Université Paris-Dauphine **2015-2016**
An introduction to evolution PDEs January 2016

1 Problem I

We consider the evolution PDE

$$\partial_t f = \mathrm{div}(A \nabla f), \tag{1.1}$$

on the unknown $f = f(t,x)$, $t \geq 0$, $x \in \mathbb{R}^d$, with $A = A(x)$ a symmetric, uniformly bounded and coercive matrix, in the sense that

$$\nu \, |\xi|^2 \leq \xi \cdot A(x) \xi \leq C \, |\xi|^2, \quad \forall \, x, \xi \in \mathbb{R}^d.$$

It is worth emphasizing that we do not make any regularity assumption on A. We complement the equation with an initial condition

$$f(0,x) = f_0(x).$$

1) Existence. What strategy can be used in order to exhibit a semigroup $S(t)$ in $L^p(\mathbb{R}^d)$, $p = 2$, $p = 1$, which provides solutions to (1.1) for initial date in $L^p(\mathbb{R}^d)$? Is the semigroup positive? mass conservative?

In the sequel we will not try to justify rigorously the a priori estimates we will establish, but we will carry on the proofs just as if there do exist nice (smooth and fast decaying) solutions. We denote by C or C_i some constants which may differ from line to line.

2) Uniform estimate. a) Prove that any solution f to (1.1) satisfies

$$\|f(t)\|_{L^2} \leq C \, t^{-d/4} \, \|f_0\|_{L^1}, \quad \forall \, t > 0.$$

b) We define the dual semigroup $S^*(t)$ by

$$\langle S^*(t) g_0, f_0 \rangle = \langle g_0, S(t) f_0 \rangle, \quad \forall \, t \geq 0, \ f_0 \in L^p, \ g_0 \in L^{p'}.$$

Identify $S^*(t)$ and deduce that any solution f to (1.1) satisfies

$$\|f(t)\|_{L^\infty} \leq C \, t^{-d/4} \, \|f_0\|_{L^2}, \quad \forall \, t > 0.$$

c) Conclude that any solution f to (1.1) satisfies

$$\|f(t)\|_{L^\infty} \leq [C^2 2^{d/2}] \, t^{-d/2} \, \|f_0\|_{L^1}, \quad \forall \, t > 0.$$

3) Entropy and first moment. For a given and (nice) probability measure f, we define the (mathematical) entropy and the first moment functional by

$$H := \int_{\mathbb{R}^d} f \log f \, dx, \quad M := \int_{\mathbb{R}^d} f \, |x| \, dx.$$

a) Prove that for any $\lambda \in \mathbb{R}$, there holds

$$\min_{s \geq 0} \{ s \log s + \lambda \, s \} = -e^{-\lambda - 1}.$$

Deduce that there exists a constant $D = D(d)$ such that for any (nice) probability measure f and any $a \in \mathbb{R}_+$, $b \in \mathbb{R}$, there holds

$$H + aM + b \geq -e^{-b-1} \, a^{-d} \, D.$$

b) Making the choice $a := d/M$ and $e^{-b} := (e/D) \, a^d$, deduce that

$$M \geq \kappa \, e^{-H/d}, \tag{1.2}$$

for some $\kappa = \kappa(d) > 0$.

From now on, **we restrict ourself** to consider an initial datum which is a (nice) probability measure:

$$f_0 \geq 0, \quad \int_{\mathbb{R}^d} f_0 \, dx = 1,$$

and we denote by $f(t)$ the nonnegative and normalized solution to the evolution PDE (1.1) corresponding to f_0. We also denote by $H = H(t)$, $M = M(t)$ the associated entropy and first moment.

3) Dynamic estimate on the entropy. Deduce from 2) that f satisfies

$$H(t) \leq K - \frac{d}{2} \log t, \quad \forall \, t > 0, \tag{1.3}$$

for some constant $K \in \mathbb{R}$ (independent of f_0).

4) Dynamic estimate on the entropy and the first moment. a) Prove that

$$\Big| \frac{d}{dt} M(t) \Big| \leq C \int |\nabla f(t)|,$$

for some positive constant $C = C(A)$.

b) Deduce that there exists a constant $\theta = \theta(C, \nu) > 0$ such that

$$\Big| \frac{d}{dt} M(t) \Big| \leq \theta \Big(- \frac{d}{dt} H(t) \Big)^{1/2}, \quad \forall \, t > 0. \tag{1.4}$$

c) Prove that for the heat equation (when $A = I$), we have

$$\frac{d}{dt} \int f |x|^2 \, dx = 2,$$

and next

$$M(t) \le C\, \langle t \rangle^{1/2}, \quad t \ge 0. \tag{1.5}$$

From now on, **we will always restrict ourself** to consider the Dirac mass initial datum

$$f_0 = \delta_0(dx),$$

and our goal is to establish a similar estimate as (1.5) (and in fact, a bit sharper estimate than (1.5)) for the corresponding solution.

5) Dynamic estimate on the first moment. a) Deduce from 1) that there exists (at least) one function $f \in C((0,\infty); L^1) \cap L^\infty_{loc}((0,\infty); L^\infty)$ which is a solution to the evolution PDE (1.1) associated to the initial datum δ_0. Why does that solution satisfy the same above estimates for positive times?

b) We define

$$R = R(t) := K/d - H(t)/d - \frac{1}{2} \log t \ge 0,$$

where K is defined in (1.3). Observing that $M(0) = 0$, deduce from the previous estimates that

$$C_1\, t^{1/2}\, e^R \le M \le C_2 \int_0^t \Big(\frac{1}{2s} + \frac{dR}{ds} \Big)^{1/2} ds, \quad \forall\, t > 0.$$

c) Observe that for $a > 0$ and $a + b > 0$, we have $(a+b)^{1/2} \le a^{1/2} + b/(2a^{1/2})$, and deduce that

$$C_1\, e^R \le M t^{-1/2} \le C_2 (1 + R), \quad \forall\, t > 0.$$

d) Deduce from the above estimate that R must be bounded above, and then

$$C_1\, t^{1/2} \le M \le C_2\, t^{1/2}, \quad \forall\, t > 0.$$

You have recovered one of the most crucial step of Nash's article "Continuity of solutions of parabolic and elliptic equations", Amer. J. Math. (1958).

2 Problem II -

We consider the fractional Fokker-Planck equation

$$\partial_t f = \mathcal{L} f := I[f] + \mathrm{div}_x(Ef) \quad \text{in } (0,\infty)\times\mathbb{R}, \tag{2.1}$$

where

$$I[f](x) = \int_{\mathbb{R}} k(y-x)\,(f(y)-f(x))\,dy, \quad k(z) = \frac{1}{|z|^{1+\alpha}}, \quad \alpha\in(0,1),$$

and where E is a smooth vectors field such that

$$\forall\, |x|\geq 1, \quad |E(x)| \leq C\,\langle x\rangle, \quad \mathrm{div}E(x)\leq C, \quad x\cdot E \geq |x|^2.$$

We complement the equation with an initial condition

$$f(0,x) = \varphi(x) \quad \text{in} \quad \mathbb{R}. \tag{2.2}$$

We denote by $\mathcal{F}$ the Fourier transform operator, and next $\hat{f} = \mathcal{F} f$ for a given function f on the real line.

Question 1. Preliminary issues (if not proved, theses identities can be accepted). Here all the functions (f, φ, β) are assumed to be suitably nice so that all the calculations are licit.
a) - Establish the formula

$$\int (I[f])\,\beta'(f)\,\varphi\,dx = \int \beta(f)\,I[\varphi]\,dx - \iint k(y-x) J(x,y)\,\varphi(x)\,dydx$$

with

$$J(x,y) := \beta(f(y)) - \beta(f(x)) - (f(y)-f(x))\beta'(f(x)).$$

b) - Prove that there exists a positive constant C_1 such that

$$\mathcal{F}(I[f])(\xi) = \int_{\mathbb{R}} f(x)\,e^{-ix\xi}\Big\{\int_{\mathbb{R}} \frac{\cos(z\xi)-1}{|z|^{1+\alpha}}\,dz\Big\}dx = C_1\,|\xi|^\alpha \hat{f}(\xi), \quad \forall\,\xi\in\mathbb{R}.$$

c) - Denote $s := \alpha/2$. Prove that

$$|||f|||^2_{\dot{H}^s} := \int_{\mathbb{R}^2} \frac{|f(x)-f(y)|^2}{|x-y|^{1+2s}}\,dxdy = \int_{\mathbb{R}} \Big\| \frac{f(z+\cdot)-f(\cdot)}{|z|^{s+1/2}}\Big\|^2_{L^2(\mathbb{R})} dz,$$

and then that there exists a positive constant C_2 such that

$$|||f|||^2_{\dot{H}^s} = C_2 \int_{\mathbb{R}} |\xi|^{2s}\,|\hat{f}(\xi)|^2\,d\xi =: C_2 \|f\|^2_{\dot{H}^s}.$$

d) - Prove that

$$\int_{\mathbb{R}} I[f]\,f\,dx = -\frac{1}{2}|||f|||^2_{\dot{H}^s}.$$

From questions 2 to 4, we consider f (and g) a solution to the fractional Fokker-Planck equation (2.1) and we establish formal a priori estimates.

Question 2. Moment estimates. For any $k \geq 0$, we define

$$M_k = M_k(t) = M_k(f(t)), \quad \text{with} \quad M_k = M_k(f) := \int_{\mathbb{R}} f(x) \, \langle x \rangle^k \, dx.$$

Prove that the solution f satisfies

$$M_0(t) \equiv M_0(\varphi), \quad \forall \, t \geq 0.$$

Prove that for any $k \in (0, \alpha)$, there exists $C > 0$ such that

$$C^{-1} \, |\langle y \rangle^k - \langle x \rangle^k| \leq \left| |y|^2 - |x|^2 \right|^{k/2} \leq C \, \left(|y-x|^{k/2} \, |x|^{k/2} + |y-x|^k \right),$$

and deduce that there exist $C_1, C_2 > 0$ such that the solution f satisfies

$$\frac{d}{dt} M_k \leq C_1 \, M_{k/2} - C_2 \, M_k.$$

Conclude that there exists $A_k = A_k(M_0(\varphi))$ such that the solution f satisfies

$$\sup_{t \geq 0} M_k(t) \leq \max(M_k(0), A_k). \tag{2.3}$$

Question 3. Fractional Nash inequality and L^2 estimate. Prove that there exists a constant $C > 0$ such that

$$\forall \, h \in \mathcal{D}(\mathbb{R}), \quad \|h\|_{L^2} \leq C \, \|h\|_{L^1}^{\frac{\alpha}{1+\alpha}} \, \|h\|_{\dot{H}^s}^{\frac{1}{1+\alpha}}.$$

Deduce that the square of the L^2-norm $u := \|f(t)\|_{L^2}^2$ of the solution f satisfies

$$\frac{d}{dt} u \leq -C_1 u^{1+\alpha} + C_2,$$

for some constants $C_i = C_i(M_0(\varphi)) > 0$. Conclude that there exists $A_2 = A_2(M_0(\varphi))$ such that

$$\sup_{t \geq 0} u(t) \leq \max(u(0), A_2). \tag{2.4}$$

Question 4. Around generalized entropies and the L^1-norm. Consider a convex function β and define the entropy $\mathcal{H}$ and the associated dissipation of entropy $\mathcal{D}$ by

$$\mathcal{H}(f|g) := \int_{\mathbb{R}} \beta(X) \, g \, dx,$$

$$\mathcal{D}(f|g) := \int_{\mathbb{R}} \int_{\mathbb{R}} k \, g_* \left\{ \beta(X_*) - \beta(X) - \beta'(X)(X_* - X) \right\} dx dx_*,$$

where $k = k(x - x_*)$, $g_* = g(x_*)$, $X = f(x)/g(x)$ and $X_* = f(x_*)/g(x_*)$. Why do two solutions f and g satisfy

$$\frac{d}{dt}\mathcal{H}(f(t)|g(t)) \leq -\mathcal{D}(f(t)|g(t))? \tag{2.5}$$

Deduce that for $\beta(s) = s_+$ and $\beta(s) = |s|$, any solution f satisfies

$$\int \beta(f(t,\cdot))\,dx \leq \int \beta(\varphi)\,dx, \quad \forall\, t \geq 0. \tag{2.6}$$

Question 5. Well-posedness. Explain briefly how one can establish the existence of a weakly continuous semigroup $S_{\mathcal{L}}$ defined in the space $X = L^1_s \cap L^2$ such that it is a contraction for the L^1 norm and such that for any $\varphi \in X$ the function $f(t) := S_{\mathcal{L}}(t)\varphi$ is a (weak) solution to the fractional Fokker-Planck equation (2.1). Why is $S_{\mathcal{L}}$ mass and positivity preserving and why does any associated trajectory satisfy (2.3), (2.4) and (2.6)?

(*Ind.* One may consider the sequence of kernels $k_n(z) := k(z)\,_{n^{-1}<|z|<n})$.

Question 6. Prove that there exists a constant C such that the set

$$\mathcal{Z} := \{f \in L^1(\mathbb{R});\ f \geq 0,\ \|f\|_{L^1} = 1,\ \|f\|_X \leq C\}$$

is invariant under the action of $S_{\mathcal{L}}$. Deduce that there exists at least one function $G \in X$ such that

$$IG + \mathrm{div}_x(E\,G) = 0, \quad G \geq 0, \quad M_0(G) = 1. \tag{2.7}$$

Question 7. We accept that for any convex function β, any nonnegative solution $f(t)$ and any nonnegative stationary solution G the following inequality holds

$$\mathcal{H}(f(t)|G) + \int_0^t \mathcal{D}(f(s)|G)\,ds \leq \mathcal{H}(\varphi|G). \tag{2.8}$$

Deduce that $\mathcal{D}(g|G) = 0$ for any (other) stationary solution g. *(Ind. First consider the case when $\beta \in W^{1,\infty}(\mathbb{R})$ and next use an approximation argument).* Deduce that

$$G(y)\left(\frac{g(y)}{G(y)} - \frac{g(x)}{G(x)}\right)^2 = 0 \quad \text{for a.e. } x, y \in \mathbb{R},$$

and then that the solution to (2.7) is unique. Prove that for any $\varphi \in X$, there holds

$$S_{\mathcal{L}}(t)\varphi \rightharpoonup M_0(\varphi)\,G \ \text{ weakly in } X, \quad \text{as} \quad t \to \infty.$$

Question 8. (a) Introducing the splitting

$$\mathcal{A} := \lambda\, I,\ \lambda \in \mathbb{R}, \quad \mathcal{B} := \mathcal{L} - \mathcal{A},$$

explain why

$$S_{\mathcal{L}} = S_{\mathcal{B}} + S_{\mathcal{B}}\mathcal{A} * S_{\mathcal{L}},$$

and next for any $n \geq 1$

$$S_{\mathcal{L}} = S_{\mathcal{B}} + ... + (S_{\mathcal{B}}\mathcal{A})^{*(n-1)} * S_{\mathcal{L}} + (S_{\mathcal{B}}\mathcal{A})^{*n} * S_{\mathcal{L}},$$

where the convolution on $\mathbb{R}_+$ is defined by

$$(u * v)(t) := \int_0^t u(t-s)\, v(s)\, ds,$$

and the itareted convolution by $u^{*1} = u$, $u^{*k} = u^{*(k-1)} * u$ if $k \geq 2$.
(b) Prove that for $\lambda > 0$, large enough, there holds

$$\|S_{\mathcal{B}}\|_{Y \to Y} \leq e^{-t}, \quad Y = L^1_k,\ k \in (0, \alpha),\ Y = L^2,$$

as well as

$$\|S_{\mathcal{B}}(t)\|_{L^1 \to L^2} \leq \frac{C}{t^{1/(2\alpha)}}\, e^{-t}, \quad \forall\, t \geq 0, \quad \int_0^\infty \|S_{\mathcal{B}}(s)\|^2_{L^2 \to \dot{H}^s}\, ds \leq C,$$

and deduce that for n large enough

$$\int_0^\infty \|(\mathcal{A}S_{\mathcal{B}})^{(*n)}(s)\|_{L^1 \to \dot{H}^s}\, ds \leq C.$$

(c) Establish that for any $\varphi \in L^1_s$, $k > 0$, the associated solution $f(t) = S_{\mathcal{L}}(t)\varphi$ splits as

$$f(t) = g(t) + h(t), \quad \|g(t)\|_{L^1} \leq e^{-t}, \quad \|h(t)\|_{L^1_k \cap \dot{H}^s} \leq C(M_0(\varphi)).$$

(d) Conclude that

$$\forall\, \varphi \in L^1(\mathbb{R}), \quad \|S_{\mathcal{L}}(t)\varphi - M_0(\varphi) G\|_{L^1} \to 0 \quad \text{as} \quad t \to \infty.$$

Question 9. Justification of (2.8). We define the regularized operator

$$\mathcal{L}_{\varepsilon,n} f := \varepsilon \Delta_x f + I_n[f] + \mathrm{div}_x(Ef)$$

with $\varepsilon > 0$, $n \in \mathbb{N}^*$ and I_n associated to the kernel k_n (introduced in question 5). Why is there a (unique) solution $G_{\varepsilon,n}$ to the stationary problem

$$G_{\varepsilon,n} \in X, \quad \mathcal{L}_{\varepsilon,n} G_{\varepsilon,n} = 0, \quad G_{\varepsilon,n} \geq 0, \quad M_0(G_{\varepsilon,n}) = 1,$$

and why does a similar inequality as (2.8) hold? Prove that there exist $G_\varepsilon, G \in X$, $\varepsilon > 0$, such that (up to the extraction of a subsequence) $G_{n,\varepsilon} \to G_\varepsilon$ and $G_\varepsilon \to G$ strongly in L^1. Prove that a similar strong convergence result holds for the familly $S_{\mathcal{L}_{\varepsilon,n}}(t)\varphi$, $\varphi \in X$. Conclude that (2.8) holds.