Keller - Segel eq

2. Identities

$$\frac{d}{dt}\int f\,dx = 0$$

$$\frac{d}{dt}\int f\,x\,dx = 0$$

$$\frac{d}{dt}\int f|x|^2dx = 4M\left(1-\frac{M}{8\pi}\right)$$

$$\frac{d}{dt}\mathcal{F}(f) = -\mathcal{D}(f)$$

with $\mathcal{F}(f) = \int f\log f + \frac{1}{2}\iint K(x-y)f(x)\,f(y)\,dx\,dy$

$$\mathcal{D}(f) = \int f\,|\nabla(\log f + K*f)|^2$$

3. A priori bounds (Entropy).

$$\mathcal{F}(f) = H(f) + \frac{1}{4\pi}\iint f(x)f(y)\log|x-y|$$

$$= \left(1-\frac{M}{8\pi}\right)H(f) + \frac{M}{8\pi}\left(H(f) + \frac{2}{M}\iint f(x)f(y)\log(x-y)\right)$$

$$\geq \left(1-\frac{M}{8\pi}\right)H(f) - \frac{M}{8\pi}C(M)$$

If $M \in (0, 8\pi)$ we have

$$\left(1-\frac{M}{8\pi}\right)\left[\int f_t \log f_t + \int f_t |x|^2 + \int f_t\right]$$

$$(HLS_{lg}) \quad \leq \mathcal{F}(f_t) + \alpha \int f_t |x|^2 + \gamma \int f_t$$

$$\leq \mathcal{F}(f_0) - \int_0^t \mathcal{D}_{\mathcal{F}}(f_s)\, ds + \alpha' \int f_0 |x|^2 + \gamma M + \beta' t$$

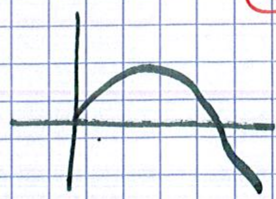
Now, on the one hand:

$$\mathcal{F}(f_0) \simeq \int f_0 \log f_0 + \int f_0(x) f_0(y) \log|x-y|$$

$$\lesssim \int f_0 \log f_0 + \int f_0(x) f_0(y) |x-y|^2$$

$$\lesssim \int f_0 \log f_0 + \int f_0 |x|^2 + \int f_0 .$$

On the other hand



$$\int f|\log f| = f(\log f)\mathbb{1}_{f>1} + f|\log f|\mathbb{1}_{0<f<e^{-|x|^2/2}}$$
$$+ f|\log f|\mathbb{1}_{e^{-|x|^2/2}<f<1}$$
$$\le f(\log f)_+ + \sqrt{f}\,\mathbb{1}_{f<e^{-|x|^2/2}}$$
$$+ f\,|\log e^{-|x|^2/2}|$$
$$\le f(\log f)_+ + e^{-|x|^2/4} + f\frac{|x|^2}{2}.$$

Lemma : $\forall f \ge 0$.

$$\int f\log f + \int f|x|^2 + \int f$$
$$\sim \int |f|\log f| + \int f|x|^2 + \int f$$
$$\sim \int f(\log f)_+ + \int f|x|^2 + \int f =: \mathcal{F}_+(f)$$

Conclusion if $M < 8\pi$

$$\mathcal{F}_+(f_t) + d\int_0^t \mathcal{D}_+(f_s)\,ds \le$$
$$\lesssim \mathcal{F}_+(f_0).$$

that gives a ~~...~~ unif control $\forall T$ on $[0,T]$ on a equi-integrability of the ~~function~~ f -

~~because~~

$$\cancel{\mathcal{D}_{\mathcal{F}}(f) = \int f \left| \frac{\nabla f}{f} - K * f \right|^2}$$

~~and~~

$$\cancel{\int \left| \frac{\nabla f}{f} - K * f \right|} \leq$$

$\boxed{\text{Def}}$: A weak solution f is a function

$$f \in C([0,T]; \mathcal{D}'(\mathbb{R}^2))$$

$$\mathcal{F}(f_t) + \int_0^t \mathcal{D}_{\mathcal{F}}(f_s)\, ds \leq \mathcal{F}_0$$

$$\Rightarrow \left.\begin{aligned} &\mathcal{F}_+(f_t) \leq C_T \\ &\int_0^T \mathcal{D}_{\mathcal{F}}(f_s)\, ds \leq C_T \end{aligned}\right\} \forall T$$

and $\partial_t f = \Delta f + \mathrm{div}((K * f) f)$ in the weak sense: $= \nabla(f \nabla \log f + f\, K * f)$

$$\int f_0 \varphi(0) = \iint f(-\partial_t \varphi + \nabla\varphi \cdot (\nabla \log f + K * f))$$

and $\int |\nabla \log f + K * f| f \leq \left(\int f\right)^{1/2} (\mathcal{D}_{\mathcal{F}}(f)) \in L^1(0,T)$.

$\rightarrow$ th

Theorem: $\forall f_0 \underset{\geq 0}{\in} L^1_2 \cap L^1 \log L^1$, $\int f_0 = M < 8\pi$

$\exists!$ f global weak solution to the KS and

$g(t,x) := R(t)^{-2} f(\log R(t), R(t)^{-1} x)$, $R(t) = \sqrt{1+2t}$.

satisfies

$$g(t,\cdot) \longrightarrow G \quad \text{in } L^1$$

where (unique) $\begin{cases} \Delta G + \operatorname{div}(xG) + \operatorname{div}(G K * G) = 0 \\ 0 \leq G \in C^\infty, \; G(x) \underset{x\to\infty}{\sim} e^{-|x|^2}, \; \int G = M. \end{cases}$

$\boxed{\text{proof}}$.

- existence $\begin{cases} \underline{\text{weak stability principle}} \\ + \text{approximation} \end{cases}$ BDP (2006)

- uniqueness $\begin{cases} \text{Carrillo} + \ldots \sim 2013 & L^1 \cap L^\infty \\ \sim \text{Masmoudi} + \ldots \sim 2013. & \text{"strong sol"} \\ \sim \text{Egaña, SM} \sim 2013 & \text{weak sol.} \end{cases}$

$$f(t) = S_\Delta(t) f_0 + \int_0^t S_\Delta(t-s) * (\operatorname{div}(f_s K * f_s)) \, ds$$

and ~~from~~ "contraction" for the norm $\sup_{(0,T)} t^{1/4} \| f(t) - g(t) \|_{L^{4/3}}$

- large time behavior. $\begin{cases} \text{Blanchet DP} \\ \text{Dolbeault} \sim 2012 \\ \quad\text{Carpio} \\ \text{Egaña, SM} \sim 2013. \end{cases}$

$$\frac{d}{dt} \mathcal{E}(g) = -\mathcal{D}_\mathcal{E}(g)$$

$$\mathcal{E}(g) := \int g \log g + \frac{1}{2} \int g(x) g(y) K(x-y) + \frac{1}{2} \int g |x|^2$$

$$\mathcal{D}_\mathcal{E}(g) = \int g \left| \nabla \left(\log g + K * g + \frac{|x|^2}{2} \right) \right|^2$$

$g(t) \to g_\infty$ weak $\mathcal{D}_\mathcal{E}(g_\infty) = 0 \Rightarrow g_\infty = G$

$\boxed{\text{Prop.}}$ Consider a sequence of weak solution (f_n) s.t.

$$0 \le f_n \ , \quad M(f_n) = M$$

$$\sup_{(0,T)} \mathcal{H}^{\#}_{+}(f_n) \le C_T \quad \forall T$$

$$\sup_{(0,T)} \mathcal{H}(f_n(t)) + \int_0^T \mathcal{D}_{\mathcal{H}}(f_n(s))\, ds \le \cancel{\mathcal{H}(f_{n0})} \le C$$

Then, we may extract a subsequence, still denoted as (f_n) and there exists f s.t.

(1) $f_n \longrightarrow f$ weakly $L^1_{t,x}$

and in fact

(2) $f_n \longrightarrow f$ strongly $L^p_{t,x}$ $\forall 1 \le p < 2$

s.t. f is a weak solution.

Pb: (1) ~~product~~ a priori bound gives (1) ok.

(2) product $f_n K * f_n \to f\, K*f$? strong cvge will help

(3) $f K * f \in$ better than L^1 (but it is not necessary)

$\boxed{\text{We need more estimates}}$

A priori bound Fisher

Lemma For any $0 \le f \in L^1(\mathbb{R}^2)$:

(1) $\forall p \in [1,\infty)$ $\quad \|f\|_{L^p} \le C_p\, M^{1/p}\, I(f)^{1-1/p}$

(2) $\forall q \in [1,2)$ $\quad \|\nabla f\|_{L^q} \le C_q\, M^{1/q - 1/2}\, I(f)^{3/2 - 1/q}$

proof: of (2)

(*1) $\|\nabla f\|_{L^q}^q = \int \left| \frac{\nabla f}{\sqrt{f}} \right|^q f^{q/2}$

$$\le I(f)^{q/2} \left(\int f^{q/2-q} \right)^{\frac{2-q}{2}} \qquad \frac{q}{2} + \underbrace{\left(1-\frac{q}{2}\right)}_{\frac{2-q}{2}} = 1$$

$$= I(f)^{q/2}\, \|f\|_{L^{\frac{q}{2-q}}}^{q/2}$$

$q^* := \frac{2q}{2-q}$ Sobolev associated to $q \in [1,2)$

(*2) $\|f\|_{L^{q/2-q}} \le \|f\|_{L^{q^*/2}} \le \|f\|_{L^1}^{\frac{1}{q^*-1}}\, \|f\|_{L^{q^*}}^{\frac{q^*-2}{q^*-1}}$

$$\le \|f\|_{L^1}^{\frac{1}{q^*-1}}\, C\, \|\nabla f\|_{L^q}^{\frac{q^*-2}{q^*-1}}$$

(*1) + (*2) together:

$$\|\nabla f\|_{L^q}^q \le I(f)^{q/2}\, \|f\|_{L^1}^{\frac{1}{2(q^*-1)}}\, \cancel{\|\nabla f\|_{L^q}^{\frac{q^*-2}{2(q^*-1)}}}$$

which gives (1).

of (1) $\quad p = q^*/2 = \frac{q}{2-q}$ with $q = \frac{2p}{1+p} \in [1,2)$

$$\|f\|_{L^p} \le \|f\|_{L^1}^{\cdot}\, \|f\|_{L^{q^*}}^{1-\cdot} \le \|f\|_{L^1}^{\cdot}\, \|\nabla f\|_{q}^{1-\cdot} \le \text{—} .$$

Lemma. Fn

$$\frac{1}{2} I(f) \le \mathcal{D}_{\mathcal{F}}(f) + 2M\exp(16 H_+(f) C M)$$
$$\in L^1(0,T) \quad \forall T.$$

proof.
$$\mathcal{D}_{\mathcal{F}}(f) = \int f\,|\nabla \log f + \nabla K * f|^2$$
$$= \int \frac{|\nabla f|^2}{f} + 2\int \cancel{f}\,\frac{\nabla f}{\cancel{f}}\cdot \nabla K * f + \int f\,|\quad|^2$$
$$\ge I(f) - 2\int f\,(\Delta K * f)$$
$$= I(f) - 2\int f^2$$

$\forall A>1$. by Cauchy-Schwarz

$$\int f^2 1_{f>A} \le \left(\int f\, 1_{f\ge A}\right)^{1/2} \Big(\underbrace{\int f^3}_{M.\, I(f)^2}\Big)^{1/2}$$
$$\le \left(\int f \frac{\log f_+}{(\log A)_+}\right)^{1/2} M^{1/2}\, I(f)$$
$$\le \frac{H_+(f)^{1/2}}{(\log A)^{1/2}} M^{1/2} I(f) \le \frac{1}{4} I(f).$$

$$\Rightarrow \cancel{I(f)} \le \mathcal{D}_{\mathcal{F}}(f) + 2\int f^2 1_{f<A} + \underbrace{\cancel{2\int f^2 1_{f>A}}}_{\cancel{\frac12 I(f)}}$$
$$\frac{1}{2} I(f) \le \mathcal{D}_{\mathcal{F}}(f) + \dots \qquad \square$$

A priori bound L^p, $W^{1,q}$, ...

lemma

$$\boxed{\begin{array}{l} f \in L^{p'}(0,T;L^p) \quad \forall p \in (1,\infty) \\ K*f \in L^{p'}(0,T;L^{\frac{2p}{2-p}}) \quad \forall p \in (1,2) \quad \text{(or almost that)} \end{array}}$$

proof.

(1) $f \in L^\infty(0,T;L^1)$

~~$f \in$~~

$$\|f_t\|_{L^p}^{\frac{p}{p-1}} \leq C_p \cdot M^\theta \cdot I(f) \in L^1_t$$

(2) It is a consequence of the HLS ineq which says that

$$\left\| \frac{1}{|x|} * f \right\|_{L^r} \leq C \|f\|_{L^p}$$

with $\frac{1}{r} = \frac{1}{p} + \frac{1}{2} - 1 = \frac{1}{p} - \frac{1}{2} = \frac{2-p}{2p}$.

Young inequality applied to $\frac{1}{|x|} \in L^\alpha + L^\beta$ $\forall \alpha < 2 < \beta$

says $K*f \in L^{p'}(0,T;L^{\tilde\alpha} + L^{\tilde\beta})$

$\forall \tilde\alpha < \frac{2p}{2-p} < \tilde\beta$.

(2) $\boxed{p = 3/2 \quad K*f \in L^3(0,T;L^6)}$

(1) $\boxed{p = 2 \qquad f \in L^2(0,T;L^2).}$ and also $L^3(0,T;L^q)$ $\forall q \leq 3/2$

$p = 3/2$

~~Cor~~ Weak and strong compactness.

~~F~~ Lemma 1 $\exists f\ \exists (f_n)$ s.t.

$$f_n \longrightarrow f \quad L^2\ \cancel{\text{weak}}\ ((0,T)\times\mathbb{R}^2)\ \text{weak}.$$

Lemma 2 : $\forall \varphi \in C_c^{\infty}(\mathbb{R}^2)$

$$\int f_n \varphi \longrightarrow \int f \varphi \quad \text{strongly } L^{\infty}(0,T)\ \cancel{\forall p \in [1,\infty[}.$$

proof. We write

$$\frac{d}{dt}\int f_n \varphi = \underbrace{\int f_n \Delta\varphi}_{\in L^{\infty}(0,T)} + \int \underbrace{f_n (K * f_n) \nabla\varphi}_{\in L^{3/2}_t(L^1_x)}$$

$L^{3/2}$

$L^3(L^q)$, $q \le 3/2$, $L^3(L^{5/4})$

$K * f_n \in L^3(L^{6+} + L^{6-})$, $\cap\ L^3(L^5_{loc})$, $\nabla\varphi \in L^{\infty}$ comp

$$\in L^{3/2}(0,T)$$

$$\int f_n \varphi \in \cancel{W^{3/2,1}}\ W^{1,3/2}(0,T) \subset C^{0,1/3}$$

by Ascoli

$$\int f_n \varphi \longrightarrow u \quad \text{uniformly}$$

but also

$$\int f_n \varphi \longrightarrow \int f \varphi \quad L^2_w.$$

Lemma 3 (Aubin's lemma).

$$f_n \xrightarrow[n\to\infty]{} f \text{ strongly in } L^a(0,T;L^b)$$

$$\forall\ a<p' \quad b<p$$

$$\forall p \in (1,\infty).$$

proof: Step 1 If ρ_ε ~~(x)~~ $\in \mathcal{D}(\mathbb{R}^d)$ ~~and~~ ρ_ε.

then

$$f_n * \rho_\varepsilon \xrightarrow[n\to\infty]{} f*\rho_\varepsilon \text{ strongly in } L^r((0,T)\times\mathbb{R}^2)$$

$$2 > r \geq 1$$

First

$$\|f_n * \rho_\varepsilon\|_{L^r} \leq \|f_n\|_{L^r}\|\rho_\varepsilon\|_{L^1} \in L^{r'} \subset L^r(0,T).$$

$$\exists\, \varphi_\alpha = \sum_j \varphi^1_{\alpha,j}(x)\,\varphi^2_{\alpha,j}(y) \longrightarrow \rho_\varepsilon(x-y)\chi(y) \text{ unif in } x,y$$

$$\mathcal{D}(\mathbb{R}^2)\otimes\mathcal{D}(\mathbb{R}^2) \text{ dense in } \mathcal{D}(\mathbb{R}^2\times\mathbb{R}^2)$$

$$\left\| \int f_n\, \rho_\varepsilon(x-y)\chi(y) - \int f_n \varphi_\alpha(x,y) \right\|_{L^r((0,T)\times\mathbb{R}^2)}$$

$$\leq \|f_n\|_{L^r}\,\|\rho_\varepsilon\chi - \varphi_\alpha\| \longrightarrow 0.$$

but

$$\int f_n \varphi_\alpha \in \text{strongly compact in } L^r((0,T)\times\mathbb{R}^2)$$

$$\| \quad \sum_j \int f_n(y)\,\varphi^2_{\alpha,j}(y)\,dy\ \varphi^1_{\alpha,j}(x)$$

and $(u_{n,\alpha}) \in$ strongly compact in X $\forall \alpha$

$$\sup_n \|u_{n,\alpha} - u_n\|_X \xrightarrow[\alpha\to 0]{} 0$$

then u_n is strongly compact in X.

precompact + complet $\Rightarrow$ compact.

As a consequence

$$\int f_m \rho_\varepsilon(x-y)\chi_R(y)\,dy \text{ is strongly cvg in } L^r_{r=1}$$

and again

$$\int f_m \rho_\varepsilon(x-y)(1-\chi_R(y)) \longrightarrow 0 \quad \text{in } L^1.$$

and finally $\int f_m \rho_\varepsilon(x-y) \in L^2_{t,x} \cap L^1_{t,x}(|x|^2)$

$$\int f_m \rho_\varepsilon(x-y)\,dy \longrightarrow \int f \rho_\varepsilon(x-y)\,dy \quad \text{a.e.}$$

~~Step 2~~.

$$\Rightarrow \text{ cvgce in } L^r_{t,x} \quad \forall\, 1 \leq r < 2.$$

$\boxed{\text{Step 2}}$ – We just write:

$$f_m = f_m - f_m * \rho_\varepsilon + f_m * \rho_m$$

with

$$\|f_m - f_m * \rho_\varepsilon\|_{L^1} \leq \iint |f_m(x+y) - f_m(x)|\,\rho_\varepsilon(y)\,dy\,dx$$

$$= \iint_{\mathbb{R}^d} \int_0^1 |\nabla f_m(x)\cdot \underbrace{y}\, s|\,\rho_\varepsilon(y)\,\underbrace{\mathbb{1}_{y\leq\varepsilon}}\,dy\,dx\,ds$$

$$\leq 2\varepsilon \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\nabla f_m(x)|\,\rho_\varepsilon(y)\,dy\,dx.$$

$$\leq 2\varepsilon\, \|\nabla f_m\|_{L^1}$$

$$\leq 2\varepsilon\, M^{1/2}\, I(f_m)^{1/2} \in L^1_t.$$

$$\|f_m - f_m * \rho_\varepsilon\|_{L^1_{t,x}} \leq \varepsilon \to 0$$

$$f_n = \underbrace{f_n - f_n * \rho_\varepsilon}_{\text{small in } L^r_{t,x}} + \underbrace{f_n * \rho_\varepsilon}_{\text{compact in } L^r}$$

$1 \leq r < r_0 \leq 2$

$r_0 \overset{?}{=}$

$\Rightarrow$ f_n is compact in L^r

and then L^r $\forall\, 1 \leq r < 2$

$\square$

<u>Conclusion</u>.

$f_n \cdot \underbrace{K * f_n}$ $\longrightarrow$ $f \cdot K * f$ $L^1_{t,x}$.

$\downarrow$

f $L^r_{t,x}$ or $L^\alpha_t(L^\beta_x)$

$\forall\, \alpha < p'$, $\forall\, \beta < p$ $\forall\, p \in [1, \infty[$

$\downarrow$

strongly also.