Université Paris-Dauphine **2014-2015**
An introduction to evolution PDEs January 2015

Exercise I - the wave equation

1) Prove that any solution $f = f(t,x)$, $t > 0$, $x \in \mathbb{R}$, to the wave equation

$$\partial^2_{tt} f = \partial^2_{xx} f, \quad f(0,.) = f_0, \ \partial_t f(0,.) = g_0,$$

formally satisfies

$$\int_{\mathbb{R}} [(\partial_t f)^2 + (\partial_x f)^2]\, dx = \int_{\mathbb{R}} [g_0^2 + (\partial_x f_0)^2]\, dx \quad \forall\, t \ge 0.$$

2) Define $H := H^1 \times L^2$, $V := H^2 \times H^1$ and then for any $F = (f_1, f_2)$, $G = (g_1, g_2) \in V$

$$\begin{aligned} a_\varepsilon(F,G) \;\; &:= \;\; \varepsilon \int \partial_x f_1\, \partial_x g_1 - \int f_2\, g_1 + \varepsilon \int \partial^2_{xx} f_1 \partial^2_{xx} g_1 \\ &\quad - \int \partial_x f_2 \partial_x g_1 + \int \partial_x f_1 \partial_x g_2 + \varepsilon \int \partial_x f_2 \partial_x g_2. \end{aligned}$$

Prove that the bilinear form a is continuous in V and it is *"coercive+dissipative"* in the sense

$$\forall\, F \in V \qquad a_\varepsilon(F,F) \ge \varepsilon \|F\|^2_V - (1+\varepsilon)\|F\|^2_H.$$

3) Prove that there exists a unique variational solution to the system of equations

$$\partial_t f_\varepsilon = g_\varepsilon + \varepsilon \partial^2_{xx} f_\varepsilon, \quad \partial_t g_\varepsilon = \partial^2_{xx} f_\varepsilon + \varepsilon \partial^2_{xx} g_\varepsilon,$$

for any $(f_0, g_0) \in H$.
4) By pass to the limit $\varepsilon \to 0$, deduce that there exists at list one solution $f \in C([0,\infty); L^2 - w)$ to the wave equation for any $(f_0, g_0) \in H$.

Exercise II - the fragmentation equation

We consider the fragmentation equation

$$\partial_t f(t,x) = (\mathcal{F} f(t,.))(x)$$

on the density function $f = f(t,x) \ge 0$, $t, x > 0$, where the fragmentation operator is defined by

$$(\mathcal{F} f)(x) := \int_x^\infty b(y,x)\, f(y)\, dy - B(x) f(x).$$

We assume that the total fragmentation rate B and the fragmentation rate b satisfy

$$B(x) = x^\gamma,\ \gamma > 0, \quad b(x,y) = x^{\gamma-1}\wp(y/x),$$

with

$$0 < \wp \in C((0,1]), \ \int_0^1 z\,\wp(z)\,dz = 1, \quad \int_0^1 z^k\,\wp(z)\,dz < \infty,\ \forall\, k < 1.$$

1) Prove that for any $f, \varphi : \mathbb{R}_+ \to \mathbb{R}$, the following identity

$$\int_0^\infty (\mathcal{F}f)(x)\varphi(x)\,dx = \int_0^\infty f(x) \int_0^x b(x,y)\left(\varphi(y) - \frac{y}{x}\varphi(x)\right) dydx$$

holds, whenever the two integrales at the RHS are absolutely convergent.

1) We define the moment function

$$M_k(f) = \int_0^\infty x^k\,f(x)\,dx, \quad k \in \mathbb{R}.$$

Prove that any solution f $(\in C^1([0,T); L^1_k) \cap L^1(0,T; L^1_{k+\gamma}),\ \forall T > 0)$ to the fragmentation equation formally (rigorously) satisfies

$$M_1(f(t)) = \text{cst}, \quad M_k(f(t,.)) \nearrow \text{ if } k < 1, \quad M_k(f(t,.)) \searrow \text{ if } k > 1.$$

Deduce that any solution $f \in C([0,\infty); L^1_{k+\gamma})$ to the fragmentation equation asymptotically satisfies

$$f(t,x)\,x \ \rightharpoonup\ M_1(f(0,.))\,\delta_{x=0} \ \text{ weakly in } \ (C_c([0,\infty))' \ \text{ as } \ t \to \infty.$$

2) For which values of $\alpha, \beta \in \mathbb{R}$, the function

$$f(t,x) = t^\alpha\,G(t^\beta\,x)$$

is a (self-similar) solution of the fragmentation equation such that $M_1(f(t,.)) = \text{cst}$. Prove then that the profile G satisfies the stationary equation

$$\gamma\,\mathcal{F}G = x\,\partial_x G + 2\,G.$$

3) Prove that for any solution f to the fragmentation equation, the rescalled density

$$g(t,x) := e^{-2t}\,f\left(e^{\gamma\,t} - 1, x\,e^{-t}\right)$$

solves the fragmentation equation in self-similar variable

$$\partial_t g + x\,\partial_x g + 2\,g = \gamma\mathcal{F}\,g.$$

Problem III

In all the problem, we consider the discrete Fokker-Planck equation

$$\partial_t f = \mathcal{L}_\varepsilon f := \Delta_\varepsilon f + \mathrm{div}_x(xf) \quad \text{ in } (0,\infty)\times\mathbb{R}, \tag{0.1}$$

where

$$\Delta_\varepsilon f = \frac{1}{\varepsilon^2}(k_\varepsilon * f - f) = \int_{\mathbb{R}} \frac{1}{\varepsilon^2} k_\varepsilon(x-y)(f(y)-f(x))\,dx,$$

and where $k_\varepsilon(x) = 1/\varepsilon k(x/\varepsilon)$, $0 \le k \in W^{1,1}(\mathbb{R}) \cap L^1_3(\mathbb{R})$,

$$\int_{\mathbb{R}} k(x) \begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix} dx = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

that we complement with an initial condition

$$f(0,x) = \varphi(x) \quad \text{ in } \mathbb{R}. \tag{0.2}$$

Question 1

Establish the formula

$$\int (\Delta_\varepsilon f)\,\beta'(f)\,m\,dx = \int \beta(f)\,\Delta_\varepsilon m\,dx - \iint \frac{1}{\varepsilon^2} k_\varepsilon(y-x) J(x,y)\,m(x)\,dydx$$

with

$$J(x,y) := \beta(f(y)) - \beta(f(x)) - (f(y)-f(x))\beta'(f(x)).$$

Formally prove that the discrete Fokker-Planck equation is mass conservative and satisfies the (weak) maximum principle. Explain (quickly) why for $\varphi \in L^2_k(\mathbb{R}^d)$, $k \ge 0$, the equation (0.1)-(0.2) has a (unique) solution $f(t)$ in some functional space to be specified. We recall that

$$\|g\|_{L^p_k} := \|g\,\langle\cdot\rangle^k\|_{L^p}, \quad \langle x\rangle := (1+|x|^2)^{1/2}.$$

Establish that if $L^2_k(\mathbb{R}^d) \subset L^1(\mathbb{R}^d)$, then the solution satisfies

$$\sup_{t\ge 0} \|f(t)\|_{L^1} \le \|\varphi\|_{L^1}.$$

Question 2

We define

$$M_k(t) = M_k(f(t)), \quad M_k(f) := \int_{\mathbb{R}} f(x)\,\langle x\rangle^k\,dx.$$

Prove that if $0 \le \varphi \in L^2_k(\mathbb{R}^d) \subset L^1(\mathbb{R}^d)$, then the solution satisfies

$$M_0(t) \equiv M_0(\varphi) \quad \forall\, t \ge 0.$$

Prove that when $L^2_k(\mathbb{R}^d) \subset L^1_2(\mathbb{R}^d)$, then the solution satisfies

$$\frac{d}{dt} M_2(t) = M_0(0) - M_2(t).$$

Question 3 (discrete Nash inequality)

Prove that there exist $\theta, \eta \in (0,1)$, $\rho > 0$ such that

$$|\hat{k}(r)| < \theta \ \forall \, |r| > \rho, \quad 1 - \hat{k}(r) > \eta r^2 \ \forall \, |r| < \rho.$$

Deduce that for any $f \in L^1 \cap L^2$ and any $R > 0$, there holds

$$\|k_\varepsilon * f\|_{L^2}^2 \le \theta \|f\|_{L^2}^2 + R \|f\|_{L^1}^2 + \frac{1}{\eta R^2} \, I_\varepsilon[f]$$

with

$$I_\varepsilon[f] := \int_{\mathbb{R}} \frac{1 - \hat{k}(\varepsilon \xi)}{\varepsilon^2} \, |\hat{f}|^2 \, d\xi.$$

Question 4

Prove that for any $\alpha > 0$, the solution satisfies

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} f^2 \quad &= \quad -(\frac{1}{\varepsilon^2} - \alpha) \iint k_\varepsilon(y - x) \, (f(y) - f(x))^2 \, dx dy \\ &\quad +2\alpha \int (k_\varepsilon * f) f - 2\alpha \int f^2 + \int f^2, \end{aligned}$$

and then

$$\frac{d}{dt} \int_{\mathbb{R}} f^2 \quad \le \quad -(1 - \alpha \varepsilon^2) \, I_\varepsilon[f] + \alpha \|k_\varepsilon * f\|_{L^2}^2 - (\alpha - 1) \|f\|_{L^2}^2.$$

Deduce that for $\varepsilon \in (0, \varepsilon_0)$, $\varepsilon_0 > 0$ small enough, the solution satisfies

$$\frac{d}{dt} \|f\|_{L^2}^2 \le -\frac{1}{2} \, I_\varepsilon[f] - C_1 \|f\|_{L^2}^2 + C_2 \|f\|_{L^1}^2.$$

Also prove that

$$\frac{d}{dt} \int_{\mathbb{R}} f^2 |x|^2 \le \int_{\mathbb{R}} f^2 - \int_{\mathbb{R}} f^2 |x|^2.$$

Question 5

Prove that there exists a constant C such that the set

$$\mathcal{Z} := \{ f \in L^2_1(\mathbb{R}); \ f \ge 0, \ \|f\|_{L^1} = 1, \ \|f\|_{L^2_1} \le C \}$$

is invariant under the action of the semigroup associated to discrete Fokker-Planck equation. Deduce that there exists a unique solution G_ε to

$$0 < G_\varepsilon \in L^2_1, \quad \Delta_\varepsilon G_\varepsilon + div(x G_\varepsilon) = 0, \quad M_0(G_\varepsilon) = 1.$$

Prove that for any $\varphi \in L^2_1$, $M_0(\varphi) = 1$, the associated solution satisfies

$$f(t) \rightharpoonup G \quad \text{as} \quad t \to \infty.$$

Problem IV

In all the problem, we consider the Fokker-Planck equation

$$\partial_t f = \Lambda f := \Delta_x f + \mathrm{div}_x(f\,\nabla a(x)) \quad \text{in } (0,\infty) \times \mathbb{R}^d \tag{0.3}$$

for the confinement potential

$$a(x) = \frac{\langle x \rangle^\gamma}{\gamma}, \quad \gamma \in (0,1), \quad \langle x \rangle^2 := 1 + |x|^2,$$

that we complement with an initial condition

$$f(0,x) = \varphi(x) \quad \text{in } \mathbb{R}^d. \tag{0.4}$$

Question 1

Exhibit a stationary solution $G \in \mathbf{P}(\mathbb{R}^d)$. Formally prove that this equation is mass conservative and satisfies the (weak) maximum principle. Explain (quickly) why for $\varphi \in L^p_k(\mathbb{R}^d)$, $p \in [1,\infty]$, $k \geq 0$, the equation (0.3)-(0.4) has a (unique) solution $f(t)$ in some functional space to be specified. Establish that if $L^p_k(\mathbb{R}^d) \subset L^1(\mathbb{R}^d)$ then the solution satisfies

$$\sup_{t \geq 0} \|f(t)\|_{L^1} \leq \|\varphi\|_{L^1}.$$

Can we affirm that $f(t) \to G$ as $t \to \infty$? and that convergence is exponentially fast?

Question 2

We define

$$\mathcal{B} f := \Lambda f - M \chi_R f$$

with $\chi_R(x) := \chi(x/R)$, $\chi \in \mathcal{D}(\mathbb{R}^d)$, $0 \leq \chi \leq 1$, $\chi(x) = 1$ for any $|x| \leq 1$, and with $M, R > 0$ to be fixed.
We denote by $f_\mathcal{B}(t) = S_\mathcal{B}(t)\varphi$ the solution associated to the evolution PDE corresponding to the operator $\mathcal{B}$ and the initial condition (0.4).
(1) Why such a solution is well defined (no more than one sentence of explanation)?
(2) Prove that there exists $M, R > 0$ such that for any $k \geq 0$ there holds

$$\frac{d}{dt} \int_{\mathbb{R}^d} f_\mathcal{B}(t) \langle x \rangle^k \, dx \leq -c_k \int_{\mathbb{R}^d} f_\mathcal{B}(t) \langle x \rangle^{k+\gamma-2} \leq 0,$$

for some constant $c_k \geq 0$, $c_k > 0$ if $k > 0$, and

$$\|S_\mathcal{B}(t)\|_{L^1_k \to L^1_k} \leq 1.$$

(3) Establish that if $u \in C^1(\mathbb{R}_+)$ satisfies

$$u' \le -c\, u^{1+1/\alpha}, \quad c, \alpha > 0,$$

there exists $C = C(c, \alpha, u(0))$ such that

$$u(t) \le C/t^\alpha \quad \forall\, t > 0.$$

(4) Prove that for any $k_1 < k < k_2$ there exists $\theta \in (0,1)$ such that

$$\forall\, f \ge 0 \quad M_k \le M_{k_1}^\theta\, M_{k_2}^{1-\theta}, \quad M_\ell := \int_{\mathbb{R}^d} f(x)\, \langle x \rangle^\ell\, dx$$

and write θ as a function of k_1, k and k_2.
(5) Prove that if $\ell > k > 0$ there exists $\alpha > 0$ such that

$$\|S_\mathcal{B}(t)\|_{L^1_\ell \to L^1} \le \|S_\mathcal{B}(t)\|_{L^1_\ell \to L^1_k} \le C/\langle t \rangle^\alpha,$$

and that $\alpha > 1$ if ℓ is large enough (to be specified).
(6) Prove that

$$S_\mathcal{L} = S_\mathcal{B} + S_\mathcal{B} * (\mathcal{A} S_\mathcal{L}),$$

and deduce that for k large enough (to be specified)

$$\|S_\mathcal{L}\|_{L^1_k \to L^1_k} \le C.$$

Remark. You have recovered (with a simpler and more general proof) a result established by Toscani and Villani in 2001.

Question 3 (difficult)

Establish that there exists $\kappa > 0$ such that for any $h \in \mathcal{D}(\mathbb{R}^d)$ satisfying

$$\langle h \rangle_\mu := \int_{\mathbb{R}^d} h\, d\mu = 0, \quad \mu(dx) := G(x)\, dx,$$

there holds

$$\int_{\mathbb{R}^d} |\nabla h|^2\, d\mu \ge \kappa \int_{\mathbb{R}^d} h^2\, \langle x \rangle^{2(\gamma-1)}\, d\mu.$$

Question 4

Establish that for any $\varphi \in \mathcal{D}(\mathbb{R}^d)$ and any convex function $j \in C^2(\mathbb{R})$ the associated solution $f(t) = S_\Lambda(t)\varphi$ satisfies

$$\frac{d}{dt} \int_{\mathbb{R}^d} j(f(t)/G)\, G\, dx = -\mathcal{D}_j \le 0$$

and give the expression of the functional $\mathcal{D}_j$. Deduce thgat

$$\|f(t)/G\|\|_{L^\infty} \le \|f_0/G\|\|_{L^\infty} \quad \forall\, t \ge 0.$$

Question 5 (difficult)

Prove that for any $\varphi \in \mathcal{D}(\mathbb{R}^d)$ and for any $\alpha > 0$ there exists C such that

$$\|f(t) - \langle \varphi \rangle \, G\|_{L^2} \leq C/t^{\alpha}.$$