CHAPTER 3 - EVOLUTION EQUATION AND SEMIGROUP

In this chapter we make the link between the existence theory for evolution PDEs we have presented in the two first chapters and the theory of continuous semigroup of linear and bounded operators. In that unified framework we may establish the Duhamel formula and the extension of the existence theory by perturbation argument. We also briefly present the Hille-Yosida-Lumer-Phillips existence theory for m-dissipative operators.

Contents

1. From linear evolution equation to semigroup

1.1. Semigroup.

We state the definition of a continuous semigroup of linear and bounded operators.

Definition 1.1. *We say that $(S_t)_{t\ge 0}$ is a continuous semigroup of linear and bounded operators on a Banach space X, or we just say that S_t is C_0-semigroup (or a semigroup) on X, we also write $S(t) = S_t$, if the following conditions are fulfilled:*

(i) one parameter family of operators: $\forall t \ge 0$, $f \mapsto S_t f$ is linear and continuous on X;

(ii) continuity of trajectories: $\forall f \in X$, $t \mapsto S_t\, f \in C([0,\infty), X)$;

(iii) semigroup property: $S_0 = I$; $\quad \forall\, s, t \ge 0$, $S_{t+s} = S_t\, S_s$;

(iv) growth estimate: $\exists\, b \in \mathbb{R}$, $\exists\, M \ge 1$,

$$\|S_t\|_{\mathscr{B}(X)} \le M\, e^{bt} \quad \forall\, t \ge 0. \tag{1.1}$$

We then define the growth bound $\omega(S)$ by

$$\omega(S) := \limsup_{t\to\infty} \frac{1}{t} \log \|S(t)\| = \inf\{b \in \mathbb{R};\ (1.1)\ \textit{holds}\}.$$

We say that (S_t) is a semigroup of contractions if (1.1) *holds with $b = 0$ and $M = 1$.*

Remark 1.2. *The two continuity properties (i) and (ii) can be understood both in the same sense of*
- the strong topology of X, and we will say that S_t is a strongly continuous semigroup;
- the weak $*$ topology $\sigma(X, Y)$ with $X = Y'$, Y a (separable) Banach space, and we will say that S_t is a weakly $*$ continuous semigroup.

In the sequel, the semigroups we consider are strongly continuous except when we specify it. Anyway, many of the results are also true for weakly $*$ continuous semigroups.

Remark 1.3. *It is worth mentioning (and we refer to the sections 7.1 & 7.3 for more details) that*
- For a given one parameter family (S_t), the growth property (iv) is automatically satisfied when (i), (ii) and (iii) hold.
- The continuity property (ii) can be replaced by the following (right) continuity assumption in $t = 0$:
(ii′) $S(t)f \to f$ when $t \searrow 0$, for any $f \in X$.
- When the continuity properties (i) & (ii) hold for the weak topology $\sigma(X, X')$ then they also hold for the strong topology in X: we do not need to make any difference between strongly and weakly continuous semigroups.
- There do exist weakly $*$ continuous semigroups which are not strongly continuous because the continuity property (ii) for the weak $*$ topology $\sigma(X, Y)$ does not implies the analogous strong continuity. Classical examples are the heat semigroup and the translation semigroup

$$S_t f = \gamma_t * f, \quad \gamma_t = (2\pi t)^{-1/2}\, e^{-\frac{|x|^2}{2t}}, \quad \textit{and} \quad (S_t f)(x) = f(\Phi_t^{-1}(x)), \quad \Phi_t^{-1}(x) := x - t,$$

in the Lebesgue space $L^\infty(\mathbb{R})$ and in the space $M^1(\mathbb{R}) := (C_0(\mathbb{R}))'$ of bounded Radon measures.
- For a given semigroup (S_t) on X, one may define the new semigroup (T_t) and the new norm $|||\cdot|||$ on X by

$$T(t) := e^{-\omega\, t}\, S(t) \qquad \textit{and} \qquad |||f||| := \sup_{t \ge 0} \|T(t)\, f\|, \tag{1.2}$$

and then show that $|||\cdot|||$ is equivalent to $\|\cdot\|$ and $T(t)$ is a semigroup of contractions for that new norm. However, that trick is not so useful because the new norm $|||\cdot|||$ does not satisfy the same nice regularity properties as the initial norm $\|\cdot\|$ often satisfies.

Exercise 1.4. *(1) Prove that (i), (ii′) and (iii) imply (iv). (Hint. Use a contradiction argument and the Banach-Steinhaus Theorem or see Proposition 7.1).*
(2) Prove that (i)-(ii′)-(iii) implies (i)-(ii)-(iii). (Hint. Use (1) or see Proposition 7.2).
(3) Prove that in Remark 1.3, the two norms are equivalent and that $T(t)$ is a semigroup of contractions for the new norm. (Hint. See Proposition 7.3).
(4) Prove that if S_t satisfies (iii) as well as the continuity properties (i) & (ii) in the sense of the weak topology $\sigma(X, X')$, then S_t is a strongly continuous semigroup. (Hint. See Theorem 7.4).

1.2. **From well-posedness to semigroup.** Given a linear operator Λ acting on a Banach space X (or on a subspace of X) and a initial datum g_0 belonging to X (or to a subspace of X), we consider the (abstract) linear evolution equation

$$\frac{d}{dt} g = \Lambda g \ \text{ in } \ (0, \infty) \times X, \quad g(0) = g_0 \ \text{ in } \ X. \tag{1.3}$$

We explain how we may associate a C_0-semigroup to the evolution equation as a mere consequence of the linearity of the equation and of the existence and uniqueness result.

Definition 1.5. *Consider three Banach spaces X, Y, Z such that $Z \subset X \subset Y'$, with continuous and dense embedding, and a linear and bounded operator $\Lambda : Z \to Y'$. We denote $\Lambda^* : Y \subset Y'' \to Z'$ the adjoint operator. We say that a function*

$$g = g(t) \in \mathscr{E}_T := C([0, T]; X_\bullet) \cap L^r(0, T; Z), \ \ 1 \le r \le \infty,$$

with $X_\bullet = X$ or $X_\bullet = X - \sigma(X, Y)$, is a weak solution to the evolution equation (1.3) *associated to the initial datum $g_0 \in X$ if*

$$[\langle g, \varphi\rangle_{X,Y}]_0^T - \int_0^T \langle g, \partial_t \varphi\rangle_{X,Y}\, dt = \int_0^T \langle g, \Lambda^* \varphi\rangle_{Z,Z'}\, dt, \tag{1.4}$$

for any test function $\varphi \in \mathscr{Y}_T := C^1([0, T]; Y_\bullet)$. In the case $X_\bullet = X$ we can take $Y_\bullet = Y - \sigma(Y, X)$, while in the case $X_\bullet = X - \sigma(X, Y)$ we must take $Y_\bullet = Y$. We do insist on that $\psi \in C([0, T); \mathcal{X} - \sigma(\mathcal{X}, \mathcal{Y}))$, with $\mathcal{X} \subset \mathcal{Y}'$ or $\mathcal{Y} \subset \mathcal{Y}'$, means that the mapping $t \mapsto \langle \psi(t), \varphi\rangle_{\mathcal{X},\mathcal{Y}}$ is continuous for any $\varphi \in \mathcal{Y}$. We note

$$\mathscr{E}_\infty := \{g : \mathbb{R}_+ \to X;\ g_{|[0,T]} \in \mathscr{E}_T,\ \forall\, T > 0\}.$$

We give two examples. For a variational solution to an abstract parabolic equation, we take $X = X_\bullet = H$, $Z = Y = V$ and $r = 2$, with the notations of Chapter 1. For a weak (and thus renormalized) solution to a transport equation, we take $X = X_\bullet = Z = L^p$, $1 \le p < \infty$, $Y = W^{1,p'}$ and $r = \infty$.

Definition 1.6. *We say that the evolution equation* (1.3) *is well-posed in the sense of Definition 1.5 of weak solutions, if for any $g_0 \in X$ there exists a unique function $g \in \mathscr{E}_\infty$ which satisfies* (1.4)*, and for any $R_0, T > 0$ there exists $R_T := C(T, R_0) > 0$ such that*

$$\|g_0\|_X \le R_0 \quad \textit{implies} \quad \sup_{[0,T]} \|g(t)\|_X \le R_T. \tag{1.5}$$

Proposition 1.7. *To an evolution equation* (1.3) *which is well-posed in the sense of Definition 1.6, we may associate a continuous semigroup of linear and bounded operators (S_t) in the following way. For any $g_0 \in X$ and any $t \ge 0$, we set $S(t)g_0 := g(t)$, where $g \in \mathscr{E}_\infty$ is the unique weak solution to the evolution equation* (1.3) *with initial datum g_0.*

Corollary 1.8. *To the time autonomous parabolic equation considered in Chaper 1 and to the time autonomous transport equation considered in Chapter 2, we can associate a strongly continuous semigroup of linear and bounded operators.*

Proof of Proposition 1.7. We just check that S as defined in the statement of the Proposition fulfilled the conditions (i), (ii) and (iii) in Definition 1.1.

• *S satisfies (i).* By linearity of the equation and uniqueness of the solution, we clearly have

$$S_t(g_0 + \lambda f_0) = g(t) + \lambda f(t) = S_t g_0 + \lambda S_t f_0$$

for any $g_0, f_0 \in X$, $\lambda \in \mathbb{R}$ and $t \ge 0$. Thanks to estimate (1.5) we also have $\|S_t g_0\| \le C(t,1)\,\|g_0\|$ for any $g_0 \in X$ and $t \ge 0$. As a consequence, $S_t \in \mathscr{B}(X)$ for any $t \ge 0$.

• *S satisfies (ii)* since by definition $t \mapsto S_t g_0 \in C(\mathbb{R}_+; X_\bullet)$ for any $g_0 \in X$.

• *S satisfies (iii).* For $g_0 \in X$ and $t_1, t_2 \ge 0$ denote $g(t) = S_t g_0$ and $\tilde g(t) := g(t + t_1)$. Making the difference of the two equations (1.4) written for $t = t_1$ and $t = t_1 + t_2$, we see that $\tilde g$ satisfies

$$\begin{aligned} \langle \tilde g(t_2), \tilde\varphi(t_2)\rangle &= \langle g(t_1+t_2), \varphi(t_1+t_2)\rangle \\ &= \langle g(t_1), \varphi(t_1)\rangle + \int_{t_1}^{t_1+t_2} \big\{ \langle \Lambda g(s), \varphi(s)\rangle + \langle \varphi'(s), g(s)\rangle \big\}\, ds \\ &= \langle \tilde g(0), \tilde\varphi(0)\rangle + \int_0^{t_2} \big\{ \langle \Lambda \tilde g(s), \tilde\varphi(s)\rangle + \langle \tilde\varphi'(s), \tilde g(s)\rangle \big\}\, ds, \end{aligned}$$

for any $\varphi \in \mathscr{Y}_{t_1+t_2}$ with the notation $\tilde\varphi(t) := \varphi(t + t_1) \in \mathscr{Y}_{t_2}$. We thus obtain

$$S_{t_1+t_2} g_0 = g(t_1+t_2) = \tilde g(t_2) = S_{t_2}\tilde g(0) = S_{t_2} g(t_1) = S_{t_2} S_{t_1} g_0,$$

where for the third equality we have used that the equation on the functions $\tilde g$ and $\tilde\varphi$ is nothing but the weak formulation associated to the equation (1.3) for the initial datum $\tilde g(0)$. □

2. Semigroup and generator

On the other way round, in this section, we explain how we can associate a generator and then a solution to a differential linear equation to a given semigroup.

Definition 2.1. *An unbounded operator Λ on X is a linear mapping defined on a linear submanifold called the domain of Λ and denoted by $D(\Lambda)$ or $dom(\Lambda) \subset X$; $\Lambda : D(\Lambda) \to X$. The graph of Λ is*

$$G(\Lambda) = \textit{graph}(\Lambda) := \{(f, \Lambda f);\ f \in D(\Lambda)\} \subset X \times X.$$

We say that Λ is closed if the graph $G(\Lambda)$ is a closed set in $X \times X$: for any sequence (f_k) such that $f_k \in D(\Lambda)$, $\forall\, k \ge 0$, $f_k \to f$ in X and $\Lambda f_k \to g$ in X then $f \in D(\Lambda)$ and $g = \Lambda f$. We denote $\mathscr{C}(X)$ the set of unbounded operators with closed graph and $\mathscr{C}_D(X)$ the set of unbounded operators which domain is dense and graph is closed.

Definition 2.2. *For a given semigroup (S_t) on X, we define*

$$D(\Lambda) \quad := \quad \big\{ f \in X;\ \lim_{t \searrow 0} \frac{S(t)\, f - f}{t} \ \textit{exists in}\ X \big\},$$
$$\Lambda\, f \quad := \quad \lim_{t \searrow 0} \frac{S(t)\, f - f}{t} \quad \textit{for any}\ \ f \in D(\Lambda).$$

Clearly $D(\Lambda)$ is a linear submanifold and Λ is linear: Λ is an unbounded operator on X. We call $\Lambda : D(\Lambda) \to X$ the (infinitesimal) generator of the semigroup (S_t), and we sometimes write $S_t = S_\Lambda(t)$. We denote $\mathscr{G}(X)$ the set of operators which are the generator of a semigroup.

We present some fundamental properties of a semigroup S and its generator Λ that one can obtain by simple differential calculus arguments from the very definitions of S and Λ.

Proposition 2.3. *(Differentiability property of a semigroup). Let $f \in D(\Lambda)$.*
(i) $S(t)f \in D(\Lambda)$ and $\Lambda\, S(t)f = S(t)\, \Lambda s f$ for any $t \ge 0$, so that the mapping $t \mapsto S(t)\, f$ is $C([0,\infty); D(\Lambda))$.
(ii) The mapping $t \mapsto S(t)f$ is $C^1([0,\infty); X)$, $\dfrac{d}{dt} S(t)f = \Lambda S(t) f$ for any $t > 0$, and then

$$S(t)f - S(s)f = \int_s^t S(\tau)\, \Lambda f \, d\tau = \int_s^t \Lambda\, S(\tau) f\, d\tau, \qquad \forall\, t > s \ge 0.$$

Sketch of the proof of Proposition 2.3. Let $f \in D(\Lambda)$.
Proof of (i). We fix $t \ge 0$ and we compute

$$\lim_{s \to 0^+} \frac{S(s)S(t)f - S(t)f}{s} = \lim_{s \to 0^+} S(t) \frac{S(s)f - f}{s} = S(t)\Lambda f,$$

which implies $S(t)f \in D(\Lambda)$ and $\Lambda S(t) f = S(t)\Lambda f$.
Proof of (ii). We fix $t > 0$ and we compute (now) the left differential

$$\begin{aligned} &\lim_{s \to 0^-} \Big\{ \frac{S(t+s)f - S(t)f}{s} - S(t)\Lambda f \Big\} = \\ &= \lim_{s \to 0^-} \Big\{ S(t+s) \Big(\frac{S(-s)f - f}{-s} - \Lambda f \Big) + \Big(S(t+s)\Lambda f - S(t)\Lambda f \Big) \Big\} = 0, \end{aligned}$$

using that the two terms within parenthesis converge to 0 and that $\|S(t+s)\| \le M\, e^{\omega t}$ for any $s \le 0$. Together with step 1, we deduce that $t \mapsto S(t)f$ is differentiable for any $t > 0$, with derivative $\Lambda S(t)f$. We conclude to the C^1 regularity by observing that $t \mapsto S(t)\Lambda f$ is continuous. Last, we have

$$S(t)f - S(s)f = \int_s^t \frac{d}{d\tau} [S(\tau)\, f]\, d\tau = \int_s^t S(\tau)\, \Lambda f\, d\tau = \int_s^t \Lambda\, S(\tau) f\, d\tau$$

and in particular

$$\|S(t)f - S(s)f\| \le (t-s)\, M\, e^{bt}\, \|\Lambda f\|,$$

for any $t > s \ge 0$. □

Definition 2.4. *Consider a Banach space X and an (unbounded) operator Λ on X. We say that $g \in C([0,\infty); X)$ is a "classical" (or Hille-Yosida) solution to the evolution equation* (1.3) *if $g \in C((0,\infty); D(\Lambda)) \cap C^1((0,\infty); X)$ so that* (1.3) *holds pointwise.*

In it worth emphasizing that Proposition 2.3 provides a "classical" solution to the evolution equation (1.3) for any initial datum $f_0 \in D(\Lambda)$ by the mean of $t \mapsto S_\Lambda(t) f_0$.

Lemma 2.5. *For any $f \in X$ and $t \ge 0$, there hold*

$$(i) \quad \lim_{h \to 0} \frac{1}{h} \int_t^{t+h} S(s)\, f\, ds = S(t) f,$$

and

$$(ii) \quad \int_0^t S(s) f\, ds \in D(\Lambda), \qquad (iii) \quad \Lambda \left(\int_0^t S(s) f\, ds \right) = S(t) f - f.$$

Sketch of the proof of Lemma 2.5. The first point is just a consequence of the fact that $s \mapsto S(s)f$ is a continuous function. We then deduce

$$\frac{1}{h}\Big\{S(h)\int_0^t S(s)f\,ds - \int_0^t S(s)f\,ds\Big) = \frac{1}{h}\Big\{\int_h^{t+h} S(s)\,f\,ds - \int_0^t S(s)f\,ds\Big\}$$
$$= \frac{1}{h}\Big\{\int_t^{t+h} S(s)\,f\,ds - \int_0^h S(s)f\,ds\Big\} \underset{h\to 0}{\longrightarrow} S(t)f - f,$$

which implies the two last points. □

In the next result we prove that $\mathscr{G}(X) \subset \mathscr{C}_D(X)$.

Definition 2.6. *We say that $\mathcal{C} \subset X$ is a core for the generator Λ of a semigroup S if*

$$\mathcal{C} \subset D(\Lambda), \quad \mathcal{C} \textit{ is dense in } X \quad \textit{and} \quad S(t)\,\mathcal{C} \subset \mathcal{C},\ \forall\, t \geq 0.$$

Proposition 2.7. *(Properties of the generator) Let $\Lambda \in \mathscr{G}(X)$.*

(i) The domain $D(\Lambda)$ is dense in X. In particular, $D(\Lambda)$ is a core.

(ii) Λ is a closed operator.

(iii) The mapping which associates to a semigroup its generator is injective. More precisely, if S_1 and S_2 are two semigroups with generators Λ_1 and Λ_2 and there exists a core $\mathcal{C} \subset D(\Lambda_1) \cap D(\Lambda_2)$ such that $\Lambda_{1|\mathcal{C}} = \Lambda_{2|\mathcal{C}}$, then $S_1 = S_2$. In other words, $S_1 \neq S_2$ implies $\Lambda_1 \neq \Lambda_2$.

Sketch of the proof of Proposition 2.7. For any $f \in X$ and $t > 0$, we define $f^t := t^{-1}\int_0^t S(s)f\,ds$. Thanks to Lemma 2.5-(i) & (ii), we see that $f^t \in D(\Lambda)$ and $f^t \to f$ as $t \to 0$. In other words, $D(\Lambda)$ is dense in X.

We prove (ii). Consider a sequence (f_k) of $D(\Lambda)$ such that $f_k \to f$ and $\Lambda f_k \to g$ in X. For $t > 0$, we write

$$S(t)f_k - f_k = \int_0^t S(s)\Lambda f_k\,ds,$$

and passing to the limit $k \to \infty$, we get

$$t^{-1}(S(t)f - f) = t^{-1}\int_0^t S(s)g\,ds.$$

We may now pass to the limit $t \to 0$ in the RHS term, and we obtain

$$t^{-1}(S(t)f - f) \to g.$$

That proves $f \in D(\Lambda)$ and $\Lambda f = g$.

We prove (iii). We observe that the mapping $t \mapsto S_i(f)f$, $i = 1, 2$, are C^1 for any $f \in \mathcal{C}$, thanks to Proposition 2.3, and

$$\begin{aligned}\frac{d}{ds}S_1(s)\,S_2(t-s)f &= \frac{dS_1(s)}{ds}\,S_2(t-s)\,f + S_1(s)\,\frac{dS_2(t-s)}{ds}\,f\\ &= S_1(s)\,\Lambda_1\,S_2(t-s)\,f + S_1(s)\,\Lambda_2\,S_2(t-s)\,f = 0.\end{aligned}$$

That implies $S_2(t)f = S_1(0)S_2(t-0)f = S_1(t)S_2(t-t)f = S_1(t)\,f$ for any $f \in \mathcal{C}$, and then $S_2 \equiv S_1$. □

Exercise 2.8. *We define recursively*

$$D(\Lambda^n) := \{f \in D(\Lambda^{n-1}),\ \Lambda f \in D(\Lambda^{n-1})\},$$

for $n \geq 2$. Prove that $D(\Lambda^n)$ is a core.

3. Duhamel formula and mild solution

Consider the evolution equation

$$\frac{d}{dt}g = \Lambda g + G \text{ on } (0,T), \quad g(0) = g_0, \tag{3.1}$$

for an unbounded operator Λ on X, an initial datum $g_0 \in X$ and a source term $G : (0,T) \to X$, $T \in (0,\infty)$. For $G \in C((0,T);X)$, a classical solution g is a function

$$g \in X_T := C([0,T);X) \cap C^1((0,T);X) \cap C((0,T);D(\Lambda)) \tag{3.2}$$

which satisfies (3.1) pointwise. For $U \in L^1(0,T;\mathscr{B}(\mathcal{X}_1,\mathcal{X}_2))$ and $V \in L^1(0,T;\mathscr{B}(\mathcal{X}_2,\mathcal{X}_3))$, we define the time convolution $V * U \in L^1(0,T;\mathscr{B}(\mathcal{X}_1,\mathcal{X}_3))$ by setting

$$(V*U)(t) := \int_0^t V(t-s)\,U(s)\,ds = \int_0^t V(s)\,U(t-s)\,ds, \quad \text{for a.e. } t \in (0,T).$$

Lemma 3.1 (Variation of parameters formula)**.** *Consider the generator Λ of a semigroup S_Λ on X. For $G \in C((0,T);X) \cap L^1(0,T;X)$, $\forall T > 0$, there exists at most one classical solution $g \in X_T$ to* (3.1) *and this one is given by*

$$g = S_\Lambda g_0 + S_\Lambda * G. \tag{3.3}$$

Proof of Lemma 3.1. Assume that $g \in X_T$ satisfies (3.1). For any fiexd $t > 0$, we define $s \mapsto u(s) := S_\Lambda(t-s)g(s) \in C^1((0,t);X) \cap C([0,t];X)$. On the one hand, we have

$$g(t) - S_\Lambda(t)g_0 = u(t) - u(0) = \int_0^t u'(s)\,ds.$$

On the other hand, we compute

$$u'(s) = -\Lambda S_\Lambda(t-s)g(s) + S_\Lambda(t-s)g'(s) = S_\Lambda(t-s)G(s),$$

for any $s \in (0,t)$. We conclude by putting together the two identities. ☐

When $G \in C((0,T);X) \cap L^1(0,T;D(\Lambda))$ and $g_0 \in D(\Lambda)$, we observe that $\bar g := S_\Lambda g_0 + S_\Lambda * G$ belongs to X_T and

$$\frac{d}{dt}\bar g(t) = \Lambda S_\Lambda(t) g_0 + \Lambda(S_\Lambda * G)(t) + S_\Lambda(0)G(t) = \Lambda \bar g(t) + G(t),$$

so that $\bar g$ is a classical solution to the evolution equation (3.1).

When $G \in L^1(0,T;X)$ and $g_0 \in X$, we observe that $\bar g \in C([0,T];X)$, $\bar g(0) = g_0$ and it is the limit of classical solutions by a density argument. We say that $\bar g$ is a mild solution to the evolution equation (3.1).

Lemma 3.2 (Duhamel formula)**.** *Consider two semigroups S_Λ and $S_\mathcal{B}$ on the same Banach space X, assume that $D(\Lambda) = D(\mathcal{B})$ and define $\mathcal{A} := \Lambda - \mathcal{B}$. If $\mathcal{A}S_\mathcal{B}, S_\mathcal{B}\mathcal{A} \in L^1(0,T;\mathscr{B}(X))$ for any $T \in (0,\infty)$, then*

$$S_\Lambda = S_\mathcal{B} + S_\Lambda * \mathcal{A}S_\mathcal{B} = S_\mathcal{B} + S_\mathcal{B}\mathcal{A} * S_\Lambda \quad \textit{in} \quad \mathscr{B}(X).$$

Proof of Lemma 3.1. Take $f \in D(\Lambda) = D(\mathcal{B})$, $t > 0$, and define $s \mapsto u(s) := S_\Lambda(s)S_\mathcal{B}(t-s)f \in C^1([0,t];X) \cap C([0,t];D(\Lambda))$. We observe that

$$\begin{aligned} u'(s) &= S_\Lambda(s)\Lambda S_\mathcal{B}(t-s)f - S_\Lambda(s)\mathcal{B}S_\mathcal{B}(t-s)f \\ &= S_\Lambda(t-s)\mathcal{A}S_\mathcal{B}(s)f, \end{aligned}$$

for any $s \in (0,t)$, from which we deduce

$$S_\Lambda(t)f - S_\mathcal{B}(t)f = \int_0^t u'(s)\,ds = \int_0^t S_\Lambda(t-s)\mathcal{A}S_\mathcal{B}(s)f\,ds.$$

By density and continuity, we deduce that the same holds for any $f \in X$, and that establishes the first version of the Duhamel formula. The second version follows by reversing the role of S_Λ and $S_\mathcal{B}$. ☐

From the above second version of Duhamel formula, we observe that for any $g_0 \in D(\Lambda)$, the function $\bar g(t) := S_\Lambda(t) g_0 \in X_T$ is a classical solution to the evolution equation (3.1) and satisfies the following functional equation

$$g = S_{\mathcal B} g_0 + S_{\mathcal B}\mathcal A * g. \tag{3.4}$$

On the other way round, we observe that if $g \in X_T$ is a solution to the functional equation (3.4), then

$$\begin{aligned} g'(t) &= \mathcal B S_{\mathcal B}(t) g_0 + \mathcal B (S_{\mathcal B}\mathcal A * g)(t) + S_{\mathcal B}(0)\mathcal A g(t) \\ &= \mathcal B g(t) + \mathcal A g(t) = \Lambda g(t), \end{aligned}$$

so that g is a classical solution to the evolution equation (3.1). More generally, when $S_{\mathcal B}\mathcal A \in L^1(0,T;\mathscr B(X))$, we say that $g \in C([0,T];X)$ is a mild solution to the evolution equation (3.1) if $g(0) = 0$ and g is a solution to the functional equation (3.4).

4. Dual semigroup and weak solution

Consider a Banach space X and an operator $A \in \mathscr C_D(X)$, with X endowed with the topology norm, and we denote $Y = X'$ in that case, or with $X = Y'$ endowed with the weak $*$ topology $\sigma(X,Y)$ for a separable Banach space Y. We define the subspace

$$D(A^*) := \big\{\varphi \in Y;\ \exists C \ge 0,\ \forall f \in D(A),\ |\langle \varphi, Af\rangle| \le C\,\|f\|_X\big\}$$

and next the adjoint operator A^* on Y by

$$\langle A^*\varphi, f\rangle = \langle \varphi, Af\rangle, \quad \forall \varphi \in D(A^*),\ f \in D(A).$$

Because $D(A) \subset X$ is dense, the operator A^* is well and uniquely defined and it is obviously linear. Because A has a closed graph, the operator A^* has also a closed graph. When A is a bounded operator, then A^* is also a bounded operator. When X is reflexive, then the domain $D(A^*)$ is always dense into X', so that $A^* \in \mathscr C_D(X')$. For a general Banach space X and a general operator A, then $D(A^*)$ is dense into X' for the weak $*\sigma(X',X)$ topology, but it happens that $D(A^*)$ is not dense into X' for the strong topology.

Consider now a semigroup S with generator Λ and $f_0 \in D(\Lambda)$. Multiplying by $\varphi \in C^1_c([0,T);D(\Lambda^*))$ the equation (1.3) satisfied by $g(t) := S(t) f_0$ and integrating in time, we get

$$\langle f_0, \varphi(0)\rangle_{X,X'} + \int_0^T \langle S(t) f_0, \partial_t\varphi(t) + \Lambda^*\varphi(t)\rangle_{X,X'}\, dt = 0.$$

Because the mapping $f_0 \mapsto S(t) f_0$ is continuous in X and the inclusion $D(\Lambda) \subset X$ is dense from Proposition 2.7, we see that the above formula is also true for any $f_0 \in X$. In other words, the semigroup $S(t)$ provides a weak solution (in the above sense) to the evolution equation (1.3) for any $f_0 \in X$.

We aim to show now that the semigroup theory provides an answer to the well-posedness issue of weak solutions to that equation for any generator Λ. More precisely, given a semigroup, we introduce its dual semigroup and we then establish that the initial semigroup provides the unique weak solution to the associated homogeneous and inhomogeneous evolution equations.

Proposition 4.1. *Consider a strongly continuous semigroup $S = S_\Lambda$ on a Banach space X with generator Λ and the dual semigroup S^* as the one-parameter family $S^*(t) := S(t)^*$ for any $t \ge 0$. Then the following hold:*
(1) S^* is a weakly $*$ continuous semigroup on X' with same growth bound as S.
(2) The generator of S^* is Λ^*. In other words, $(S_\Lambda)^* = S_{\Lambda^*}$.
(3) The mapping $t \mapsto S^*(t)\varphi$ is $C([0,\infty);X')$ (for the strong topology) for any $\varphi \in D(\Lambda^*)$. Similarly, $t \mapsto S^*(t)\varphi$ is $C^1([0,\infty);X') \cap C([0,\infty);D(\Lambda^*))$ for any $\varphi \in D(\Lambda^{*2})$.

Proof of Proposition 4.1. (1) We just write

$$\langle S^*(t)\varphi, f\rangle = \langle \varphi, S(t) f\rangle =: T_f(t,\varphi) \quad \forall t \ge 0,\ f \in X,\ \varphi \in X',$$

and we see that $(t,\varphi) \mapsto T_f(t,\varphi)$ is continuous for any $f \in X$.

(2) Denoting by $D(L)$ and L the domain and generator of S^* as defined as in section 2, for any $\varphi \in D(L)$ and $f \in D(\Lambda)$ we have

$$\begin{aligned}\langle L\varphi, f\rangle &:= \lim_{t\to 0}\Big\langle \frac{1}{t}(S(t)^*\varphi - \varphi), f\Big\rangle\\ &= \lim_{t\to 0}\Big\langle \varphi, \frac{1}{t}(S(t)f - f)\Big\rangle = \langle \varphi, \Lambda f\rangle,\end{aligned}$$

from which we immediately deduce that $D(L) \subset D(\Lambda^*)$ and $L = \Lambda^*|_{D(L)}$. To conclude, we use that L is closed. More precisely, for a given $\varphi \in D(\Lambda^*)$, we associate the sequence (φ^ε) defined through

$$\varphi^\varepsilon := \frac{1}{\varepsilon}\int_0^\varepsilon S(t)^*\varphi\, dt.$$

We have $\varphi^\varepsilon \rightharpoonup \varphi$ in the weak $*\sigma(X', X)$ sense, $\varphi^\varepsilon \in D(L)$ and, for any $f \in D(\Lambda)$,

$$\begin{aligned}\langle L\varphi^\varepsilon, f\rangle &= \langle \Lambda^*\varphi^\varepsilon, f\rangle = \langle \varphi^\varepsilon, \Lambda f\rangle\\ &= \langle \varphi, \frac{1}{\varepsilon}\int_0^\varepsilon S(t)\Lambda f\, dt\rangle \to \langle \varphi, \Lambda f\rangle,\end{aligned}$$

so that $L\varphi^\varepsilon \rightharpoonup \Lambda^*\varphi$ in the weak $*\sigma(X', X)$ sense. The graph $G(L)$ of L being closed, we have $(\varphi, \Lambda^*\varphi) \in G(L)$, which in turns implies $\varphi \in D(L)$ and finaly $L = \Lambda^*$.
(3) From Proposition 2.3, we have

$$\|S^*(t)\varphi - S^*(s)\varphi\|_{X'} = \Big\|\int_s^t S^*(\tau)\Lambda^*\varphi\, d\tau\Big\|_{X'} \le M e^{bt}(t-s)\|\Lambda^*\varphi\|_{X'}$$

for any $t > s \ge 0$ and $\varphi \in D(\Lambda^*)$, so that $t \mapsto S^*(t)\varphi$ is Lipschitz continuous from $[0, \infty)$ into X' endowed with the strong topology. □

Proposition 4.2. *Consider a weakly $*$ continuous semigroup $T = S_{\mathcal{L}}$ on a Banach space $X = Y'$ with generator $\mathcal{L}$, and the dual semigroup T^* as the one-parameter family $T^*(t) := T(t)^*$ of bounded operator on Y for any $t \ge 0$. Then the following hold:*
(1) $S = T^*$ is a strongly continuous semigroup on Y with same growth bound as T.
(2) The generator Λ of S satisfies $\mathcal{L} = \Lambda^*$.

Proof of Proposition 4.2. Just as in the proof of Proposition 4.1, we have $(t, f) \mapsto \langle \varphi, S(t)f\rangle$ is continuous for any $\varphi \in X'$. That means that $S(t)$ is a weakly $\sigma(X, X')$ continuous semigroup in X and therefore a strongly continuous semigroup in X thanks to Theorem 7.4. The rest of the proof is unchanged with respect to the proof of Proposition 4.1. □

For any $g_0 \in X$ and $G \in L^1(0, T; X)$, we say that $g \in C([0, T]; X)$ is a weak solution to the inhomogeneous initial value problem (3.1) if

$$\langle \varphi(T), g(T)\rangle - \langle \varphi(0), g_0\rangle = \int_0^T \big\{\langle \varphi' + \Lambda^*\varphi, g\rangle + \langle \varphi, G\rangle\big\}\, dt, \tag{4.1}$$

for any $\varphi \in C^1([0, T]; X') \cap C([0, T]; D(\Lambda^*))$.

Proposition 4.3. *Assume that Λ generates a semigroup S on X. For any $g_0 \in X$ and $G \in L^1(0, T; X)$, there exists a unique weak solution to equation* (3.1)*, which is nothing but the mild solution*

$$\bar g = S_\Lambda g_0 + S_\Lambda * G. \tag{4.2}$$

Proof of Proposition 4.3. We define

$$\bar g(t) = \bar g_t := S(t)g_0 + \int_0^t S(t-s)G(s)\, ds \in C([0, T]; X).$$

For any $\varphi = \varphi_t \in C^1([0, T]; X') \cap C([0, T]; D(\Lambda^*))$, we have

$$\langle \varphi_t, \bar g_t\rangle = \langle S_t^*\varphi_t, g_0\rangle + \int_0^t \langle S_{t-s}^*\varphi_t, G_s\rangle\, ds \in C^1([0, T])$$

and then

$$\begin{aligned}\frac{d}{dt}\langle \varphi_t, \bar g_t\rangle &= \langle S_t^*(\Lambda^*\varphi_t + \varphi_t'), g_0\rangle + \int_0^t \langle S_{t-s}^*(\Lambda^*\varphi_t + \varphi_t'), G_s\rangle\, ds + \langle G_t, \varphi_t\rangle \\ &= \langle \Lambda^*\varphi_t + \varphi_t', \bar g_t\rangle + \langle \varphi_t, G_t\rangle,\end{aligned}$$

from which we deduce that $\bar g$ is a weak solution to the inhomogeneous initial value problem (3.1) in the weak sense of equation (4.1). Now, if g is another weak solution, the function $f := g - \bar g$ is then a weak solution to the homogeneous initial value problem with vanishing initial datum, namely

$$\langle \varphi(T), f(T)\rangle = \int_0^T \langle \varphi' + \Lambda^*\varphi, f\rangle\, dt, \quad \forall \varphi \in C^1([0,T];X') \cap C([0,T];D(\Lambda^*)).$$

A first way to conclude is to define

$$\varphi(s) := \int_s^T S^*(\tau - s)\, \psi(\tau)\, d\tau,$$

for any given $\psi \in C^1_c((0,T);D(\Lambda^*))$, and to observe that $\varphi \in C^1([0,T];X') \cap C([0,T];D(\Lambda^*))$ is a (backward) solution to the dual problem

$$-\varphi' = \Lambda^*\varphi + \psi \ \text{ on } \ (0,T), \quad \varphi(T) = 0.$$

For that choice of test function, we get

$$0 = \int_0^T \langle \psi, f\rangle\, dt, \quad \forall \psi \in C^1_c((0,T);D(\Lambda^*)),$$

and thus $g = \bar g$.
An alternative way to get the uniqueness result is to define $\varphi(t) := S^*(T-t)\psi$ for a given $\psi \in D(\Lambda^*)$. Observing that φ is a (backward) solution to the dual problem

$$-\varphi' = \Lambda^*\varphi, \quad \varphi(T) = \psi, \tag{4.3}$$

that choice of test function leads to

$$\langle \psi, f(T)\rangle = 0 \qquad \forall \psi \in D(\Lambda^*), \ \forall T > 0,$$

and thus again $g = \bar g$. □

Exercise 4.4. *Consider a Banach space X and an unbounded operator Λ on X. We assume that $X = Y'$ for a Banach space Y and that the dual opeartor Λ^* generates a strongly continuous semigroup T on Y.*
(1) Prove that $S := T^*$ is a (at least) weakly $*\,\sigma(X,Y)$ continuous semigroup on X with generator Λ and that it provides the unique weak solution to the associated evolution equation.
(2) Prove that for any smooth functions $a = a(x)$ and $c = c(x)$, one can define a weakly continuous semigroup $S = S_\Lambda$ on $L^\infty = L^\infty(\mathbb{R}^d)$ associated to the transport operator

$$(\Lambda f)(x) := -a(x)\cdot\nabla f(x) - c(x)\, f(x),$$

as the dual semigroup associated to the dual operator Λ^* defined on $L^1(\mathbb{R}^d)$.
(3) Prove similarly that one can define a weakly continuous semigroup on $M^1(\mathbb{R}^d) := (C_0(\mathbb{R}^d))'$, the space of Radon measures, associated to the transport operator Λ.

5. Coming back to the well-posedness issue for evolution equations

Using mainly the Duhamel formula and duality arguments, we present several ways for proving the well-posedness of evolution equation and building the associated semigroup.

5.1. **A perturbation trick.** We give a very efficient result for proving the existence of a semigroup associated to a generator which is a mild perturbation of the generator of a semigroup.

Theorem 5.5. *Consider $S_{\mathcal{B}}$ a semigroup satisfying the growth estimate $\|S_{\mathcal{B}}(t)\|_{\mathscr{B}(X)} \le M\, e^{bt}$ and $\mathcal{A}$ a bounded operator. Then, $\Lambda := \mathcal{A}+\mathcal{B}$ is the generator of a semigroup which satisfies the growth estimate $\|S_\Lambda(t)\|_{\mathscr{B}(X)} \le M\, e^{b' t}$, with $b' = b + M\|\mathcal{A}\|$.*

Proof of Theorem 5.5. Step 1. Existence. We define

$$\mathscr{E} := C([0,T];\mathscr{B}(X))$$

the space of family of operators $(\mathcal{U}(t))_{t\in[0,T]}$ such that $\mathcal{U}(t) \in \mathscr{B}(X)$ for any $t \in [0,T]$, $\mathcal{U}(0) = I$ and $t \mapsto \mathcal{U}(t)g \in C([0,T];X)$ for any $g \in X$. For any $\mathcal{U} \in \mathscr{E}$, we define

$$\mathcal{V}(t) = \Phi[\mathcal{U}](t) := S_{\mathcal{B}}(t) + (S_{\mathcal{B}}\mathcal{A} * \mathcal{U})(t),$$

so that $\mathcal{V} \in \mathscr{E}$. More precisely, defining

$$\mathscr{K} := \{\mathcal{U} \in \mathscr{E};\ \|\mathcal{U}\|_{\mathscr{E}} \le e^{b'T}\, M\},$$

we observe that $\Phi : \mathscr{K} \to \mathscr{K}$ because

$$\|\mathcal{V}\|_{\mathscr{E}} \le \|S_{\mathcal{B}}\|_{\mathscr{E}} + \int_0^T \|S_{\mathcal{B}}(s)\mathcal{A}\|_{\mathscr{B}(X)}\, ds \|\mathcal{U}\|_{\mathscr{E}} \le 2\, e^{bT}\, M,$$

for $T > 0$ small enough. Moreover, for $\mathcal{U}_1, \mathcal{U}_2 \in \mathscr{K}$, we write $\mathcal{U} := \mathcal{U}_2 - \mathcal{U}_1$ and we have

$$\mathcal{V}(t) := \mathcal{V}_2(t) - \mathcal{V}_1(t) = (S_{\mathcal{B}}\mathcal{A} * \mathcal{U})(t),$$

and then

$$\|\Phi(\mathcal{U}_2) - \Phi(\mathcal{U}_1)\|_{\mathscr{E}} \le \frac{1}{2}\|\mathcal{U}_2 - \mathcal{U}_1\|_{\mathscr{E}}.$$

From the Banach contraction Theorem, there exists a fixed point $\mathcal{U} \in \mathscr{K}$ to the mapping Φ, so that

$$\mathcal{U}(t) = S_{\mathcal{B}}(t) + (S_{\mathcal{B}} * \mathcal{A}\mathcal{U})(t) = S_{\mathcal{B}}(t) + \int_0^t S_{\mathcal{B}}(t-s)\mathcal{A}\mathcal{U}(s)\, ds, \quad \forall\, t \in [0,T].$$

We extend the function $\mathcal{U}$ to $[0,\infty)$ by iterating the construction. We may be a bit more accurate in the estimate of $\mathcal{U}(t)$. By writing

$$\|\mathcal{U}(t)\|_{\mathscr{B}(X)} \le M\, e^{bt} + M\, \|A\|_{\mathscr{B}(X)} \int_0^t e^{b(t-s)} \|\mathcal{U}(s)\|_{\mathscr{B}(X)}\, ds$$

and then applying the Gronwall lemma to the function $u(t) := \|\mathcal{U}(t)\|_{\mathscr{B}(X)}\, e^{-bt}$, we indeed obtain the announced growth rate. The very same kind of arguments tells us that there exists a unique solution $g \in C(\mathbb{R}_+;X)$ to the functional equation

$$g(t) = S_{\mathcal{B}}(t)g_0 + (S_{\mathcal{B}} * \mathcal{A}g)(t) \quad \text{in } X, \quad \forall\, t \ge 0, \tag{5.4}$$

and $g(t) = \mathcal{U}(t)g_0$.

Step 2. Weak solution. We claim that

$$\mathcal{U}'(t) = \mathcal{B}S_{\mathcal{B}}(t) + \mathcal{B}\int_0^t S_{\mathcal{B}}(t-s)\mathcal{A}\mathcal{U}(s)\, ds + S_{\mathcal{B}}(0)\mathcal{A}\mathcal{U}(t) = (\mathcal{B}+\mathcal{A})\,\mathcal{U}(t),$$

on a weak sense, what we obtain by repeating the computation we have performed just after the proof of Lemma 3.2 about the Duhamel formula. More precisely, we fix $g_0 \in X$, $\varphi = \varphi_t \in C^1(\mathbb{R}_+;X) \cap C(\mathbb{R}_+;D(\mathcal{B}^*))$, and denoting $g_t := U_t g_0$, $S^*_t := S_{\mathcal{B}}(t)^*$, we define

$$\lambda(t) = \langle \varphi_t, g_t\rangle = \langle S^*_t \varphi_t, g_0\rangle + \int_0^t \langle S^*_{t-s}\varphi_t, \mathcal{A}g_s\rangle\, ds.$$

We clearly have $\lambda \in C^1(\mathbb{R}_+)$ and

$$\begin{aligned}
\lambda'(t) &= \langle S^*_t\mathcal{B}^*\varphi_t + S^*_t\varphi'_t, g_0\rangle + \langle \varphi_t, \mathcal{A}g_t\rangle + \int_0^t \langle S^*_{t-s}\mathcal{B}^*\varphi_t + S^*_{t-s}\varphi'_t, \mathcal{A}g_s\rangle\, ds. \\
&= \Big\langle \mathcal{B}^*\varphi_t + \varphi'_t, \Big(S_{\mathcal{B}}(t) + \int_0^t S_{t-s}\mathcal{A}U_s\Big)g_0\Big\rangle + \langle \mathcal{A}^*\varphi_t, g_t\rangle \\
&= \Big\langle \Lambda^*\varphi_t + \varphi'_t, g_t\Big\rangle.
\end{aligned}$$

We conclude by writing

$$\langle \varphi_t, g_t \rangle - \langle \varphi_0, g_0 \rangle = \int_0^t \lambda'(s)\, ds,$$

and observing that this nothing but the weak formulation of the evolution equation associated to Λ. We use here that $D(\Lambda) = D(\mathcal{B})$ because $\Lambda : D(\mathcal{B}) \to X$ is a bounded operator and thus $D(\Lambda^*) = D(\mathcal{B}^*)$.

Step 3. Semigroup property. We claim that $\mathcal{U} g_0 \in Y_T := C^1([0,T];X)$ for any $g_0 \in D(\Lambda)$ and $T > 0$. The proof follows by adapting the construction we have made in Step 1. For $f \in Y_T$ and $g_0 \in D(\Lambda)$, we denote

$$g_t := \varphi_t[g_0, f] = S_{\mathcal{B}}(t) g_0 + (S_{\mathcal{B}} * \mathcal{A} f)(t),$$

and we observe that

$$\begin{aligned} \frac{1}{h}(g_{t+h} - g_t) &= \frac{1}{h}[S_{\mathcal{B}}(t+h)g_0 - S_{\mathcal{B}}(t)g_0] + \frac{1}{h}\int_t^{t+h} S_{\mathcal{B}}(s)\mathcal{A} f_{t+h-s} ds \\ &\quad + \frac{1}{h}\int_0^t S_{\mathcal{B}}(s)\mathcal{A}[f_{t+h-s} - f_{t-s}]\, ds \\ &\to S_{\mathcal{B}}(t)\mathcal{B} g_0 + S_{\mathcal{B}}(t)\mathcal{A} g_0 + \int_0^t S_{\mathcal{B}}(s)\mathcal{A} f'_{t-s}\, ds =: g'_t, \end{aligned}$$

as $h \to 0$, where the limit term belongs to $C([0,T];X)$. We introduce the norms

$$\|f\|_{Y_T} := \sup_{t \in [0,T]} \big\{ \|f_t\|_X + \|f'_t\|_X \big\}, \quad \|g_0\|_{D(\Lambda)} := \|g_0\|_X + \|\Lambda g_0\|_X$$

on Y_T and $D(\Lambda)$. From the computations made in Step 1 and the one made just above, we see that

$$\begin{aligned} \|g\|_{Y_T} &\le M e^{bT} \{ \|g_0\|_X + \|\mathcal{A}\|\, T\, \|f\|_{\mathcal{E}} \} + M e^{bT} \{ \|\Lambda g_0\|_X + \|\mathcal{A}\|\, T\, \|f'\|_{\mathcal{E}} \} \\ &\le M e^{bT} (\|g_0\|_{D(\Lambda)} + \|\mathcal{A}\|\, T\, \|f\|_{Y_T}), \end{aligned}$$

and for two images $g_i := \Phi(\mathcal{U}) f_i$, $f_i \in X_T$, $i = 1, 2$, that

$$\|g_2 - g_1\|_{Y_T} \le \|\mathcal{A}\|\, M T e^{bT}\, \|f_2 - f_1\|_{Y_T}.$$

We straightforwadly adapt the proof presented in Step 1 and we obtain the existence of a fix point $g = \varphi[g_0, g] \in Y_T$. This one satisfies the functional equation (5.4) and that proves the claim $\mathcal{U} g_0 = g \in Y_T$. Also observing that

$$\begin{aligned} \frac{1}{h}(S_{\mathcal{B}}(h) g_t - g_t) &= \frac{1}{h}(g_{t+h} - g_t) - \frac{1}{h}\int_t^{t+h} S_{\mathcal{B}}(s)\mathcal{A} f_{t+h-s}\, ds \\ &\to g'_t - S_{\mathcal{B}}(t)\mathcal{A} g_0, \end{aligned}$$

as $h \to 0$, we have $\mathcal{U} g_0 \in C([0,T]; D(\mathcal{B}))$. Performing the same analyse for the dual operator, we get that for any $\psi \in D(\Lambda^*)$, there exists a solution $\varphi \in C^1([0,T];X') \cap C([0,T];D(\Lambda^*))$ to the (backward) dual problem (4.3). Arguing as in the proof of Proposition 4.3, we deduce that $g := \mathcal{U} g_0 \in C(\mathbb{R}_+; X)$, $g_0 \in X$, is the unique solution to the weak equation

$$\langle \varphi_t, g_t \rangle - \langle \varphi_0, g_0 \rangle = \int_0^t \big\langle \Lambda^* \varphi_s + \varphi'_s, g_s \big\rangle\, ds,$$

for any $\varphi \in C^1(\mathbb{R}_+, X') \cap C(\mathbb{R}_+, D(\Lambda^*))$. We conclude that $\mathcal{U}$ satisfies the semigroup property by using Proposition 1.7. $\square$

5.2. **Semilinear evolution equation.** With very similar arguments as in the previous section, we present a possible extension of the existence theory for semilinear evolution equation.

We consider the generator Λ of a semigroup of contractions S_Λ and a function $\mathcal{Q} : X \to X$ which is Lipschitz continuous on bounded sets: for any $R > 0$ there exists C such that

$$\forall\, f, g \in B(0,R) \quad \|\mathcal{Q}(f) - \mathcal{Q}(g)\| \le C\, \|f - g\|. \tag{5.1}$$

We define $\mathcal{C}(R) := \inf\{C > 0, \text{ such that (5.1) holds}\}$. The mapping $R \to \mathcal{C}(R)$ is increasing.

We consider the semilinear equation

$$\frac{d}{dt} f = \Lambda f + \mathcal{Q}(f), \quad f(0) = f_0 \tag{5.2}$$

in classical form with $f \in C([0,T];D(\Lambda)) \cap C^1([0,T];X)$ and $f_0 \in D(\Lambda)$, as well as the associated mild formulation

$$f(t) = S_\Lambda(t) f_0 + \int_0^t S_\Lambda(t-s)\mathcal{Q}(f(s))\, ds \tag{5.3}$$

with $f \in C([0,T];X)$ and $f_0 \in X$.

Proposition 5.1 (uniqueness)**.** *If $f, g \in C([0,T];X)$ are two solutions of* (5.3) *associated to the same initial datum $f_0 \in X$, then $f = g$.*

Proof of Proposition 5.1. For any given $T > 0$, we define

$$R := \max_{0 \le t \le T} \max(\|f(t)\|, \|g(t)\|),$$

and we get that $h := f - g$ satisfies $\|h(t)\| \le e^{\mathcal{C}(R)T}\|h(0)\|$ thanks to the Gronwall lemma. □

Proposition 5.2 (local existence)**.** *For $R > 0$, we define*

$$M := 2R + \|\mathcal{Q}(0)\|, \quad T_R := \frac{M-R}{\|\mathcal{Q}(0)\| + M\mathcal{C}(M)}.$$

For any $f_0 \in B(0,R)$, there exists a unique function $f \in C([0,T_R];X)$ solution to (5.3).

Proof of Proposition 5.2. We define

$$\mathscr{E} := \{f \in C([0,T_R];X);\ \|f(t)\| \le M,\ \forall\, t \in [0,T_R]\}.$$

For any $f \in \mathscr{E}$, we define

$$\Phi(f) := S_\Lambda f_0 + S_\Lambda * \mathcal{Q}(f),$$

so that $\Phi : \mathscr{E} \to \mathscr{E}$. Indeed, we observe that

$$\begin{aligned} \|\Phi(f)(t)\| &\le R + \int_0^t \|\mathcal{Q}(f(\tau))\|\, d\tau \\ &\le R + T_R\, [\mathcal{Q}(0)\| + \mathcal{C}(M)M] \le M, \end{aligned}$$

for any $t \le T_R$. On the other hand, for any $f, g \in \mathscr{E}$ and any

$$t \le T_R := \frac{M-R}{\|\mathcal{Q}(0)\| + M\mathcal{C}(M)} < \frac{1}{\mathcal{C}(M)},$$

we have

$$\begin{aligned} \|(\Phi(f) - \Phi(g))(t)\| &\le \int_0^t \|\mathcal{Q}(f(\tau)) - \mathcal{Q}(g(\tau))\|\, d\tau \\ &\le T_R\, \mathcal{C}(M) \max_{[0,T_R]} \|f-g\|. \end{aligned}$$

Thanks to the Banach fixed point theorem for contractions, we conclude to the existence of a unique fixed point for the function Φ, and that provides a solution to the semilinear equation (5.3). □

We set

$$T_{max}(f_0) := \sup\{T > 0;\ \exists f \in C([0,T];X) \text{ solution to } (5.3)\}.$$

Theorem 5.3 (maximal solution)**.** *For any given $f_0 \in X$, there exists a unique maximal solution $f \in C([0,T_{max}(f_0));X)$ to* (5.3)*, for which the following alternative holds:*

(i) $T_{max}(f_0) = +\infty$, we say that f is a global solution;

(ii) $T_{max}(f_0) < \infty$ and $\|f(t)\| \to \infty$ as $t \to T_{max}(f_0)$, we say that f blows up in finite time.

Proof of Theorem 5.3. The result is a straightforward consequence of the estimate

$$\Big(1 + \frac{\|f(t)\|}{\|\mathcal{Q}(0)\| + \|f(t)\|}\Big)\Big(1 + \mathcal{C}(\|\mathcal{Q}(0)\| + 2\|f(t)\|)\Big) \ge \frac{1}{T_{max}(f_0) - t}, \quad \forall\, t \in [0, T_{max}(f_0)), \tag{5.4}$$

that we prove now. We may assume $T_{max}(f_0) < \infty$, and we assume by contradiction that there exists $t_0 \in [0, T_{max}(f_0))$ such that

$$\Big(1 + \frac{\|f(t_0)\|}{\|\mathcal{Q}(0)\| + \|f(t_0)\|}\Big)\Big(1 + \mathcal{C}(\|\mathcal{Q}(0)\| + 2\|f(t_0)\|)\Big) < \frac{1}{T_{max}(f_0) - t_0}.$$

We define

$$R := \|f(t_0)\|, \quad M := \|\mathcal{Q}(0)\| + 2R,$$

so that the above assumption writes

$$T_{max}(f_0) - t_0 < \frac{M - R}{M(1 + \mathcal{C}(M))}.$$

On the other hand, we have

$$T_R := \frac{M - R}{\|\mathcal{Q}(0)\| + M\mathcal{C}(M)} > \frac{M - R}{M(1 + \mathcal{C}(M))}.$$

Thanks to Proposition 5.2, there exists a unique $g \in C([0, T_R]; X)$ which satisfies

$$g(t) = S_\Lambda(t) f(t_0) + \int_0^t S_\Lambda(t - \tau) \mathcal{Q}(g(\tau)) \, d\tau.$$

We then set

$$h(t) := f(t), \ \forall\, t \in [0, t_0], \quad h(t) := g(t - t_0), \ \forall\, t \in [t_0, t_0 + T_R],$$

and we observe that h is a solution to (5.3) on the interval $[0, t_0 + T_R]$, with $t_0 + T_R > T_{max}(f_0)$, what is not possible. □

A straightforward application of Theorem 5.3 is the following global existence result.

Proposition 5.4 (global existence)**.** *Any solution is global when $\mathcal{Q}$ is globally Lipschitz.*

Exercise 5.5. *Extend all the above results to the case when S_Λ is a general (not necessarily of contractions) C_0-semigroup.*

5.3. **Dissipativity and extension trick.** For $f \in X$, we define its dual set

$$F(f) := \big\{ f^* \in X', \ \langle f^*, f \rangle = \|f\|_X^2 = \|f^*\|_{X'}^2 \big\}.$$

That set is never empty thanks to the Hahn-Banach theorem (Exercise). Observe that when X is an Hilbert space $F(f) = \{f\}$ and when $X = L^p$, $1 \le p < \infty$, $F(f) = \{f \, |f|^{p-2} \, \|f\|_{L^p}^{2-p}\}$. For a general Banach space, one can show that $F(f) \neq \emptyset$ thanks to the Hahn-Banach Theorem.

We say that an (unbounded) operator Λ is dissipative if

$$\forall\, f \in D(\Lambda), \ \exists\, f^* \in F(f), \quad \langle f^*, \Lambda f \rangle \le 0.$$

When X is an Hilbert space the dissipativity condition writes

$$\forall\, f \in D(\Lambda), \quad (f, \Lambda f) \le 0, \tag{5.5}$$

and we also say that Λ is coercive.

We say that a Banach space X is "regular" if $F(f)$ is a singleton $\{f^*\}$, $f^* \in X'$, for any $f \in X$, the mapping $\varphi : X \to \mathbb{R}$, $f \mapsto \varphi(f) := \|f\|^2/2$ is differentiable and

$$D\varphi(f) \cdot h = \langle f^*, h \rangle, \quad \forall\, f, h \in X.$$

Examples of regular spaces are the Hilbert spaces and the Lebesgues spaces L^p, $1 < p < \infty$.

Theorem 5.6. *Consider a Banach space $\mathcal{X}$ and an unbounded opeartor $\mathcal{L}$ on $\mathcal{X}$. We assume that*

- (i) *$\mathcal{X}$ is a regular space and $\mathcal{L}$ is dissipative;*
- (ii) *there exists a dense Banach space $X \subset \mathcal{X}$ and an operator L on X such that $L = \mathcal{L}_{|X}$ and L is the generator of a semigroup S_L on X.*

Then $\mathcal{L}$ is the generator of a semigroup (of contractions) $S_\mathcal{L}$ on $\mathcal{X}$ such that $S_{\mathcal{L}|X} = S_L$.

Proof of Theorem 5.6. For any $f_0 \in D(L)$, the mapping $t \mapsto f_t := S_L(t) f_0$ is C^1 on X, so that also is the mapping $t \mapsto \|f_t\|_\mathcal{X}^2$. By the chain rule and the dissipativity property, we find

$$\frac{1}{2} \frac{d}{dt} \|f_t\|_\mathcal{X}^2 = D\varphi(f_t) \cdot \frac{d}{dt} f_t = \langle f_t^*, \mathcal{L} f_t \rangle \le 0.$$

Thanks to the Gronwall lemma, we deduce

$$\|S_L(t) f_0\|_\mathcal{X} \le \|f_0\|_\mathcal{X}, \quad \forall\, t \ge 0.$$

We conclude by a extension argument using the above uniform continuity estimate and the density property $D(L) \subset \mathcal{X}$. □

We often just say that $\mathcal{L}$ is a dissipative operator in a (complex) Hilbert space H when there exists a real number $b \in \mathbb{R}$ such that

$$\forall f \in D(\mathcal{L}),\ \exists f^* \in F(f), \qquad \Re e\langle f^*, \mathcal{L} f\rangle \le b\|f\|^2. \tag{5.6}$$

For any $f \in D(\mathcal{L})$ and denoting $f_t := S_t f$, we have from (5.6)

$$\frac{d}{dt}\|f_t\|_H^2 := \frac{d}{dt}(\bar f_t, f_t)_H = 2\Re e\langle f_t, \mathcal{L} f_t\rangle \le 2b\,\|f_t\|^2.$$

Thanks to the Gronwall lemma, we deduce

$$\|S_t f\| \le e^{bt}\,\|f\| \quad \forall\, t \ge 0. \tag{5.7}$$

It is then classical to show that the "dissipative growth rate" $\omega_d(\mathcal{L})$ satisfies

$$\omega_d(\mathcal{L}) := \inf\{b \in \mathbb{R};\ (5.7)\text{ holds}\} = \inf\{b \in \mathbb{R};\ (5.6)\text{ holds}\}. \tag{5.8}$$

6. Semigroup Hille-Yosida-Lumer-Phillips' existence theory

We say that an (unbounded) operator Λ is maximal if there exists $x_0 > 0$ such that

$$R(x_0 - \Lambda) = X. \tag{6.9}$$

We say that Λ is m-dissipative if Λ is dissipative and maximal.

We present now the Lumer-Phillips' version of the Hille-Yosida Theorem which establishes the link between semigroup of contractions and dissipative operator.

Theorem 6.7 (Hille-Yosida, Lumer-Phillips). *Consider $\Lambda \in \mathscr{C}_D(X)$. The two following assertions are equivalent:*
(a) Λ is the generator of a semigroup of contractions;
(b) Λ is dissipative and maximal.

For the sake of brevity and simplicity, we only present the proof of the implication $(b) \Rightarrow (a)$ in the case of an Hilbert framework. The reverse implication is let as an exercise.

Exercise 6.8. *Consider an (unbounded) operator Λ on a Hilbert space X which is dissipative and maximal. Prove that $\Lambda \in \mathscr{C}_D(X)$. Prove the same result in the case when X is reflexive.*

Exercise 6.9. *Consider a semigroup $S = S_\Lambda$ on an regular Banach space X (as defined in section 5.3). Prove that the generator Λ is dissipative iff S is a semigroup of contractions.*
(Hint. Argue similarly as in the proof of Theorem 5.6).

Exercise 6.10. *Consider a semigroup $S = S_\Lambda$ on a Banach space X which satisfies the growth bound* (1.1)*. Prove that $\Lambda - z$ is invertible for any $z \in \Delta_b := \{z \in \mathbb{C};\ \Re e\, z > b\}$.*
(Hint. Prove that the operator

$$\mathcal{U}(z) := -\int_0^\infty S_\Lambda(t)\, e^{zt}\, dt$$

is well defined in $\mathscr{B}(X)$ for any $z \in \Delta_b$ and that

$$\mathcal{U}(z)\,(\Lambda - z) = I_{D(\Lambda)}, \quad (\Lambda - z)\,\mathcal{U}(z) = I_X).$$

Exercise 6.11. *Prove (a) $\Rightarrow$ (b) in Theorem 6.7. (Hint. Use* Excercise 6.9 *and* Excercise 6.10*).*

Lemma 6.12. *Under condition (b), the operator Λ satisfies*

$$\forall\, \varepsilon > 0, \quad R(I - \varepsilon\Lambda) = X. \tag{6.10}$$

Moreover, for any $\varepsilon > 0$, $I - \varepsilon\Lambda$ is invertible from $D(\Lambda)$ into X, and

$$\|(I - \varepsilon\Lambda)^{-1}\|_{\mathscr{B}(X)} \le 1 \tag{6.11}$$

Finally, $D(\Lambda^n)$ is dense in $D(\Lambda^{n-1})$ for any $n \ge 1$.

Proof of Lemma 6.12. Step 1. We know that (6.10) holds with $\varepsilon = \varepsilon_0 := 1/x_0$ and we prove that it also holds for any $\varepsilon > \varepsilon_0/2$. First, we observe that for any $g \in X$ there exists $f \in D(\Lambda)$ such that

$$f - \varepsilon_0 \Lambda f = g.$$

That solution $f \in D(\Lambda)$ is unique because for any other solution $h \in D(\Lambda)$, the difference $u := h - f \in D(\Lambda)$ satisfies

$$u - \varepsilon_0 \Lambda u = 0,$$

so that

$$\|u\|^2 \le \langle u^*, u\rangle - \varepsilon_0 \langle u^*, \Lambda u\rangle = 0,$$

and $u = 0$. In other words, $I - \varepsilon_0 \Lambda$ is invertible. Moreover, we also have

$$\|f\|^2 \le \langle f^*, f\rangle - \varepsilon_0 \langle f^*, \Lambda f\rangle = \langle f^*, g\rangle \le \|f^*\| \, \|g\|,$$

so that $\|f\| \le \|g\|$. In other words, $(I - \varepsilon_0\Lambda)^{-1} \in \mathscr{B}(X)$ and $\|(I - \varepsilon_0\Lambda)^{-1}\|_{\mathscr{B}(X)} \le 1$, which is nothing but (6.11) for $\varepsilon = \varepsilon_0$. Next, for a given $\varepsilon > 0$ and a given $g \in X$, we want to solve the equation

$$f \in D(\Lambda), \quad f - \varepsilon \Lambda f = g.$$

We write that equation as

$$f - \varepsilon_0 \Lambda f = (1 - \frac{\varepsilon_0}{\varepsilon})\, f + \frac{\varepsilon_0}{\varepsilon}\, g,$$

and then

$$f = \Phi(f) := (I - \varepsilon_0 \Lambda)^{-1} \Big[(1 - \frac{\varepsilon_0}{\varepsilon})\, f + \frac{\varepsilon_0}{\varepsilon}\, g\Big].$$

Finally, when $|1 - \varepsilon_0/\varepsilon| < 1$, which means $\varepsilon > \varepsilon_0/2$, we deduce from the Banach contraction fixed point Theorem that there exists a unique $f \in X$ such that $f = \Phi(f) \in D(\Lambda)$. That concludes the proof of (6.10) for any $\varepsilon > \varepsilon_0/2$. Repeating the argument, we then get (6.10) and (6.11).

Step 2. We already know (it is one of our assumptions) that $D(\Lambda)$ is dense in X. We define $V_\varepsilon := (1 - \varepsilon\Lambda)^{-1}$, and for $f \in X$, we define $f_\varepsilon = V_\varepsilon f$. We claim that

$$f_\varepsilon \to f \ \text{ in } \ X, \quad \text{as} \quad \varepsilon \to 0. \tag{6.12}$$

First, we observe that

$$V_\varepsilon - I = V_\varepsilon (I - (I - \varepsilon \Lambda)) = \varepsilon\, V_\varepsilon\, \Lambda.$$

For any $f \in X$, we may introduce a sequence $f^n \to f$ in X such that $f^n \in D(\Lambda)$. We then have

$$f_\varepsilon - f = V_\varepsilon f - V_\varepsilon f^n + V_\varepsilon f^n - f^n + f^n - f,$$

and

$$\begin{aligned} \|f_\varepsilon - f\| &\le \|V_\varepsilon f - V_\varepsilon f^n\| + \|\varepsilon\, V_\varepsilon\, \Lambda f^n\| + \|f^n - f\| \\ &\le 2\,\|f^n - f\| + \varepsilon \|\Lambda f^n\|, \end{aligned}$$

from what (6.12) immedediately follows.

Consider $f \in D(\Lambda)$ and $f_\varepsilon := V_\varepsilon f \in D(\Lambda)$. From $(I + \varepsilon\Lambda) f_\varepsilon = f$, we deduce that

$$\varepsilon \Lambda f_\varepsilon = f - f_\varepsilon \in D(\Lambda),$$

which means that $f_\varepsilon \in D(\Lambda^2)$. Moreover, we have $f_\varepsilon \to f$ in X as well as

$$f_\varepsilon \to f \quad \text{and} \quad \Lambda f_\varepsilon = V_\varepsilon \Lambda f \to \Lambda f$$

in X as $\varepsilon \to 0$ thanks to (6.12). We have proved that $D(\Lambda^2)$ is dense in $D(\Lambda)$. We prove that $D(\Lambda^n)$ is dense in $D(\Lambda^{n-1})$ for any $n \ge 2$ in a similar way. □

Proof of (b) ⇒ (a) in Theorem 6.7. Step 1. For fixed $\varepsilon > 0$, we may build by induction and thanks to Lemma 6.12, the sequence $(g_k)_{k \ge 1}$ in $D(\Lambda)$ defined by the family of equations

$$\forall\, k \ge 0 \qquad \frac{g_{k+1} - g_k}{\varepsilon} = \Lambda\, g_{k+1}. \tag{6.13}$$

Observe that from the identity

$$(g_{k+1}, g_{k+1}) - \varepsilon\, \langle \Lambda g_{k+1}, g_{k+1}\rangle = (g_k, g_{k+1}),$$

we deduce

$$\|g_k\| \le \|g_0\| \quad \forall\, k \ge 0.$$

We fix $T > 0$, $n \in \mathbb{N}^*$ and we define

$$\varepsilon := T/n, \quad t_k = k\,\varepsilon, \quad g^\varepsilon(t) := g_k \text{ on } [t_k, t_{k+1}).$$

and

$$g_\varepsilon(t) := \frac{t_{k+1} - t}{\varepsilon}\, g_k + \frac{t - t_k}{\varepsilon}\, g_{k+1} \text{ on } [t_k, t_{k+1}).$$

The previous estimate writes then

$$\sup_{[0,T]} \|g^\varepsilon\| \le \|g_0\|, \quad \sup_{[0,T]} \|g_\varepsilon\| \le \|g_0\|. \tag{6.14}$$

Step 2. We next establish that g_ε is equi-uniformly continuous in $C([0,T];X)$ when $g_0 \in D(\Lambda)$. With the above notation, we write

$$g_k = (1 - \varepsilon\Lambda)^{-1} g_{k-1} = V_\varepsilon^k g_0, \quad V_\varepsilon := (1 - \varepsilon\Lambda)^{-1}.$$

Observing that

$$V_\varepsilon^k - I = (V_\varepsilon - I) \sum_{\ell=0}^{k-1} V_\varepsilon^\ell, \quad V_\varepsilon - I = V_\varepsilon\, \varepsilon\Lambda,$$

V_ε commutes with Λ and $\|V_\varepsilon\| \le 1$, we get

$$\|g_k - g_0\| \le \sum_{\ell=0}^{k-1} \|V_\varepsilon\|^\ell\, \|(V_\varepsilon - I) g_0\| \le k\,\varepsilon\, \|\Lambda g_0\|.$$

We see then that $\|g_\varepsilon(t) - g_0\| \le t\,\|\Lambda g_0\|$ for any $t, \varepsilon > 0$, and by construction, we also have $\|g_\varepsilon(t) - g_\varepsilon(s)\| \le (t-s)\,\|\Lambda g_0\|$ for any $t > s > 0$ and $\varepsilon > 0$.

Step 3. We finally improve the bound (6.14) by showing that g_ε is a Cauchy sequence in $C([0,T];X)$ for any $T > 0$, when $\varepsilon := 2^{-n}$ and $g_0 \in D(\Lambda^2)$. We fix $t \in (0,T)$ dyadic, that means $t2^{n_t} \in \mathbb{N}$ for some $n_t \in \mathbb{N}^*$, and for any $n \ge n_t$, we write

$$h_n := g_{2^{-n}}(t) = V_n^{2^n t} g_0, \quad V_n := (1 - 2^{-n}\Lambda)^{-1}$$

and

$$h_{n+1} := g_{2^{-n+1}}(t) = U_n^{2^n t} g_0, \quad U_n := (1 - 2^{-n-1}\Lambda)^{-2}.$$

Now, we observe that

$$\begin{aligned} U_n^{2^n t} - V_n^{2^n t} &= (U_n - V_n) \sum_{\ell=0}^{2^n t - 1} U_n^{2^n - \ell} V_n^\ell \\ &= U_n\, [(1 - 2^{-n-1}\Lambda)^2 - (1 - 2^{-n}\Lambda)]\, V_n \sum_{\ell=0}^{2^n t - 1} U_n^{2^n - \ell} V_n^\ell \\ &= 2^{-2n-2} \sum_{\ell=0}^{2^n t - 1} U_n^{2^n - \ell + 1} V_n^{\ell+1}\, \Lambda^2, \end{aligned}$$

so that

$$\|h_{n+1} - h_n\| \le 2^{-n-2}\, t\, \|\Lambda^2 g_0\|.$$

As a consequence, for any $m > n \ge n_t$, we have

$$\|h_m - h_n\| \le 2^{-n-1}\, t\, \|\Lambda^2 g_0\|,$$

and (h_n) is a Cauchy sequence in X. Thanks to Step 2, we conclude that $g_{2^{-n}}$ is a Cauchy sequence in $C([0,T];X)$ for any $T > 0$.

Step 4. Consider now a test function $\varphi \in C_c^1([0,T); D(\Lambda^*))$ and define $\varphi_k := \varphi(t_k)$, so that $\varphi_n = \varphi(T) = 0$. Multiplying the equation (6.13) by φ_k and summing up from $k = 0$ to $k = n$, we get

$$-(\varphi_0, g_0) - \sum_{k=1}^{n} \langle \varphi_k - \varphi_{k-1}, g_k \rangle = \sum_{k=0}^{n} \varepsilon\, \langle \Lambda g_{k+1}, \varphi_k \rangle.$$

Introducing the two functions $\varphi^\varepsilon, \varphi_\varepsilon : [0,T) \to X$ defined by

$$\varphi^\varepsilon(t) := \varphi_{k-1} \quad \text{and} \quad \varphi_\varepsilon(t) := \frac{t_{k+1} - t}{\varepsilon}\, \varphi_k + \frac{t - t_k}{\varepsilon}\, \varphi_{k+1} \qquad \text{for} \quad t \in [t_k, t_{k+1}),$$

in such a way that

$$\varphi'_\varepsilon(t) = \frac{\varphi_{k+1} - \varphi_k}{\varepsilon} \qquad \text{for} \quad t \in (t_k, t_{k+1}),$$

the above equation also writes

$$- \langle \varphi(0), g_0 \rangle - \int_\varepsilon^T \langle \varphi'_\varepsilon, g^\varepsilon \rangle \, dt = \int_0^T \langle \Lambda g^\varepsilon, \varphi^\varepsilon \rangle \, dt. \tag{6.15}$$

On the one hand, from Step 3, we know that there exists $g \in C([0,T];X)$, for any $T > 0$, such that $g_\varepsilon \to g$ in $C([0,T];X)$ and we then deduce $g^\varepsilon \to g$ in $L^\infty(0,T;X)$. On the other hand, from the above construction, we have $\varphi'_\varepsilon \to \varphi'$ and $\varphi_\varepsilon \to \varphi$ both strongly in $L^\infty(0,T;X')$. We may then pass to the limit as $\varepsilon \to 0$ in (6.15) and we get that

$$\langle g_0, \varphi(0) \rangle + \int_0^t \langle \varphi'(s) + \Lambda^* \varphi(s), g(s) \rangle_{X',X} \, ds = 0. \tag{6.16}$$

Step 5. All together, for $g_0 \in D(\Lambda^2)$, we have proved that there exists a function $g \in C([0,\infty);X)$ which satisfies the evolution equation in the weak form (6.16) and $\|g(t)\|_X \le \|g_0\|_X$ for any $t \ge 0$. Repeating the same argument as in steps 1, 2 and 3, we find $\Lambda g \in C([0,\infty);X)$ and $\|\Lambda g(t)\|_X \le \|\Lambda g_0\|_X$, at least when $g_0 \in D(\Lambda^3)$. By a density argument and using the two above contraction estimates, we get that the same holds for any $g_0 \in D(\Lambda)$. From (6.16) and that regularity estimate, we get $g \in C^1([0,\infty);X)$ and thus g is a classical solution to the evolution equation (1.3).
In a Hilbert space, we have the uniquness of solution in such a class of functions by proving that $g_0 = 0$ implies $g \equiv 0$ thanks to a standard Gronwall argument. Indeed, if $g \in C^1([0,\infty);X) \cap C([0,\infty);D(\Lambda))$ satisfies (6.16) for g_0 and thus (1.3), we compute

$$\frac{d}{dt}\|g(t)\|^2 = 2(\Lambda g, g) \le 0, \quad \|g(0)\|^2 = 0,$$

so that $g = 0$.
In a general Banach space, we have the same existence theory for the backward dual problem for any $\varphi_T \in D(\Lambda^*)$. As a consequence, if φ is such a backward solution associated to φ_T, we have

$$\begin{aligned} \langle \varphi_T, g(T) \rangle &= \int_0^T \frac{d}{ds} \langle \varphi(s), g(s) \rangle \, ds + \langle \varphi(0), g(0) \rangle \\ &= \int_0^T [\langle \Lambda^* \varphi(s), g(s) \rangle + \langle \Lambda^* \varphi(s), g(s) \rangle] \, ds = 0, \end{aligned}$$

and we conclude again that $g(T) = 0$ because φ_T is arbitrary and $D(\Lambda^*)$ is dense (for the weak topology) into X'. We conclude thanks to Proposition 1.7. $\square$

7. Complements

7.1. **Continuity.** For the sake of completeness and in order to make the chapter as self-contained as possible, we establish the results we let as exercises in Excercise 1.4.

Proposition 7.1. *Let (S_t) satisfy (i), (ii′) and (iii) in Definition 1.1 and Remark 1.3. Then (S_t) also satisfies the growth estimate (iv) in Definition 1.1.*

Proof of Proposition 7.1. We first claim that

$$\exists\, \delta > 0, \ \exists\, C \ge 1, \ \text{such that } \|S(t)\| \le C \ \forall\, t \in [0,\delta].$$

On the contrary, there exists a sequence (t_n) such that $t_n \searrow 0$ and $\|S(t_n)\| \to \infty$. On the other hand, we know that $S(t_n) f \to f$ for any $f \in X$ which implies $\sup \|S(t_n) f\| < \infty$ for any $f \in X$ (that is a consequence of the Banach-Steinhaus Theorem for a weak $*$ continuous semigroup). Using the Banach-Steinhaus Theorem (again) that implies $\sup \|S(t_n)\| < \infty$ and a contradiction. We then obtain the growth estimate (1.1) with $M := C$ and $b := (\log C)/\delta$ thanks to an euclidian division argument. $\square$

Proposition 7.2. *Let (S_t) satisfy (i), (ii′) and (iii) in Definition 1.1 and Remark 1.3. Then (S_t) also satisfies the continuity of trajectories condition (ii) in Definition 1.1.*

Proof of Proposition 7.2. For $t > 0$, we write

$$S(t+h) - S(t) = S(t)(S(h) - I)$$

when $h > 0$,

$$S(t+h) - S(t) = S(t+h)(I - S(-h))$$

when $h < 0$. We conclude using the condition (ii′) together with the growth estimate (iv) established in Proposition 7.1. ☐

Proposition 7.3. *For a given semigroup (S_t) on X, the new norm $|||\cdot|||$ defined in* (1.2) *is equivalent to the initial norm and the new semigroup (T_t) defined in* (1.2) *is a semigroup of contractions for that new norm.*

Proof of Proposition 7.3. The two norms are equivalent because

$$\|f\| = \|T(0)f\| \le |||f||| = \sup_{t\ge 0} \|e^{-\omega t}\, S(t)\, f\| \le M\, \|f\|.$$

Moreover, for any $t \ge 0$,

$$\begin{aligned} |||T(t)f||| &= \sup_{s\ge 0} \|T(s)\, T(t) f\| \\ &= \sup_{s\ge 0} \|e^{-\omega(s+t)}\, S(t+s) f\| \\ &\le \sup_{\tau\ge 0} \|e^{-\omega\tau}\, S(\tau) f\| = |||f|||, \end{aligned}$$

which proves that $T(t)$ is a semigroup of contractions for that norm. ☐

Theorem 7.4. *Let (S_t) be a semigroup in the sense of Definition 1.1 in which conditions (i) and (ii) are understood in the sense of the weak topology $\sigma(X, X')$. Then (S_t) is a strongly continuous semigroup.*

Proof of Theorem 7.4. For any fixed $f \in X$, we define

$$f^\varepsilon := \frac{1}{\varepsilon} \int_0^\varepsilon S_t f dt. \tag{7.1}$$

Using the growth estimate, we then compute

$$\begin{aligned} \|S_h f^\varepsilon - f^\varepsilon\| &= \sup_{\|\varphi\|\le 1} \frac{1}{\varepsilon} \Big| \int_0^\varepsilon \langle S_{t+h} f, \varphi\rangle\, dt - \int_0^\varepsilon \langle S_t f, \varphi\rangle\, dt \Big| \\ &= \sup_{\|\varphi\|\le 1} \frac{1}{\varepsilon} \Big| \int_\varepsilon^{\varepsilon+h} \langle S_t f, \varphi\rangle\, dt - \int_0^h \langle S_t f, \varphi\rangle\, dt \Big| \\ &\le \frac{h}{\varepsilon}\, \|f\|\, 2M e^{|b|(\varepsilon+h)} \to 0, \end{aligned}$$

as $h \to 0$. We define now

$$\mathcal{X} := \{f \in X;\ S_t f \to f \text{ in norm as } t \to 0\}.$$

It is a norm closed linear subspace so that it is weakly closed. Because $f^\varepsilon \rightharpoonup f$ and $f^\varepsilon \in \mathcal{X}$ for any $f \in X$, it is also weakly dense. That proves that $\mathcal{X} = X$. ☐

7.2. **Nonautonomous semigroup.** We briefly present a possible extension of the semigroup theory to the nautonomous evolution equation framework.

Definition 7.5. *Let fix $T \in (0, \infty)$. We say that a two parameters family $(U_{t,s})_{T\ge t\ge s\ge 0}$ is a continuous nonautonomous semigroup of linear and bounded operators on X (or we just say a nonautonomous semigroup on X), if the following conditions are fulfilled:*

(i) family of operators: $\forall s, t \ge 0$, $f \mapsto U_{t,s} f$ is linear and continuous on X;

(ii) continuity of trajectories: $\forall f \in X$, $\{(t,s);\, 0 \le s \le t\} \ni (t,s) \mapsto U_{t,s}\, f$ is continuous;

(iii) semigroup property: $\forall\, t \ge r \ge s \ge 0$, $U_{s,s} = I$ and $U_{t,r} \circ U_{r,s} = U_{t,s}$;

(iv) growth estimate: $\exists\, b \in \mathbb{R}$, $\exists\, M \ge 1$,

$$\|U_{t,s}\|_{\mathscr{B}(X)} \le M\, e^{b(t-s)} \quad \forall\, t \ge s \ge 0. \tag{7.2}$$

To a nonautonomous semigroup, we may associate a one parameter family $(\Lambda(t))_{0\le t\le T}$ of (unbounded) operators on X and the forward nonautonomous evolution equation

$$\frac{d}{dt} f = \Lambda(t) f \quad \text{on} \quad (0,T), \quad f(0) = f_0, \tag{7.3}$$

in the following way.

Theorem 7.6. *Consider a nonautonomous semigroup $(U_{t,s})_{T\ge t\ge s\ge 0}$ on a Banach space X. For any given $t\in[0,T)$, we define the (linear unbounded) operator $\Lambda(t)$ by*

$$\begin{aligned} D(\Lambda(t)) &:= \Big\{f\in X;\ \lim_{h\searrow 0}\frac{U_{t+h,t}f-f}{h} \text{ exists in } X\Big\},\\ \Lambda(t)f &:= \lim_{h\searrow 0}\frac{U_{t+h,t}f-f}{h} \quad \text{for any } f\in D(\Lambda(t)). \end{aligned}$$

Assume that there exists $X_1\subset X$ dense such that $U_{t,s}$ is a nonautonomous semigroup on X_1 and $X_1\subset D(\Lambda(t))$ for any $t\in[0,T)$. Then

$$\frac{\partial}{\partial t}U_{t,s}f=\Lambda(t)\,U_{t,s}f \quad \forall f\in X_1,\ 0\le s\le t\le T; \tag{7.4}$$

$$\frac{\partial}{\partial s}U_{t,s}f=-U_{t,s}\,\Lambda(s)f \quad \forall f\in X_1,\ 0\le s\le t\le T. \tag{7.5}$$

In particular, for any $f_0\in X_1$, the function $t\mapsto f(t):=U_{t,0}f_0$ provides a solution to the evolution equation (7.3).

Proof of Theorem 7.6. We (at least formally) compute

$$\begin{aligned} \frac{\partial}{\partial t}U_{t,s} &= \lim_{h\to 0}\frac{1}{h}\big(U_{t+h,s}-U_{t,s}\big)=\lim_{h\to 0}\frac{1}{h}\big(U_{t+h,t}-I\big)U_{t,s}=\Lambda(t)\,U_{t,s},\\ -\frac{\partial}{\partial s}U_{t,s} &= \lim_{h\to 0}\frac{1}{h}\big(U_{t,s}-U_{t,s+h}\big)=\lim_{h\to 0}U_{t,s+h}\frac{1}{h}\big(U_{s+h,s}-I\big)=U_{t,s}\,\Lambda(t), \end{aligned}$$

and we observe that these limits can be easily rigorously justify in the space $\mathscr{B}(X_1,X)$. Finally, defining $f(t):=U_{t,0}f_0$ for any $f_0\in X_1$, we obserse that (7.3) immediatley follows from (7.4). □

Corollary 7.7. *Under the asumptions of Theorem 7.6 and for any $G\in C([0,T];X)$, any solution to the non homogeneous equation*

$$\frac{d}{dt}f=\Lambda(t)f+G \quad \text{on} \quad (0,T), \quad f(0)=f_0,$$

satisfies

$$f(t)=U_{t,0}\,f_0+\int_0^t U_{t,s}\,G(s)\,ds.$$

In the other way round, by the mean of the J.-L. Lions theory or the characteristics method one can show that we can associate a nonautonomous semigroup $U_{t,s}$ which provides solutions to the forward nonautonomous evolution equation. That is let as an exercise and we refer to chapters 1 and 2 for details. We also present a result (without proof) which is more in the spirit of the Hille-Yosida semigroup theory.

Theorem 7.8. *Consider a one parameter family $(\Lambda(\tau))_{0\le\tau\le T}$ of (unbounded) operators on a Banach space X. We assume*
(1) $\Lambda(\tau)$ generates a semigroup $S_{\Lambda(\tau)}$ on X for any $\tau\in[0,T]$ with growth bound independent of τ.
(2) $\Lambda(\tau)$ generates a semigroup on another Banach space $X_1\subset X$ for any $\tau\in[0,T]$, with growth bound independent of $\tau\in[0,T]$.
(3) The mapping $[0,T]\to\mathscr{B}(X_1,X)$, $\tau\mapsto\Lambda(\tau)$, is continuous and $X_1\subset X$ is dense.
Then there exists a unique nonautonomous semigroup $(U_{t,s})_{T\ge t\ge s\ge 0}$ in X with infinitesimal generators $\Lambda(t)_{0\le t\le T}$.

Exercise 7.9. *Establish Theorem 7.8 by first assuming that $\Lambda(\tau)$ is a piecewise constant family and next approximating a general family $\Lambda(\tau)$ by a sequence of piecewise constant families.*

Theorem 7.10. *Consider a one parameter family $(\Lambda(\tau))_{0\le\tau\le T}$ of (unbounded) operators on $X=Y'$ for a Banach space Y. We assume that there exists $Y_1\subset Y$ dense and $U^*_{t,s}$ a nonautonomous semigroup on Y such that*

$$\begin{aligned} \frac{d}{dt}U^*_{t,s} &= U^*_{t,s}\Lambda(t)^*, \quad \text{on } \{t>s\}\times\mathscr{B}(Y_1,Y),\\ \frac{d}{ds}U^*_{t,s} &= -\Lambda(s)^*U^*_{t,s}, \quad \text{on } \{t>s\}\times\mathscr{B}(Y_1,Y). \end{aligned}$$

Then $U_{t,s}$ is a nonautonomous semigroup on X which satisfies (7.4) *and* (7.5) *in the weak sense. It provides the unique weak solution to the equation to the evolution equation* (7.3) *for any $f \in X$.*

Proof of Theorem 7.10. For $f \in X$ and $\varphi \in Y_1$, we write

$$\langle \partial_t U_{t,s} f, \varphi \rangle := \langle f, \partial_t U^*_{t,s} \varphi \rangle = \langle f, U^*_{t,s} \Lambda(t)^* \varphi \rangle$$

and

$$\langle \partial_s U_{t,s} f, \varphi \rangle := \langle f, \partial_s U^*_{t,s} \varphi \rangle = -\langle f, \Lambda(s)^* U^*_{t,s} \varphi \rangle,$$

in which we recognize a weak formulation of equations (7.4) and (7.5). For $f_0 \in X$, we define $f(t) := U_{t,0} f_0$, and more precisely, we define by duality

$$\langle f(t), \varphi \rangle := \langle f_0, U^*_{t,0} \varphi \rangle, \quad \forall \varphi \in Y.$$

From the first above identity, we have

$$\partial_t \langle f(t), \varphi \rangle = \langle f_0, U^*_{t,0} \Lambda(t)^* \varphi \rangle = \langle f(t), \Lambda(t)^* \varphi \rangle \quad \text{on} \quad (0,T),$$

for any $\varphi \in Y_1$, which is a weak formulation of the evolution equation (7.3).
On the other hand, in order to prove the uniqueness of the solution, we consider a weak solution $f(t)$ to the evolution equation (7.3) associated to the initial datum $f_0 = 0$. For any $\tau \in (0,T)$ and $\varphi_\tau \in Y_1$, we define $\varphi(s) := U^*_{\tau,s} \varphi_\tau$ on $[0,\tau]$, so that

$$\partial_s \varphi(s) = \partial_s U^*_{\tau,s} \varphi_\tau = -\Lambda(s)^* U^*_{\tau,s} \varphi_\tau = -\Lambda(s)^* \varphi(s) \quad \text{on} \quad (0,\tau)$$

and $\varphi(\tau) = \varphi_\tau$. We then compute

$$\begin{aligned} \frac{d}{dt} \langle f(t), \varphi(t) \rangle &= \langle \frac{d}{dt} f(t), \varphi(t) \rangle + \langle f(t), \frac{d}{dt} \varphi(t) \rangle \\ &= \langle \Lambda(t) f(t), \varphi(t) \rangle + \langle f(t), -\Lambda(t)^* \varphi(t) \rangle = 0. \end{aligned}$$

As a consequence

$$\langle f(\tau), \varphi_\tau \rangle = \langle f_0, \varphi_0 \rangle = 0$$

for any $\varphi_\tau \in Y_1$ and that implies $f(\tau) = 0$ for any $\tau \in (0,T)$. □

7.3. **Transport equation in measures and L^∞ frameworks.** We consider the transport equation

$$\partial_t f = \Lambda f = -a(t,x) \cdot \nabla_x f - c(t,x) f, \tag{7.6}$$

with smooth cofficients a and c. Denoting by $\Phi_{t,s}$ the characterics of the associated ODE, namely for any $x_s \in \mathbb{R}^d$, $x(t) := \Phi_{t,s} x_s$ is the solution to

$$\frac{d}{dt} x = a(t,x), \quad x(s) = x_s,$$

we observe that any smooth solution f to (7.6) satisfies

$$\frac{d}{dt} \big[f(t, \Phi_{t,s}) \, e^{\int_s^t c(\tau, \Phi_{\tau,s}(x)) d\tau} \big] = 0.$$

As a consequence, for any smooth function f_s, the function

$$f(t,x) := f_s(\Phi^{-1}_{t,s}) \, e^{-\int_s^t c(\tau, \Phi^{-1}_{\tau,s}(x)) d\tau}$$

is the unique solution to the equation (7.6) corresponding to the initial condition $f(s,x) = f_s(x)$.
We now consider the dual equation

$$\partial_t \varphi = \Lambda^* \varphi := a \cdot \nabla_x \varphi + (\operatorname{div} a - c) \varphi. \tag{7.7}$$

That last equation generates a strongly continuous nonautonmous semigroup $V_{t,s}$ in $C_0(\mathbb{R}^d)$ and in $L^1(\mathbb{R}^d)$ that one can build by the above characteristics method in $C^1_c(\mathbb{R}^d)$ and next by a density and continuity argument in $L^p(\mathbb{R}^d)$, $p = \infty, 1$.
Finally, by setting $U_{t,s} := V^*_{t,s}$ and using Theorem 7.10, we build a weakly $*$ continuous nonautonmous semigroup in $M^1(\mathbb{R}^d)$ and in $L^\infty(\mathbb{R}^d)$ which is associated to the initial transport equation (7.6).

8. Discussion

8.1. **Several way to build solutions.** In the previous chapters, we have seen two ways to build a solution to an evolution equation associated to an abstract or a PDE operator. More precisely, we have built

(1) variational solutions for coercive operator,

(2) weak (and in fact renormalized) solutions for transport operator.

(3) There exist other classical ways to build solutions in some particular situations. On the one hand, we may use some explicit representation formula exactly as we did to solve the transport equation thanks to the characteristics method. The most famous example concerns the Laplacian operator and the associated heat equation which can be solved in the all space by introducing the heat kernel. More precisly, one may observe that

$$S(t)f_0 := \gamma_t * f_0, \quad \gamma_t(x) = \frac{1}{t^{d/2}}\,\gamma\Big(\frac{x}{\sqrt{t}}\Big), \quad \gamma(x) = \frac{1}{(2\pi)^{d/2}}\,\exp(-|x|^2/2),$$

which is meaningfull for $f_0 \in L^1(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$, defines a semigroup (for instance in $X = L^p(\mathbb{R}^d)$, $1 \le p \le \infty$, or $X = C_0(\mathbb{R}^d)$) and a solution to the heat equation

$$\partial_t f = \frac{1}{2}\Delta f \text{ in } (0,\infty)\times\mathbb{R}^d, \quad f(0,\cdot) = f_0 \text{ in } \mathbb{R}^d.$$

That is immediate from the explicit formulas

$$\partial_t\gamma_t(z) = -\frac{d/2}{t}\,\gamma_t(z) + \frac{|z|^2}{2t^2}\,\gamma_t(z)$$

and

$$\nabla_z\gamma_t(z) = -\frac{z}{t}\,\gamma_t(z), \quad \Delta_z\gamma_t(z) = -\frac{d}{t}\,\gamma_t(z) + \frac{|z|^2}{t^2}\,\gamma_t(z).$$

On the other hand, in some situation, we may build (a bit less explicit) representation formula by introducing convenient basis. To give an example, we consider the operator $\Lambda = \Delta$ in the space $X = C([0,2\pi])$. We observe that $\varphi_k(x) := e^{i\,kx}$ is an eigenfunction associated to the eigenvalues problem

$$\Delta\,\varphi_k = -k^2\,e^{i\,kx} = \lambda_k\varphi_k, \quad \varphi_k(0) = \varphi_k(2\pi).$$

We then define the semigroup $S(t) : C_{per}([0,2\pi]) \to C_{per}([0,2\pi])$ which to any function

$$f(x) = \sum_{k\in\mathbb{Z}} c_k\,\varphi_k(x)$$

associate the function

$$(S_t f)(x) := \sum_{k\in\mathbb{Z}} c_k\,e^{\lambda_k t}\,\varphi_k(x),$$

and we easily verify that $S_t f$ gives a solution to the heat equation in $[0,2\pi]$ (with periodic boundary conditions). The same spectral decomposition method can be generalized to the case where $\Lambda = \Delta$ (or even Λ is a general parabolic operator) is posed in a (smooth) bounded open domain with Dirichlet, Neumann or Robin conditions at the boundary. Similarly, when

$$(\Lambda f)(x) := \int_\Omega b(y,x)\,f(y)\,dy$$

with $b \in L^2(\Omega\times\Omega)$, then Λ is an Hilbert-Schmidt operator in $L^2(\Omega)$. That means that the exists a Hilbert basis (ϕ_k) of $L^2(\Omega)$ made of eigenfunctions of Λ. As in the first example, in order to solve the evolution equation associated to Λ, one just has to write the Hilbert expansion series of the initial datum on the basis (ϕ_k) and to solve (straightforwardly) the evolution equation for each coordinate.

(4) In this chapter 3, we have presented several simple but powerful tricks, as extension arguments and duality arguments, in order to build a semigroup from one other.

(5) Finally, we have briefly presented the Hille-Yosida theory (or more precisely its Lumer-Phillips' version) which provides a clear link between the semigroup theory and its abstract evolution equation counterpart.

8.2. **From Hille-Yosida theory to variational solutions.** The Hille-Yosida-Lumer-Phillips Theorem 6.7 can be seen as a generalisation of the J.-L. Lions theorem presented in chapter 1, in the sense we explain now. We consider a Hilbert space H and an operator $\Lambda : D(\Lambda) \subset H \to H$ such that the Hille-Yosida theory applies: for any $g_0 \in D(\Lambda)$, there exists $g \in C([0,\infty); D(\Lambda)) \cap C^1([0,\infty); H)$ such that

$$\frac{dg}{dt} = \Lambda g \quad \text{in } H.$$

We moreover assume that there exists a Hilbert space V such $D(\Lambda) \subset V \subset H$ and realizes the hypothesizes (i) and (ii) of Theorem I.3.2. We claim that for any $g_0 \in H$, the semigroup solution $g(t) := e^{\Lambda\,t}\,g_0$, given by the Hille-Yosida theory (and thus obtained by a uniform continuity principle from the solutions

corresponding to initial data in $D(\Lambda)$), is a variational solution (in the sense that it satifies the evolution equation in the variational sense).
We first consider $g_0 \in D(\Lambda)$. The Hille-Yosida solution $g(t)$ satisfies

$$\frac{d}{dt}|g(t)|_H^2 = 2\,\langle \Lambda g(t), g(t)\rangle \le -2\alpha\,\|g(t)\|_V^2 + 2b\,|g(t)|_H^2,$$

so that

$$|g(t)|_H^2 + 2\alpha \int_0^t \|g(s)\|_V^2\,ds \le e^{2bt}\,|g_0|_H^2.$$

We also have $g \in H^1(0,T;V')$, what comes from hypothesis (i) and the bound

$$\begin{aligned}\|\partial_t g\|_{L^2(V')} &= \|\Lambda g\|_{L^2(V')} = \sup_{\|\varphi\|_{L^2(V)}\le 1} \int_0^T \langle \Lambda g, \varphi\rangle\,dt \\ &\le \sup_{\|\varphi\|_{L^2(V)}\le 1} M\,\|g\|_{L^2(V)}\,\|\varphi\|_{L^2(V)} \le \frac{M}{2\alpha}\,e^{2bT}\,|g_0|_H^2.\end{aligned}$$

Now, for $g_0 \in H$, we may introduce a sequence of initial data $g_{0,\alpha} \in D(\Lambda)$ such that $g_{0,\alpha} \to g_0$ in H. Next, considering the associated sequence of Hille-Yoside solutions $g_\alpha(t)$, writting the variational formulation

$$[\langle g_\alpha, \varphi\rangle_{H,H}]_0^T - \int_0^T \langle \varphi', g_\alpha\rangle_{V',V}\,dt = \int_0^T \langle \Lambda g_\alpha, \varphi\rangle_{V',V}\,dt,$$

and passing to the limit $\alpha \to 0$, we get that $g = g_\alpha$ is indeed a variational solution.

8.3. **Very weak solution.**

Definition 8.1. *Let X, Y, Z be three Banach spaces such that $Z \subset X \subset Y'$ with continuous and dense embeddings. Assume furthermore that Y is seprable for its norm topology. Let $\Lambda : Z \to Y'$ be a linear and bounded opeartor and $\Lambda^* : Y \subset Y'' \to Z'$ be its adjoint operator. A function*

$$g = g(t) \in \mathscr{E}_T := L^\infty(0,T;X) \cap C([0,T);Y'_w) \cap L^1(0,T;Z)$$

is said to be a (very) weak solution to the evolution equation

$$\frac{d}{dt} g = \Lambda g + G, \quad g(0) = g_0$$

associated to the initial datum $g_0 \in X$ and the source term $G \in L^1(0,T;Y')$ if, for any test function $\varphi \in C^1([0,T];Y)$, there holds

$$[\langle g, \varphi\rangle_{Y',Y}]_0^T - \int_0^T \langle g, \partial_t\varphi\rangle_{Y',Y}\,dt = \int_0^T \{\langle g, \Lambda^*\varphi\rangle_{Z,Z'} + \langle G, \varphi\rangle_{Y',Y}\}\,dt. \tag{8.1}$$

Let us emphasize that we only assume $\varphi \in Y$, and not $\varphi \in Y'' \supset Y$, and that the continuity property $C([0,T);Y'_w)$ simply means thet the mapping $t \mapsto \langle g(t), \varphi\rangle_{X,Y}$ is continuous for any $\varphi \in Y$. The main difference with respect to Definition 1.5 is that last continuity property which is weaken here.

Proposition 8.2. *Under the assumptions of Definition 8.1, a function $g \in \mathscr{E}_T$ is a (very) weak solution (in the sense of Definition 8.1) if, and only if,*

$$\frac{d}{dt}\langle g, \varphi\rangle_{Y',Y} = \langle \Lambda g + G, \varphi\rangle_{Y',Y}, \quad \forall\,\varphi \in Y, \quad \textit{and} \quad g(0) = g_0. \tag{8.2}$$

Sketch of the proof of Proposition 8.2. The direct sense is clear and the reciprocity sense is a good exercice using the separation hypothesis made on Y and a density argument. We claim that for any $\varphi \in C_c^1((0,T);X)$ and $\varepsilon > 0$, we can find a function $\varphi_\varepsilon \in C_c^1((0,T);X)$ such that

$$\varphi_\varepsilon(t) = \sum_{k=1}^n \theta_k(t)\,\psi_k, \quad \|\varphi - \varphi_\varepsilon\|_{W^{1,\infty}} < \varepsilon, \tag{8.3}$$

for a finite familly $\theta_1, ..., \theta_n \in C_c^1((0,T);\mathbb{R})$ and $\psi_1, ..., \psi_n \in Y$. As a consequence, summing up the n corresponding equations (8.1), we get

$$-\int_0^T \langle g, \partial_t\varphi_\varepsilon\rangle_{Y',Y}\,dt = \int_0^T \{\langle g, \Lambda^*\varphi_\varepsilon\rangle_{Z,Z'} + \langle G, \varphi_\varepsilon\rangle_{Y',Y}\}\,dt.$$

Passing to the limit $\varepsilon \to 0$, we obtain the same equation with $\varphi \in C_c^1((0,T);X)$. Now, for $\varphi \in C_c^1([0,T);X)$, we define $\varphi_\varepsilon := \varphi\chi_\varepsilon \in C_c^1(0,T);X)$ with

$$\chi_\varepsilon(t) := \int_0^t \rho_\varepsilon(s)\,ds, \quad \rho_\varepsilon(s) := \varepsilon^{-1}\rho(\varepsilon^{-1}s), \quad 0 \le \rho \in C_c((0,1)), \quad \int_0^1 \rho(s)\,ds = 1.$$

We compute

$$\begin{aligned} -\int_0^T \chi_\varepsilon \, \langle g, \partial_t \varphi \rangle_{Y',Y} \, dt - \int_0^T \rho_\varepsilon \, \langle g, \varphi \rangle_{Y',Y} \, dt &= -\int_0^T \langle g, \partial_t \varphi_\varepsilon \rangle_{Y',Y} \, dt \\ &= \int_0^T \{ \langle g, \Lambda^* \varphi_\varepsilon \rangle_{Z,Z'} + \langle G, \varphi_\varepsilon \rangle_{Y',Y} \} \, dt. \end{aligned}$$

Passing to the limit $\varepsilon \to 0$, we immediately get (8.1). $\square$

Exercise 8.3. *Prove* (8.3). *Hint. Consider $\varphi \in C^1_c((0,T);X)$. Prove that for any $\varepsilon > 0$ we can find $n \in \mathbb{N}^*$ such that*

$$\sup_{t \in [t_{k-2}, t_{k+2}]} \|\varphi'(t) - \varphi'(t_k)\|_X < \varepsilon/(T \vee 1), \quad \forall \, k = 2, ..., n-2, \quad \varphi' = 0 \text{ on } [0, t_1] \cup [t_{n-1}, t_n],$$

where $t_k := k\delta$, $\delta := T/n$ and $\varphi'(t_0) = \varphi(t_1) = \varphi(t_{n-1})$. Introduce then the scalar functions

$$\tilde\theta(s) := (1 - |s|/(2\delta))_+, \quad \tilde\theta_k(s) := \tilde\theta(s - t_k), \quad \theta_k(t) := \tilde\theta_k(t)/\big(\sum_{k'=1}^{n-1} \tilde\theta_{k'}(t)\big)$$

and χ_ε defined by

$$\chi_\varepsilon(s) = s/\delta \text{ on } [0,\delta], \; \chi_\varepsilon(1) = 1 \text{ on } [\delta, T-\delta], \; \chi_\varepsilon(s) = (T-s)/\delta \text{ on } [T-\delta, T].$$

Show that the function

$$\varphi_\varepsilon(t) := \chi_\varepsilon(t) \int_0^t \phi_\varepsilon(s) \, ds, \quad \phi_\varepsilon(t) := \sum_{k=2}^{n-2} \theta_k(t) \varphi'(t_k)$$

is a convenient choice.

We give now an existence result of (very) weak solutions.

Theorem 8.4. *Let X, Y be two Banach spaces such that $X \subset Y'$ with continuous and dense embeddings. Assume furthermore that Y is separable and X is reflexive and regular. Let Λ_ε, $\varepsilon \ge 0$, be a family of unbounded operators on X such that for some positive constants M and b:*

(i) *Λ_ε is the generator of a semigroup for any $\varepsilon > 0$;*
(ii) *$\langle \Lambda_\varepsilon f, f \rangle \le b \, \|f\|_X^2$ for any $f \in D(\Lambda_\varepsilon)$, $\varepsilon \ge 0$;*
(iii) *$|\langle \Lambda_\varepsilon f, \varphi \rangle| \le M \, \|f\|_X \, \|\varphi\|_Y$ for any $f \in X$, $\varphi \in Y$ and any $\varepsilon \ge 0$, or equivalently (Λ_ε) is bounded in $\mathscr{B}(Y, X')$, and furthermore $\langle \Lambda_\varepsilon g, \varphi \rangle \to \langle \Lambda g, \varphi \rangle$ as $\varepsilon \to 0$ for any $g \in X$ and $\varphi \in Y$, where we denote $\Lambda := \Lambda_0$.*

Then, for any $g_0 \in X$, $G \in L^1(0,T;X)$, there exists a function

$$g = g(t) \in C([0,T]; X_w)$$

solution to (8.1) *for the spaces $X = Z$ and Y.*

Proof of Theorem 8.4. We only consider the case $G = 0$ and let the general case as an exercise. By assumption (i), for any $\varepsilon > 0$, there exists a unique (Hille-Yosida, weak) solution $g_\varepsilon \in C([0,T];X)$ to the evolution equation

$$\frac{d}{dt} g_\varepsilon = \Lambda_\varepsilon g_\varepsilon$$

and using assumption (ii) we get the uniform estimate

$$\sup_{[0,T]} \|g_\varepsilon\|_X \le e^{bT} \, \|g_0\|_X.$$

For any $\varphi \in Y$, we have

$$\frac{d}{dt} \langle g_\varepsilon(t), \varphi \rangle = \langle \Lambda_\varepsilon g_\varepsilon(t), \varphi \rangle, \tag{8.4}$$

where the left hand side term is bound thanks to assumption (iii). As a consequence, up to the extraction of a subsequence, there exists $g \in C([0,T];X_w)$ such that $g_\varepsilon \rightharpoonup g$ in $C([0,T];X_w)$. We conclude by passing to the limit $\varepsilon \to 0$ in (8.4). $\square$

Example 8.5. *(Viscosity method). For $a \in L^\infty(\mathbb{R}^d)$, $div\, a \in L^\infty(\mathbb{R}^d)$, we define in $X := L^2(\mathbb{R}^d)$ the operators*

$$\Lambda f := a \cdot \nabla f, \quad \Lambda_\varepsilon f := \varepsilon \Delta f + a \cdot \nabla f.$$

We set $Y := C^2_c(\mathbb{R}^d)$ and we check that the assumptions of Theorem 8.4 are fulfiled. On the one hand, we clearly have

$$\langle \Lambda_\varepsilon f, f \rangle \le b \, \|f\|_X^2 - \varepsilon \|f\|_{H^1}^2, \quad b := \frac{1}{2} \|(div\, a)_+\|_{L^\infty},$$

for any $f \in H^1(\mathbb{R}^d) \supset D(\Lambda_\varepsilon)$, $\varepsilon \ge 0$, so that assumption (i) is fullfiled. We then may apply J.-L. Lions's existence Theorem I.3.2 and we deduce that Λ_ε is the generator of a semigroup for any $\varepsilon > 0$. Assumption (iii) is obtained by performing one integration by parts. As a consequence of Theorem 8.4 and for any $g_0 \in L^2$, we deduce the existence of a weak solution $g \in C([0,T]; L^2_w)$ to the transport equation associated to a. Without additional assumption on the force field a, we cannot be sure that the solution is unique and thus that the transport equation generates a semigroup.

Example 8.6. *(Regularization trick). For $a \in L^\infty(\mathbb{R}^d)$, $div\, a \in L^\infty(\mathbb{R}^d)$, we define in $X := L^p(\mathbb{R}^d)$, $1 < p < \infty$, the operators*

$$\Lambda f := a \cdot \nabla f, \quad \Lambda_\varepsilon f := a_\varepsilon \cdot \nabla f,$$

with $a_\varepsilon = a * \rho_\varepsilon$, for a smooth mollifer (ρ_ε), so that $0 \le \rho_\varepsilon \in C^1_c(\mathbb{R}^d)$, $\|\rho_\varepsilon\|_{L^1} = 1$ and $\rho_\varepsilon \rightharpoonup \delta_0$ as $\varepsilon \to 0$. Again, we aim to apply Theorem 8.4 with the choice $X := L^p$, $Y := W^{1,p'}$. Because $a_\varepsilon \in W^{1,\infty}$, we may use the characteristics method of Chapter 2 and we obtain the existence of a solution $g_\varepsilon \in C([0,T]; L^p)$ and of a semigroup associated to the transport operator Λ_ε. From

$$\langle g^*, \Lambda_\varepsilon g \rangle \le \frac{1}{p} \|(div\, a_\varepsilon)_+\|_{L^\infty} \|g\|^2_{L^p} \le \frac{1}{p} \|div\, a\|_{L^\infty} \|g\|^2_{L^p},$$

we deduce that assumption (ii) holds. We also have

$$|\langle \varphi, \Lambda_\varepsilon g \rangle| = \Big| \int g\, div(a_\varepsilon \varphi) \Big| \le (\|a\|_{L^\infty} + \|div a\|_{L^\infty})\, \|g\|_{L^p}\, \|\varphi\|_{W^{1,p'}},$$

which is nothing but assumption (iii). We conclude again to the existence of a solution $g \in C([0,T]; L^p_w)$ to the transport equation associated to the vector field a for any initial datum $g_0 \in L^p$.

Exercise 8.7. *(1) Prove a similar result in $X = L^\infty$.*
(2) Prove a similar result in $X = M^1$ when furthermore a, $div\, a \in C(\mathbb{R}^d)$.
(3) Prove a similar result in $X = L^1$ when furtheremore $g_0 \ge 0$ and $g_0 (\log g_0)_+ + g_0 |x|^2 \in L^1$. (Hint. Use the Dunford-Pettis Theorem).
(4) Prove a similar result in $X = L^1$ without any further assumption. (Hint. Use also the De La Vallée Poussin Theorem).

Exercise 8.8 (Miyadera-Voigt perturbation theorem)**.** *Given a generator $\mathcal{B}$ on X, we say that $\mathcal{A} \in \mathscr{C}_D(X)$ is $\mathcal{B}$-bounded if*

$$\|\mathcal{A} f\| \le C(\|f\| + \|\mathcal{B} f\|) \quad \forall\, f \in D(\mathcal{B})$$

for some constant $C \in (0,\infty)$. In particular, $D(\mathcal{B}) \subset D(\mathcal{A}) \subset X$.
Consider $S_\mathcal{B}$ a semigroup satisfying the growth estimate $\|S_\mathcal{B}(t)\|_{\mathscr{B}(X)} \le M\, e^{bt}$ and $\mathcal{A}$ a $\mathcal{B}$-bounded operator such that

$$\exists\, T > 0, \quad \int_0^T \|S_\mathcal{B}(t) \mathcal{A}\|_{\mathscr{B}(X)}\, dt \le \frac{1}{2}, \qquad \sup_{t \in [0,T]} \|S_\mathcal{B}(t)\mathcal{A}\|_{\mathscr{B}(X, X_{-1})} < \infty, \tag{8.5}$$

or

$$\exists\, T > 0, \quad \int_0^T \|\mathcal{A} S_\mathcal{B}(t)\|_{\mathscr{B}(X)}\, dt \le \frac{1}{2}, \qquad \sup_{t \in [0,T]} \|\mathcal{A} S_\mathcal{B}(t)\|_{\mathscr{B}(X, X_{-1})} < \infty, \tag{8.6}$$

where the abstract Sobolev space $X_{-1} = X^\mathcal{B}_{-1}$ is defined as the closure of X for the norm

$$\|f\|_{X_{-1}} := \|(\mathcal{B} - b - 1)^{-1} f\|_X.$$

Prove that $\Lambda := \mathcal{A} + \mathcal{B}$ is the generator of a semigroup which satisfies the growth estimate $\|S_\Lambda(t)\|_{\mathscr{B}(X)} \le M'\, e^{b' t}$, with $M' = 2 e^{bT} M$ and $b' = (\log 2 e^{bT} M)/T$. (Hint. Repeat the proof of Theorem 5.5).

Exercise 8.9. *Apply the Hille-Yosida-Lumer-Phillips Theorem 6.7 on the folowing equations.*
– Heat equation

$$\partial_t u = \Delta u, \quad u(0) = u_0,$$

on the space $H := L^2(\Omega)$, with $\Lambda u := \Delta u$, $D(\Lambda) = H^1_0(\Omega) \cap H^2(\Omega)$, $\Omega \subset \mathbb{R}^d$.
– Wave equation

$$\partial^2_{tt} u = \Delta u \quad u(0) = u_0,\ \partial_t u(0) = v_0,$$

written as

$$\partial_t U = \Lambda U, \quad U = (u, \partial_t u), \quad \Lambda = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}$$

on the space $H := H^1_0(\Omega) \times L^2(\Omega)$, $D(\Lambda) = (H^1_0(\Omega) \cap H^2(\Omega)) \times H^1_0(\Omega)$, $\Omega \subset \mathbb{R}^d$.
– Scrödinger equation

$$i \partial_t u + \Delta u = 0, \quad u(0) = u_0,$$

on the space $H := L^2(\mathbb{R}^d;\mathbb{C})$, *with* $\Lambda u := i\Delta u$, $D(\Lambda) = H^2(\mathbb{R}^d)$
– *Stokes equation*

$$\partial_t u = \Delta u, \quad div\, u = 0, \quad u(0) = u_0,$$

on the space $H := \{u \in (L^2(\mathbb{R}^d))^d;\ div\, u = 0\}$, *with* $\Lambda u := \Delta u$ *and*

$$D(\Lambda) = \{u \in (H^2(\mathbb{R}^d))^d \cap H,\ \Delta u \in H\}.$$

9. Bibliographic discussion

Most of the material presented in this chapter can be found in

- [1] Engel, K.-J. and Nagel, R. *One-parameter semigroups for linear evolution equations.* Graduate Texts in Mathematics, Vol 194. Springer-Verlag, New York, 2000.
- [2] Pazy, A. *Semigroups of linear operators and applications to partial differential equations.* Applied Mathematical Sciences. Vol. 4. Springer-Verlag, New York, 1983.

Section 5.2 is adapted from the Master course notes of O. Kavian (personal communication) and the proof of Theorem 6.7 has been suggested to me by O. Kavian.
The definition of "regular" space in Section 5.3 is maybe original. It is motivated by the fact that it enables to establish a priori estimates in a very simple way, just using ordinary differential inequality and Gronwall lemma.