left(\frac{f_*}{F_*} - \frac{f}{F} \right)^2 dv dv_*. \tag{5.7}$$

We then say that

$$\mathcal{H}_2(f) := \int_{\mathcal{V}} f^2\, \frac{\phi}{F}\, dv$$

is a Lyapunov (or generalized relative entropy) for the equation (5.2).

To prove (5.6) in the case $\phi = 1$, we perform the following computations

$$\begin{aligned}(\mathcal{L}f, f/F) &= \iint b_* F_* \frac{f_*}{F_*}\frac{f}{F} - \frac{1}{2}\iint b\,F\,\frac{f^2}{F^2} - \frac{1}{2}\iint b\,F\,\frac{f^2}{F^2} \\ &= \iint b_* F_* \frac{f_*}{F_*}\frac{f}{F} - \frac{1}{2}\iint b_*\,F_*\,\frac{(f_*)^2}{(F_*)^2} - \frac{1}{2}\iint b_*\,F_*\,\frac{f^2}{F^2} \\ &= -\frac{1}{2}\iint b_* F_* \left(\frac{f_*}{F_*} - \frac{f}{F}\right)^2,\end{aligned}$$

where in order to pass from the first to the second line we have just changed the name of the variables in the second term

$$\iint b\,F\,\frac{f^2}{F^2} = \iint b_*\,F_*\,\frac{(f_*)^2}{(F_*)^2}$$

and we have used the fact that F is a stationary solution in the third term

$$\int b\,F\,dv_* = \int b_*\,F_*\,dv_*.$$

For a general law of conservation ϕ, the computation is almost the same

$$\begin{aligned}(\mathcal{L}f, \phi\,f/F) &= \iint b_*\,\phi\,F_* \frac{f_*}{F_*}\frac{f}{F} - \frac{1}{2}\iint b\,\phi_*\,F\,\frac{f^2}{F^2} - \frac{1}{2}\iint b\,\phi_*\,F\,\frac{f^2}{F^2} \\ &= \iint b_*\,\phi\,F_* \frac{f_*}{F_*}\frac{f}{F} - \frac{1}{2}\iint b_*\,\phi\,F_*\,\frac{(f_*)^2}{(F_*)^2} - \frac{1}{2}\iint b_*\,\phi\,F_*\,\frac{f^2}{F^2} \\ &= -\frac{1}{2}\iint b_*\,\phi\,F_* \left(\frac{f_*}{F_*} - \frac{f}{F}\right)^2.\end{aligned}$$

A theorem. We now consider the same situation as in example 1, and we assume furthermore that there exist some constants $0 < k_0 \le k_1 < \infty$ such that

$$\forall\, v, v_* \in \mathcal{V}, \qquad k_0 \le k(v, v_*) \le k_1.$$

We consider the scattering equation (5.1) in that case, that we complement with an initial condition

$$f(0, v) = f_0(v) \quad \forall\, v \in \mathcal{V}.$$

Theorem 5.1. *Assume $f_0 \in L^1(\mathcal{V})$, $\mathcal{V} = \mathbb{R}^d$.*
(1) There exists a unique global solution $f \in C([0,\infty); L^1(\mathcal{V}))$ to the scattering equation (5.1). That solution is mass conserving

$$\int_{\mathcal{V}} f(t, v)\,dv = \int_{\mathcal{V}} f_0(v)\,dv =: \langle f_0 \rangle$$

and satisfies the maximum principle

$$f_0 \ge 0 \quad \Rightarrow \quad f(t, .) \ge 0 \quad \forall\, t \ge 0.$$

(2) In the large time asymptotic, the solution converges to the unique stationary solution with same mass

$$\|f(t,.) - \langle f_0 \rangle\, F\|_E \le e^{-k_0 t/2}\, \|f_0 - \langle f_0 \rangle\, F\|_E,$$

where $\|\cdot\|_E$ is the Hilbert norm defined by

$$\|f\|_E^2 := \int_{\mathcal{V}} f^2\, F^{-1}\, dv.$$

For the proof of point (1) we refer to the precedent chapters where the needed arguments have been introduced. We are going to give now the (formal) proof of point (2).
Functional inequality and long time behaviour. The following functional inequality holds true: for any function $f \in E$, we have

$$D_2(f) \ge k_0 \|f - \langle f \rangle\, F\|_E^2. \tag{5.8}$$

It is worth observing that the Cauchy-Schwarz inequality implies

$$|\langle f \rangle| \le \int_{\mathcal{V}} (|f| F^{-1/2})\, F^{1/2} \le \Big(\int_{\mathcal{V}} f^2 F^{-1}\Big)^{1/2} \Big(\int_{\mathcal{V}} F\Big)^{1/2} = \|f\|_E,$$

so that the mass $\langle f\rangle$ is well defined if $f\in E$. Let us accept for a while the inequality (5.8) and let us prove then the convergence result (2) in Theorem 5.1. Thanks to (5.6), the fact that F is a stationary solution, the fact that f is mass conserving and (5.8), we have

$$\frac{d}{dt}\|f-\langle f\rangle F\|_E^2=-D_2(f)\le -k_0\|f-\langle f\rangle\, F\|_E^2,$$

and we conclude by applying the Gronwall lemma.

Let us prove now the functional inequality (5.8). From the lower bound assumption made on k, the following first inequality holds

$$D_2(f):=\iint b_*\,F_*\left(\frac{f_*}{F_*}-\frac{f}{F}\right)^2\ge k_0\iint F\,F_*\left(\frac{f_*}{F_*}-\frac{f}{F}\right)^2.$$

On the other hand, by integrating (in the v_* variable) the identity

$$f\,F_*-f_*\,F=\left(\frac{f}{F}-\frac{f_*}{F_*}\right)F\,F_*,$$

we get

$$g=F\int_{\mathcal V}\left(\frac{f}{F}-\frac{f_*}{F_*}\right)F_*\,dv_*$$

with $g=f-\langle f\rangle F$. Thanks to the Cauchy-Schwarz inequality, we deduce

$$g^2\le\int_{\mathcal V}\left(\frac{f}{F}-\frac{f_*}{F_*}\right)^2F\,F_*\,dv_*\times\int_{\mathcal V}F\,F_*\,dv_*,$$

so that we get the second inequality

$$\int_{\mathcal V}\frac{g^2}{F}\,dv\quad\le\quad\int_{\mathcal V}\int_{\mathcal V}\left(\frac{f}{F}-\frac{f_*}{F_*}\right)^2F\,F_*\,dv_*\,dv.$$

We conclude by gathering these two estimates. □

Exercise 5.2. *Consider the mass conservative scattering equation* (5.1) *and assume that*

$$K_1\le B(v)\le K_2,\quad 0\le b(v,v_*)\le K_3\qquad\forall\,v,v_*\in\mathcal V:=\mathbb R^d,$$

as well as

$$\int_{\mathbb R^d}(|v_*|^2-|v|^2)\,b(v,v_*)\le K_4\quad\forall\,v\in\mathbb R^d,$$

for some constants $K_i\in(0,\infty)$. Show that there exists a positive and unit mass steady state $f_1\in L^2\cap L^1_2$ (Hint. Use Theorem 1.2) and that any unit mass solution converges to that steady state (Hint. Use the GRE method).

6. Third example: the growth fragmentation equation

In the two previous sections we have considered some equations which have a clear law of conservation (in most of the cases which are mass conservative). Here we present a situation where the conservation law exists as a consequence of the Krein-Rutman we have presented in section 2.

We consider the growth-fragmentation equation

$$\partial_t f=\mathcal L f=-\partial_x f-K(x)f+\int_x^\infty k(y,x)f(y)\,dy\quad\text{on }(0,\infty)\times(0,\infty),$$

on the density $f=f(t,x)$ of particles (or cells) of size $x>0$ and time $t\ge0$, that we complement with the boundary condition

$$f(t,0)=0\quad\forall t>0.$$

We recognize the fragmentation operator

$$\mathcal F f:=-K(x)f+\mathcal F^+f,\quad \mathcal F^+f=\int_x^\infty k(y,x)f(y)\,dy$$

with

$$K(x)=\int_0^x\frac{y}{x}\,k(x,y)\,dy,\quad k(x,y)=K(x)\wp(y/x)/x,\quad\int_0^1 z\wp(z)\,dz=1.$$

As a consequence of the identity

$$(\mathcal{F}^{+*}\phi)(x) = \int_0^x k(x,y)\phi(y)\,dy, \quad \forall\,\phi \in L^1_{loc}(\mathbb{R}_+),$$

and the above relation between K and k, we then have

$$\langle \mathcal{F} f, x\rangle = \langle f, K(x)x - \mathcal{F}^{+*}x\rangle = 0$$

but $\psi(x) := x$ is not an invariant for the all equation since that for any $f \in L^1_x(\mathbb{R}_+)$, $f \not\equiv 0$,

$$\langle \mathcal{L} f, \psi\rangle = \langle -\partial_x f, x\rangle = \langle f\rangle \ge 0$$

We assume that

$$\wp_* \le \wp(z) \le \wp^* \quad \forall\, z \in (0,1), \qquad K_*\langle x\rangle^\gamma \le K(x) \le K^*\langle x\rangle^\gamma \quad \forall\, x \ge 0$$

for some real numbers $\wp_*, \wp^*$, γ, K_*, $K^* \in (0,\infty)$.

Proposition 6.1. *There exist $r > 2$ and $b^* \in \mathbb{R}_+$ such that $\mathcal{L} - b^*$ is dissipative in $L^2_{r/2}$.*

Proof of Proposition 6.1. For $\ell \in \mathbb{R}_+$, we compute

$$(\mathcal{L}\, g, g)_{L^2(x^{\ell/2})} = T_1 + T_2 + T_3$$

with

$$\begin{aligned}
T_1 &:= \int_0^\infty (-\partial_x g)\, g\, x^\ell = \int_0^\infty \frac{\partial_x x^\ell}{2}\, g^2 = \frac{\ell}{2}\int g^2\, x^{\ell-1}\,dx,\\
T_2 &:= -\int_0^\infty K(x)\, g\, x^\ell\, g \le -K_* \int_0^\infty g^2\, x^\ell \langle x\rangle^\gamma,\\
T_3 &:= \int_0^\infty (\mathcal{F}^+ g)\, x^\ell\, g \le \wp^* K^* \int_0^\infty x^\ell g(x) \left(\int_x^\infty g(y)\,\langle y\rangle^\gamma\, y^{-1}\,dy\right) dx.
\end{aligned}$$

Introducing the notation $\mathcal{G}(x) = \int_x^\infty g(y)\,\langle y\rangle^\gamma\, y^{-1}\,dy$, we have

$$\begin{aligned}
T_3 &\le -\frac{\wp^* K^*}{2}\int_0^\infty 2\,\mathcal{G}\,\mathcal{G}'\, x^{\ell+1}\langle x\rangle^{-\gamma}\,dx\\
&= \frac{\wp^* K^*}{2}\int_0^\infty \mathcal{G}^2(x)\,\partial_x(x^{\ell+1}\langle x\rangle^{-\gamma})\,dx.
\end{aligned}$$

Thanks to the Cauchy-Schwarz inequality, we get for any $a > 1$

$$\mathcal{G}^2(x) \le \int_x^\infty \langle y\rangle^{2\gamma} y^{a-2}\, g^2(y)\,dy \int_x^\infty y^{-a}\,dy \le \frac{x^{1-a}}{a-1}\int_x^\infty \langle y\rangle^{2\gamma} y^{a-2}\, g^2(y)\,dy.$$

All together, we obtain

$$T_3 \le \frac{\wp^* K^*}{2}\,\frac{\ell+1}{a-1}\int_0^\infty \langle y\rangle^{2\gamma} y^{a-2}\, g^2(y)\int_0^y x^{1-a+\ell}\,\langle x\rangle^{-\gamma}\,dx\,dy.$$

With the choice $\ell = 0$, $a \in (\max(1, 2-\gamma), 2)$, we deduce

$$T_3 \le C(\gamma)\wp^* K^* \int_0^\infty \langle y\rangle^{2\gamma}\, g^2(y)\,dy.$$

With the choice $\ell > \gamma$, $a \in (1, 2+\ell-\gamma)$, we deduce

$$T_3 \le \nu \int_0^\infty \langle y\rangle^\gamma\, y^\ell\, g^2(y)\,dy, \quad \nu := \frac{\wp^* K^*}{2}\,\frac{\ell+1}{(\ell+1)-\gamma-(a-1)} \times \frac{1}{a-1}.$$

More precisely, choosing $\ell = r := \gamma + 2a - 1$, we have

$$\nu = \frac{\wp^* K^*}{2}\,\frac{\gamma+2a}{a+1} \times \frac{1}{a-1} \le \frac{1}{2} K_*$$

for $a > 1$ large enough, and we fix such a real number a such that furthermore $r > 2$.

As a conclusion, we have proved that for the above definition of r, we have

$$\begin{aligned}(\mathcal{L}\, g, g)_{L^2((1+x^r)^{1/2})} &\le -K_* \int_0^\infty g^2 \langle x\rangle^\gamma \, dx + C(\gamma)\wp^* K^* \int_0^\infty g^2 \langle x\rangle^{2\gamma} \, dx \\ &\quad + \frac{r}{2} \int g^2 \, x^{r-1} \, dx - \frac{K_*}{2} \int_0^\infty g^2 \langle x\rangle^\gamma \, x^r \, dx \\ &\le C \int_0^\infty g^2 \, dx - \frac{K_*}{4} \int_0^\infty g^2 \langle x\rangle^{\gamma+r} \, dx,\end{aligned}$$

because the dominant term for large values of $x > 0$ is the last term. We next fix $b^* > 0$ large enough and we deduce that

$$((\mathcal{L} - b^*)\, g, g)_{L^2_{r/2}} \le -\alpha \, \|g\|^2_{L^2_{(r+\gamma)/2}}.$$

for some constant $\alpha > 0$. □

Proposition 6.2. *With the notation of teh previous proposition, there exists a constant C such that for any functions f, $g \in L^2_{r/2}$ satisfying the resolvent equation $(\mathcal{L} - b^*)f = g$, there holds*

$$\|f\|^2_{D(\mathcal{L})} := \|f\|^2_{L^2_{r/2}} + \|\mathcal{L} f\|^2_{L^2_{r/2}} \le C\, \|g\|^2_{L^2_{r/2}}. \tag{6.1}$$

Moreover, for any $f \in D(\mathcal{L})$, there holds

$$\|f\|^2_{L^2_{(r+\gamma)/2}} + \|\partial_x f\|^2_{L^2} \le C\, \|f\|^2_{D(\mathcal{L})}. \tag{6.2}$$

Proof of Proposition 6.4. Step 1. Consider $f, g \in L^2_{r/2}$ such that $(\mathcal{L} - b^*)f = g$. From the dissipativity estimate, we have

$$\begin{aligned}\alpha \, \|f\|^2_{L^2_{(r+\gamma)/2}} &\le -((\mathcal{L} - b)f, f)_{L^2_{r/2}} = -(g, f)_{L^2_{r/2}} \\ &\le \frac{\alpha}{2} \|f\|^2_{L^2_{r/2}} + \frac{1}{2\alpha} \|g\|^2_{L^2_{r/2}},\end{aligned}$$

from which we deduce

$$\|f\|_{L^2_{(r+\gamma)/2}} \le \alpha^{-1} \|g\|_{L^2_{r/2}}. \tag{6.3}$$

Moreover, from the resolvent equation again, there holds

$$\|\mathcal{L} f\|^2_{L^2_{r/2}} = (b^* f + g, \mathcal{L} f)_{L^2_{r/2}} \le (b^* \, \|f\|_{L^2_{r/2}} + \|g\|_{L^2_{r/2}}) \, \|\mathcal{L} f\|_{L^2_{r/2}},$$

from which we deduce (6.1) thanks to (6.3).

Step 2. On the one hand, from the dissipativity estimate, we have

$$\alpha \, \|f\|^2_{L^2_{(r+\gamma)/2}} \le -((\mathcal{L} - b)f, f)_{L^2_{r/2}} \le (b\|f\|_{L^2_{r/2}} + \|\mathcal{L} f\|_{L^2_{r/2}}) \|f\|_{L^2_{r/2}},$$

from which we deduce

$$\|f\|_{L^2_{(r+\gamma)/2}} \le C \|f\|_{D(\mathcal{L})}. \tag{6.4}$$

On the other hand, from the very definition of $\mathcal{L}$ and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\int (\partial_x f)^2 &= \int ((\mathcal{F}^+ f) - Bf - \mathcal{L} f)\, \partial_x f \\ &\le (\|\mathcal{F}^+ f\|_{L^2} + C\|f\|_{L^2_{\gamma/2}} + \|\mathcal{L} f\|_{L^2}) \, \|\partial_x f\|_{L^2},\end{aligned}$$

or equivalently (using the Young inequality)

$$\|\partial_x f\|_{L^2} \le C \, (\|\mathcal{F}^+ f\|_{L^2} + \|f\|_{L^2_{\gamma/2}} + \|\mathcal{L} f\|_{L^2}).$$

Using the Cauchy-Schwarz inequality, we have

$$(\mathcal{F}^+ f)(x)^2 \le (K^* \wp^*)^2 \int_x^\infty f^2 \langle y\rangle^{2\gamma} y^{a-2} \, dy \, \frac{x^{1-a}}{a-1},$$

and with the choice $a = 3/2$, we deduce

$$\begin{aligned}\|\mathcal{F}^+ f\|_{L^2}^2 &\le 2(K^*\wp^*)^2 \int_0^\infty f^2 \langle y\rangle^{2\gamma} y^{-1/2} \Big(\int_0^y x^{-1/2}\,dx\Big)\,dy \\ &\le (K^*\wp^*)^2 \int_0^\infty f^2 \langle y\rangle^{2\gamma}\,dy.\end{aligned}$$

Putting together the two last estimates, we get

$$\|\partial_x f\|_{L^2} \le C\,(\|f\|_{L^2_\gamma} + \|\mathcal{L} f\|_{L^2}),$$

and we conclude to (6.2) thanks to (6.4) and recalling that $r > \gamma$. □

Corollary 6.3. *With the notation of teh previous propositions, for any $b \ge b_*$, $\mathcal{L} - b$ is dissipative in $H := L^2_{r/2}$ and $R_\mathcal{L}(b)$ is compact.*

Proof of Corollary 6.3. The main point is to explain why the resolvent operator $R_\mathcal{L}(b)$ is well define for any $b \ge b^*$. We do not give all the details of the proof, but just sketch some possible strategies. Finally, the fact that $R_\mathcal{L}(b)$ is compact just comes from Proposition 6.4 and the compact embedding $D(\mathcal{L}) \subset H$.

- *Strategy 1.* We define $\mathcal{B} f := -\partial_x f - K(x) f$, and we observe that the semigroup

$$[S_\mathcal{B}(t) f_0](x) := f_0(x-t) e^{\mathcal{K}(x-t) - \mathcal{K}(x)}, \quad \mathcal{K}(y) := \int_0^y K(z)\,dz,$$

is well defined for $f_0 \in C_c((0,\infty))$, its generator is $\mathcal{B}$ and it extends as a bounded semigroup in H. More precisely, it is a contraction for an equivalent norm, because if $f_\mathcal{B}(t) = S_\mathcal{B}(t) f_0$, we have

$$\begin{aligned}\frac{d}{dt} \int f_\mathcal{B}^2 (1 + \varepsilon x^r)\,dx &= \int [-\partial_x f_\mathcal{B}^2 - 2K(x) f_\mathcal{B}^2](1+\varepsilon x^r)\,dx \\ &\le \int f_\mathcal{B}^2\,[\varepsilon r x^{r-1} - 2K_* \langle x\rangle^\gamma - 2\varepsilon\,\langle x\rangle^\gamma x^r]\,dx \le 0\end{aligned}$$

for $\varepsilon > 0$ small enough. Thanks to the Duhamel formula and the previous estimates on the operator $\mathcal{F}^+$ in H, we can build the semigroup $S_\mathcal{L}$ on H which satisfies the growth estimate $\|S_\mathcal{L}(t)\|_{\mathscr{B}(H)} \le e^{b^* t}$. We conclude thanks to Lemma 2.7.

- *Strategy 2.* We can use the same characteristics method and fixed point theorem directly at the level of the resolvent equation $(\mathcal{L} - b) f = g$ in order to prove that this one has a unique solution for any $g \in H$ and any $b > b_*$.

- *Strategy 3.* We can use a vanishing viscosity method by adding an $\varepsilon \partial^2_{xx}$ term in the definition of $\mathcal{L}$ and prove the well-posedness of the associated resolvent problem thanks to the Lax-Milgram Theorem or the existence of a variational solution (and thus a semigroup) thanks to the J.-L. Lions' Theorem. We then pass the limit $\varepsilon \to 0$. □

Proposition 6.4. *The growth-fragmentation operator $\mathcal{L}$ satisfies the complex Kato's inequality in the sense that for any smooth function $f \in L^2_{r/2}$, there holds*

$$\mathcal{L}|f| \ge (\mathit{sign} f) \cdot (\mathcal{L} f). \tag{6.5}$$

Proof of Proposition 6.4. For any $f \in L^2_{(r+\gamma)/2} \subset L^1_\gamma$, there holds

$$\theta'(f) \cdot \mathcal{F}^+ f - \mathcal{F}^+ \theta(f) = \int_x^\infty k(y,x) [\theta'(f(x)) \cdot f(y) - \theta(f(y))]\,dy \le 0,$$

because $k \ge 0$ and

$$\theta'(a) b - \theta(b) \le 0 \quad \forall\, a, b \in \mathbb{R}$$

when $\theta(s) = s_+$, as well as

$$\theta'(a) \cdot b - \theta(b) = \frac{\bar a b + a \bar b}{2|a|} - |b| \le 0 \quad \forall\, a, b \in \mathbb{C}$$

when $\theta(s) = |s|$. Since we also have

$$(-\partial_x f - K(x) f) \cdot \theta'(f) = -\partial_x \theta(f) - K(x) \theta(f),$$

we immediately deduce that (6.5) holds. □

Proposition 6.5. *The growth-fragmentation operator $\mathcal{L}$ satisfies the strong maximum principle in the sense that for any smooth function $f \in L^2_{r/2}$, there holds*

$$f \ge 0, \quad \mathcal{L} f \le 0 \quad \textit{implies} \quad f \equiv 0 \textit{ or } f > 0, \tag{6.6}$$

and $D(\mathcal{L}) \subset C([0,\infty))$.

Proof of Proposition 6.5. Since $D(\mathcal{L}) \subset H^1(\mathbb{R}_+) \subset C(\mathbb{R}_+)$, we have $f \in C(\mathbb{R}_+)$ and we may assume that there exists $x_0 \ge 0$ such that $f(x_0) > 0$. From the assumed inequality satisfies by f, we have

$$\partial_x f + K\, f \ge \mathcal{F}^+ f \ge 0 \quad \text{on} \quad (x_0, \infty), \tag{6.7}$$

and then $f(x) \ge f(x_0) \exp(-\int_{x_0}^x K(z)\, dz) \mathbf{1}_{x \ge x_0} =: g(x)$ for any $x \ge x_0$. Coming back to (6.7), we have

$$f(0) \ge 0, \qquad \partial_x f + K\, f \ge \mathcal{F}^+ g > 0 \quad \text{on} \quad (0, x_0),$$

which in turn implies $f > 0$. □

Exercise 6.6. *With the help of that previous propositions, investigate the first eigenvalue problem (thanks to the Krein-Rutman Theorem in section 2) and the long-time asymptotics of the solutions (thanks to the GRE principle in section 3).*

7. Appendix

Theorem 7.1 (Brouwer-Schauder-Tychonoff). *Consider a locally convex topological vector space X and $\mathcal{Z} \subset X$ a convex set which is metrizable and compact for the induced topology. Then, any continuous function $\varphi : \mathcal{Z} \to \mathcal{Z}$ has a least one fixed point.*

Remark 7.2. *The examples we have in mind are the following:*

1. A Banach space X endowed with its norm $\| \cdot \|_X$ and a convex and bounded set $\mathcal{Z} \subset X$ which is furthermore compact for the strong topology. Typically $X = L^p$ and $\mathcal{Z} := \{f \in W^{1,p} \cap L^p_1;\ \|f\|_{W^{1,p} \cap L^p_1} \le 1\}$.

2. A separable and reflexive Banach space X endowed with the weak topology $\sigma(X, X')$ and a bounded, closed and convex set $\mathcal{Z} \subset X$. Because X' is separable, the topology $\sigma(X, X')$ on the bounded set $\mathcal{Z}$ is metrizable, and the set $\mathcal{Z}$ is both topologically and sequentially compact.

3. $X = L^1(\Omega)$, $\Omega \subset \mathbb{R}^d$ open set, endowed with the weak topology $\sigma(L^1, L^\infty)$ and a bounded, closed and convex set $\mathcal{Z} \subset X$ such that $\mathcal{Z}$ is uniformly equi-integrable both locally and at the infinity. For instance there exist $\omega : \Omega \to [1,\infty)$, $\omega(x) \to \infty$ when $|x| \to \infty$, $\Phi : \mathbb{R} \to \mathbb{R}_+$, $\Phi(s)/s \to \infty$ when $|s| \to \infty$ and $C \in \mathbb{R}_+$, such that

$$\mathcal{Z} \subset \Big\{ f \in L^1(\Omega),\ \int_\Omega (\Phi(f) + |f|\, \omega)\, dx \le C \Big\}.$$

As a consequence, $\mathcal{Z}$ is both topologically and sequentially compact for the weak topology $\sigma(L^1, L^\infty)$.

4. A Banach space X such that $X = Y'$ for a separable Banach space Y endowed with the weak $*$ topology $\sigma(X, Y)$ and a convex and bounded set $\mathcal{Z} \subset X$ which is furthermore closed for the weak $*$ topology $\sigma(X, Y)$. Because $\mathcal{Z} \subset \{f \in X,\ \|f\|_X \le C\}$, for some $C \in \mathbb{R}_+$, and that last set is topologically and sequentially compact for the weak $*$ topology $\sigma(X, Y)$, the same is true for $\mathcal{Z}$.

Proof of Theorem 7.1. By assumption $\mathcal{Z}$ is endowed with a metrizable topology associated to a family of seminorms $(p_i)_{i \in I}$ with $I = \{0\}$ or $I = \mathbb{N}$. We assume that we are in the second case, the first case being simpler, and we also assume without restriction that (p_i) is increasing. We split the proof into two steps.

Step 1. By compactness of $\mathcal{Z}$, for any $\varepsilon > 0$ and $n \in I$, there exists a finite set J and some vectors $e_j \in \mathcal{Z}$, $j \in J$, such that

$$\mathcal{Z} \subset \bigcup_{j \in J} \{p_n(x - e_j) < \varepsilon/2\}. \tag{7.1}$$

We then define φ_ε by

$$\varphi_\varepsilon(x) := \sum_i \theta_i(x)\, e_i, \quad \theta_i(x) = \frac{q_i(x)}{\sum_{j \in J} q_j(x)}, \quad q_i(x) := \max(\varepsilon - p_n(\varphi(x) - e_i), 0).$$

For any $i \in J$, the mapping $x \mapsto q_i(x)$ is continuous and moreover, for any $x \in \mathcal{Z}$, there exists at least one $i_x \in J$ such that $q_{i_x}(x) \ge \varepsilon/2$. As a consequence, $x \mapsto \sum q_j(x)$ is continuous and larger than $\varepsilon/2$, which in turn imply that φ_ε is a continuous mapping. Because

$$0 \le \theta_i(x) \quad \text{and} \quad \sum_{i \in I} \theta_i(x) = 1,$$

we get that $\varphi_\varepsilon : \mathcal{Z}_\varepsilon \subset \mathcal{Z} \to \mathcal{Z}_\varepsilon$, where $\mathcal{Z}_\varepsilon$ is the convex hull of the points $(e_j)_{j \in J}$. In particular, $\mathcal{Z}_\varepsilon$ is a convex and compact subset of the finite dimensional space $\mathrm{Vect}(e_j;\ j \in J)$ endowed with the topology induced by the family of seminorms $(p_i)_{i \in I}$, which therefore is a normable topology (it is associated to a seminorm p_m, m large, which is a norm on that finite dimensional space). We may apply the Brouwer theorem and we get the existence of at least a fixed point. Namely, there exists $x_\varepsilon \in \mathcal{Z}_\varepsilon$ such that $\varphi_\varepsilon(x_\varepsilon) = x_\varepsilon$. We next observe that for any $x \in \mathcal{Z}$, there holds

$$\begin{aligned} p_n(\varphi(x) - \varphi_\varepsilon(x)) &= p_n\big(\sum_{j\in I} \theta_j(x)(\varphi(x) - e_j)\big) \\ &\le \sum_{j\in I} \theta_j(x) p_n(\varphi(x) - e_j) \le \varepsilon \end{aligned}$$

because $p_n(\varphi(x) - e_j) \le \varepsilon$ when $\theta_j(x) \neq 0$.

Step 2. For any $n \in \mathbb{N}^*$, we take $\varepsilon_n = 1/n$ in the previous construction, and we write φ_n instead of φ_{ε_n} as well instead x_n instead of x_{ε_n}. With this notation, we have for any $n \ge m \ge 1$

$$\varphi_n(x_n) = x_n \quad \text{and} \quad p_m(\varphi(x) - \varphi_n(x)) \le \frac{1}{n}, \tag{7.2}$$

because (p_n) is an increasing sequence. By compactness of $\mathcal{Z}$ there exist a subsequence, still denoted as (x_n), and $\bar x \in \mathcal{Z}$ such that $x_n \to \bar x$. By continuity of φ and thanks to (7.2), we deduce

$$p_m(\varphi(\bar x) - \bar x) \le p_m(\varphi(\bar x) - \varphi(x_n)) + p_m(\varphi(x_n) - \varphi_n(x_n)) + p_m(\varphi_n(x_n) - x_n) + p_m(x_n - \bar x) \to 0$$

as $n \to \infty$, for any $m \ge 1$. Because (p_n) separates points, we conclude with $\varphi(\bar x) = \bar x$. □

8. Bibliographic discussion

Theorem 1.2 in section 1.1 is an abstract version and generalization of a technical lemma classically used in the proof of the Poincaré-Bendixson Theorem about the qualitative behaviour of solutions to a 2d system of ode. I learned the material of sections 1.2 and 1.4 in Haraux's book [5]. The result in section 1.4 belongs to folklore (it has been used several times in order to prove the convergence of the solution of the Boltzmann equation to the corresponding Maxwellian equilibrium).

The material of Section 2 (at least of sections 2.1 and 2.2) is very classical and it can be found in many textbooks. I learn most of the results presented in sections 2.1 and 2.2 in [1]. The proof of Theorem 2.11 is taken from [1, Proposition VI.9 & Corollaire VI.10], the proof of Theorem 2.12 is taken from [1, Théorème VI.11] to which we refer for more details. I read the proof of Proposition 2.6 somewhere in Kato's book [6]. Example 2.10 is classical, I learned it from O. Kavian. Section 2.3 about the Krein-Rutman theorem is a simplified presentation of some result obtained recently in [9] but probably can be found elsewhere (for instance in some of the many references quoted in [9]).

The computations presented in section 3 and leading to the General Relative Entropy are taken from [7]. The case $\phi = 1$ corresponds to the usual probability framework and then can be found in many earlier papers of the probability community.

The material of section 4 on the Fokker-Planck equation is a simplified presentation of more or less recent results on this very active line of research. The dissipativity estimate established in Proposition 4.3 is taken from [4] (see also [8]). The most general case when no structure assumption is made on E is inspired from a Master degree work by M. Ndao [10].

The material of section 5 on the scattering belongs to folklore. I learned it from A. Mellet.

The material of section 6 on the growth-fragmentation equation is inspired from recent research papers on the subject. In particular, the estimate established in Proposition 6.1 is taken from [3, 2].

References

[1] Brezis, H. *Analyse fonctionnelle.* Collection Mathématiques Appliquées pour la Maîtrise. [Collection of Applied Mathematics for the Master's Degree]. Masson, Paris, 1983. Théorie et applications. [Theory and applications].

[2] Cáceres, M. J., Cañizo, J. A., and Mischler, S. Rate of convergence to self-similarity for the fragmentation equation in L^1 spaces. *Commun. Appl. Ind. Math. 1*, 2 (2010), 299–308.

[3] Cáceres, M. J., Cañizo, J. A., and Mischler, S. Rate of convergence to an asymptotic profile for the self-similar fragmentation and growth-fragmentation equations. *J. Math. Pures Appl. (9) 96*, 4 (2011), 334–362.

[4] Gualdani, M. P., Mischler, S., and Mouthot, C. Factorization of non-symmetric operators and exponential H-Theorem. hal-00495786.

[5] Haraux, A. *Systèmes dynamiques dissipatifs et applications*, vol. 17 of *Recherches en Mathématiques Appliquées [Research in Applied Mathematics].* Masson, Paris, 1991.

[6] Kato, T. *Perturbation theory for linear operators.* Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.

[7] Michel, P., Mischler, S., and Perthame, B. General relative entropy inequality: an illustration on growth models. *J. Math. Pures Appl. (9) 84*, 9 (2005), 1235–1260.

[8] Mischler, S., and Mouthot, C. Exponential stability of slowly decaying solutions to the kinetic-fokker-planck equation. work in progress.

[9] Mischler, S., and Scher, J. Spectral analysis of semigroups and growth-fragmentation equations. hal-00877870.

[10] Ndao, M. Etude asymptotique d'une équation de Fokker-Planck générale. work in progress.