Université Paris-Dauphine **2013-2014**
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An introduction to evolution PDEs

Problem I

Question 1 - the linear problem

(1) Under which assumption on $E = E(x)(\neq 0!)$ the problem

$$\frac{\partial}{\partial t} f = \Delta f + \nabla \cdot (E\, f) \quad \text{in } (0,\infty) \times \mathbb{R}^d \tag{0.1}$$

$$f(0,x) = \varphi(x) \quad \text{in } \mathbb{R}^d \tag{0.2}$$

admits a variational solution for any $\varphi \in L^2(\mathbb{R}^d)$. What does that mean?
(2) Explain why for $E = E(x) \in W^{1,\infty}(\mathbb{R}^d)$ and $\varphi \in L^2(\mathbb{R}^d) \cap L^1_2(\mathbb{R}^d)$ there exists a unique function

$$f \in X_T := C([0,T); L^2) \cap L^\infty(0,T; L^1_2) \cap L^2(0,T; H^1) \cap H^1(0,T; H^{-1}), \quad \forall\, T,$$

which is variational and renormalized solution to the problem (0.1)-(0.2).
(3) Explain (briefly) why the same result holds for a vector field which is time constant by steps and then for any $E = E(t,x) \in L^\infty(0,T; W^{1,\infty}(\mathbb{R}^d))$.

Question 2 - a smooth nonlinear problem

We denote by S_Δ the semigroup associated to the heat equation.
(1) Give a integral representation of S_Δ and prove that

$$\|\nabla S_\Delta(t)\psi\|_{L^2} \le \frac{C}{t^{1/2}} \|\psi\|_{L^2} \quad \forall\, \psi \in L^2(\mathbb{R}^d).$$

We fixe $a \in W^{1,\infty}(\mathbb{R}^d)$ a vector field. For $g \in X_T$ we denote $E_g := g *_x a$. For any $\varphi \in L^2(\mathbb{R}^d) \cap L^1_2(\mathbb{R}^d)$, we define the operator $\mathcal{Q} : Y_T \to Y_T$, $Y_T := C([0,T]; L^2)$, by $\mathcal{Q}g := f$ for any $g \in X_T$ where f satisfies

$$f(t) := S_\Delta(t)\varphi + \int_0^t S_\Delta(t-s)\Big(\mathrm{div}_x(E_g(s) f(s))\Big)\, ds. \tag{0.3}$$

(2) What is the PDE associated to the functional equation (0.3)? Why does such function $f \in X_T$ exist?

We define

$$\mathcal{C} = \mathcal{C}_A := \{f \ge 0,\ \|f\|_{L^1} = M,\ \sup_{[0,T]} \|f\|_{L^2 \cap L^1_2} \le A\}.$$

(3) Prove that for $A := A(T, \varphi)$ large enough $\mathcal{Q} : \mathcal{C} \to \mathcal{C}$.
For $g_1, g_2 \in \mathcal{C}$ we denote $f_1 := \mathcal{Q} g_1$, $f_2 := \mathcal{Q} g_2$, as well as $f := f_2 - f_1$, $g := g_2 - g_1$.
(4) Prove that

$$f(t) = \int_0^t \mathrm{div}_x \Big(S_\Delta(t-s)(E_g(s) f_1(s) + E_{g_2}(s) f(s)) \Big)\, ds$$

and then

$$\|f(t)\|_{L^2} \le \int_0^t \frac{C}{(t-s)^{1/2}} \left(\|g(s)\|_{L^2} + \|f(s)\|_{L^2} \right) ds.$$

(5) Deduce that for T small enough (which only depends on φ and a) there exists $\alpha \in (0, 1/2)$ such that

$$(1-\alpha) \sup_{[0,T]} \|f\|_{L^2} \le \alpha \sup_{[0,T]} \|g\|_{L^2}.$$

(6) Conclude that for any $\varphi \in L^2(\mathbb{R}^d) \cap L^1_2(\mathbb{R}^d)$ and $a \in W^{1,\infty}(\mathbb{R}^d)$ there exists a unique (global) variational solution $f \in X_T$, $\forall T > 0$, to the nonlinear PDE

$$\frac{\partial}{\partial t} f = \Delta f + \nabla \cdot ((a * f)\, f) \quad \text{ in } (0,\infty) \times \mathbb{R}^d \tag{0.4}$$

$$f(0,x) = \varphi(x) \quad \text{ in } \mathbb{R}^d. \tag{0.5}$$

Question 3 - Existence of solution for the Keller-Segel equation

We assume from now on that $d = 2$. We accept that there exists a sequence (κ_ε) of potentials on $\mathbb{R}^2$ such that (abusing notations) $\kappa_\varepsilon(z) = \kappa_\varepsilon(|z|)$, κ_ε is a smooth (say $W^{2,\infty}(\mathbb{R}_+)$) non-increasing function such that

$$\kappa_\varepsilon(r) = -\frac{1}{2\pi} \log r \quad \forall r \in [\varepsilon, 1/\varepsilon]$$

and

$$|\kappa_\varepsilon(r)| \le \frac{1}{2\pi} |\log r|, \quad |\nabla \kappa_\varepsilon(z)| \le \frac{1}{2\pi |z|}, \quad \rho_\varepsilon := -\Delta \kappa_\varepsilon(z) \ge 0,$$

with $\|\rho_\varepsilon\|_{L^1(\mathbb{R}^2)} \le 1$. We define $a_\varepsilon := \nabla \kappa_\varepsilon$.
(1) Prove that for any $0 \le \varphi \in L^2(\mathbb{R}^d) \cap L^1_2(\mathbb{R}^d)$ and $\varepsilon > 0$, there exists a renormalized solution $0 \le f_\varepsilon \in L^\infty(0,T; L^1_2) \cap X_T$ to the nonlinear PDE

$$\frac{\partial}{\partial t} f_\varepsilon = \Delta f_\varepsilon + \nabla \cdot ((a_\varepsilon * f_\varepsilon)\, f_\varepsilon) \quad \text{ in } (0,\infty) \times \mathbb{R}^2 \tag{0.6}$$

$$f_\varepsilon(0,x) = \varphi(x) \quad \text{ in } \mathbb{R}^2. \tag{0.7}$$

(2) We assume that

$$\int_{\mathbb{R}^2} \varphi(x)\, dx = M \in (0, 8\pi).$$

Prove that (f_ε) satisfies

$$\sup_{[0,T]} \int_{\mathbb{R}^2} f_\varepsilon \{1 + |x|^2 + |\log f_\varepsilon|\}\, dx + \int_0^T \int_{\mathbb{R}^2} |f_\varepsilon \nabla(\log f_\varepsilon + \kappa_\varepsilon * f_\varepsilon)|^2\, dxdt \le C_T$$

for some constant C_T independent of $\varepsilon \in (0,1)$. Deduce that $I(f_\varepsilon)$ is uniformly bounded in $L^1(0,T)$.

(3) Prove that there exist $f \in C([0,T]; L^1(\mathbb{R}^2))$ and a sequence $\varepsilon_k \to 0$ such that

$$f_k := f_{\varepsilon_k} \rightharpoonup f \quad \text{weakly in} \quad L^1((0,T)\times\mathbb{R}^2),$$

and that f satisfies

$$\sup_{[0,T]} \int_{\mathbb{R}^2} f \{1 + |x|^2 + |\log f|\}\, dx + \int_0^T I(f) dt < \infty$$

as well as the Keller-Segel equation.

Problem II

In all the problem, we consider the Fokker-Planck equation

$$\partial_t f = \Lambda f := \Delta_x f + \mathrm{div}_x(f\,\nabla a(x)) \quad \text{in } (0,\infty)\times\mathbb{R}^d \tag{0.8}$$

for the confinement potential

$$a(x) = \frac{\langle x\rangle^\gamma}{\gamma}, \quad \gamma\in(0,1), \quad \langle x\rangle^2 := 1+|x|^2,$$

that we complement with an initial condition

$$f(0,x) = \varphi(x) \quad \text{in } \mathbb{R}^d. \tag{0.9}$$

Question 1

Exhibit a stationary solution $G\in\mathbf{P}(\mathbb{R}^d)$. Formally prove that this equation is mass conservative and satisfies the (weak) maximum principle. Explain (quickly) why for $\varphi\in L^p_k(\mathbb{R}^d)$, $p\in[1,\infty]$, $k\geq 0$, the equation (0.8)-(0.9) has a (unique) solution $f(t)$ in some functional space that must be specified. Establish that if $L^p_k(\mathbb{R}^d)\subset L^1(\mathbb{R}^d)$ then the solution satisfies

$$\sup_{t\geq 0}\|f(t)\|_{L^1}\leq\|\varphi\|_{L^1}.$$

Can we affirm that $f(t)\to G$ as $t\to\infty$? and that convergence is exponentially fast?

Question 2

We define

$$\mathcal{B}f := \Lambda f - M\chi_R f$$

with $\chi_R(x) := \chi(x/R)$, $\chi\in\mathcal{D}(\mathbb{R}^d)$, $0\leq\chi\leq 1$, $\chi(x)=1$ for any $|x|\leq 1$, and with $M,R>0$ to be fixed.
We denote by $f_\mathcal{B}(t) = S_\mathcal{B}(t)\varphi$ the solution associated to the evolution PDE corresponding to the operator $\mathcal{B}$ and the initial condition (0.9).
(1) Why such a solution is well defined (no more than one sentence of explanation)?
(2) Prove that there exists $M,R>0$ such that for any $k\geq 0$ there holds

$$\frac{d}{dt}\int_{\mathbb{R}^d} f_\mathcal{B}(t)\langle x\rangle^k\,dx \leq -c_k\int_{\mathbb{R}^d} f_\mathcal{B}(t)\langle x\rangle^{k+\gamma-2}\leq 0,$$

for some constant $c_k\geq 0$, $c_k>0$ if $k>0$, and

$$\|S_\mathcal{B}(t)\|_{L^1_k\to L^1_k}\leq 1.$$

(3) Establish that if $u\in C^1(\mathbb{R}_+)$ satisfies

$$u'\leq -c\,u^{1+1/\alpha}, \quad c,\alpha>0,$$

there exists $C = C(c, \alpha, u(0))$ such that

$$u(t) \le C/t^\alpha \quad \forall\, t > 0.$$

(4) Prove that for any $k_1 < k < k_2$ there exists $\theta \in (0,1)$ such that

$$\forall\, f \ge 0 \quad M_k \le M_{k_1}^{\theta}\, M_{k_2}^{1-\theta}, \quad M_\ell := \int_{\mathbb{R}^d} f(x)\, \langle x \rangle^\ell \, dx$$

and write how θ as a function of k_1, k and k_2.
(5) Prove that if $\ell > k > 0$ there exists $\alpha > 0$ such that

$$\|S_{\mathcal{B}}(t)\|_{L^1_\ell \to L^1} \le \|S_{\mathcal{B}}(t)\|_{L^1_\ell \to L^1_k} \le C/\langle t \rangle^\alpha,$$

and that $\alpha > 1$ if ℓ is large enough (to be specified).
(6) Prove that

$$S_{\mathcal{L}} = S_{\mathcal{B}} + S_{\mathcal{B}} * (\mathcal{A} S_{\mathcal{L}}),$$

and deduce that for k large enough (to be specified)

$$\|S_{\mathcal{L}}\|_{L^1_k \to L^1_k} \le C.$$

Remark. You have recovered (with a simpler proof) a result established by Toscani and Villani in 2001.

Question 3 (difficult)

Establish that there exists $\kappa > 0$ such that for any $h \in \mathcal{D}(\mathbb{R}^d)$ satisfying

$$\langle h \rangle_\mu := \int_{\mathbb{R}^d} h \, d\mu = 0, \quad \mu(dx) := G(x)\, dx,$$

there holds

$$\int_{\mathbb{R}^d} |\nabla h|^2 \, d\mu \ge \kappa \int_{\mathbb{R}^d} h^2 \, \langle x \rangle^{2(\gamma-1)} \, d\mu.$$

Question 4

Establish that for any $\varphi \in \mathcal{D}(\mathbb{R}^d)$ and any convex function $j \in C^2(\mathbb{R})$ the associated solution $f(t) = S_\Lambda(t)\varphi$ satisfies

$$\frac{d}{dt} \int_{\mathbb{R}^d} j(f(t)/G)\, G \, dx = -\mathcal{D}_j \le 0$$

and give the expression of the functional $\mathcal{D}_j$. Deduce thgat

$$\|f(t)/G\|\|_{L^\infty} \le \|f_0/G\|\|_{L^\infty} \quad \forall\, t \ge 0.$$

Question 5 (difficult)

Prove that for any $\varphi \in \mathcal{D}(\mathbb{R}^d)$ and for any $\alpha > 0$ there exists C such that

$$\|f(t) - \langle \varphi \rangle\, G\|_{L^2} \le C/t^\alpha.$$