Université Paris-Dauphine **2016-2017**
An introduction to evolution PDEs January 2017

1 Problem I

We consider the evolution PDE

$$\partial_t f = \Delta f + \mathrm{div}(Ef), \tag{1.1}$$

on the unknown $f = f(t, x)$, $t \geq 0$, $x \in \mathbb{R}^d$, with $E = E(x)$ a given smooth force field which satisfies for some $\gamma \geq 1$

$$\forall\, |x| \geq 1, \quad |E(x)| \leq C\, |x|^{\gamma-1}, \quad \mathrm{div}E(x) \leq C|x|^{\gamma-2}, \quad x \cdot E \geq |x|^\gamma.$$

We complement the equation with an initial condition

$$f(0, x) = f_0(x).$$

Question 1. Which strategy can be used in order to exhibit a semigroup $S(t)$ in $L^p(\mathbb{R}^d)$, which provides solutions to (2.1) for initial data in $L^p(\mathbb{R}^d)$? Is the semigroup positive? mass conservative? Explain briefly why there exists a function $G = G(x)$ such that

$$0 \leq G \in L^2(m), \quad \langle G \rangle := \int G = 1, \quad \mathcal{L}G = 0.$$

We accept that $G > 0$. For any nice function $f : \mathbb{R}^d \to \mathbb{R}$ we denote $h := f/G$ and, reciprocally, for any nice function $h : \mathbb{R}^d \to \mathbb{R}$ we denote $f := Gh$.

In the sequel we will not try to justify rigorously the a priori estimates we will establish, but we will carry on the proofs just as if there do exist nice (smooth and fast decaying) solutions. We denote by C or C_i some constants which may differ from line to line.

Question 2. Prove that for any weight function $m : \mathbb{R}^d \to [1, \infty)$ and any nice function $f : \mathbb{R}^d \to \mathbb{R}$, there holds

$$\int (\mathcal{L}f) f m = -\int |\nabla f|^2 m + \frac{1}{2} \int f^2 \mathcal{L}^* m,$$

where we will make explicit the expression of $\mathcal{L}^*$.

Question 3. Prove that there exist $w : \mathbb{R}^d \to [1, \infty)$, $\alpha > 0$ and $b, R_0 \geq 0$ such that

$$\mathcal{L}^* w \leq -\alpha\, w + b \mathbf{1}_{B_{R_0}}.$$

Question 4. For some constant $\lambda \geq 0$ to be specified later, we define $W := w + \lambda$. Deduce from the previous question that

$$\int h^2 w\, G \leq \frac{1}{\alpha} \int h^2 \,(b\, \mathbf{1}_{B_{R_0}} - \mathcal{L}^* W)\, G,$$

for any nice function $h : \mathbb{R}^d \to \mathbb{R}$.

Question 5. Take a nice function $h : \mathbb{R}^d \to \mathbb{R}$ such that $\langle hG \rangle = 0$ and denote $G(\Omega) := \langle G\, \mathbf{1}_\Omega \rangle$. Prove that for any $R \geq R_0$ there exists $\kappa_R \in (0, \infty)$ such that

$$\int h^2\, \mathbf{1}_{B_R}\, G \leq \kappa_R \int_{B_R} |\nabla h|^2\, G + \frac{1}{G(B_R)} \Big(\int_{B_R^c} h\, G \Big)^2$$

and deduce that

$$\int h^2\, \mathbf{1}_{B_R}\, G \leq \frac{\kappa_R}{1 + \lambda} \int |\nabla h|^2\, W\, G + \frac{G(B_R^c)}{G(B_R)} \int h^2\, w\, G.$$

Question 6. Establish finally that there exist some constants $\lambda, K_1 \in (0, \infty)$ such that

$$\frac{2}{K_1} \int h^2 w G \leq \int \Big(W |\nabla h|^2 - \frac{1}{2} h^2 \mathcal{L}^* W \Big) G = \int (-\mathcal{L} f) f G^{-1} W,$$

for all nice function h such that $\langle hG \rangle = 0$.

Question 7. Consider a nice solution f to (2.1) associated to an initial datum f_0 such that $\langle f_0 \rangle = 0$. Establish that f satisfies

$$\frac{1}{2} \frac{d}{dt} \int h_t^2\, W G = - \int |\nabla h|^2\, W G + \frac{1}{2} \int h^2 G \mathcal{L}^* W.$$

Deduce that there exists $K_2 \in (0, \infty)$ such that f satisfies the decay estimate

$$\int f_t^2\, W G^{-1}\, dx \leq e^{-K_2 t} \int f_0^2\, W G^{-1}\, dx, \quad \forall\, t \geq 0.$$

2 Problem II

We consider the Keller-Segel equation

$$\partial_t f = \Delta f + \mathrm{div}(\mathcal{K}_f f), \tag{2.1}$$

on the unknown $f = f(t,x)$, $t \geq 0$, $x \in \mathbb{R}^2$, with

$$\mathcal{K}_f := \mathcal{K} * f, \quad \mathcal{K} := \nabla \kappa = \frac{1}{2\pi}\frac{z}{|z|^2}.$$

We complement the equation with an initial condition

$$f(0,x) = f_0(x).$$

We accept that for any initial datum $f_0 \geq 0$ with finite mass $M > 0$, finite moment of order 2 and finite entropy, there exists at least one nonnegative solution $f \in C([0,T); L^1)$ for some $T > 0$, and that any weak solution furthermore satisfies

$$f \in C^1((0,T); W^{1,p}(\mathbb{R}^2)), \quad \forall\, p \in [1,\infty].$$

Question 1. Establish that any weak solution satisfies

$$\frac{d}{dt}\|f\|_{L^2}^2 + 2\|\nabla_x f\|_{L^2}^2 = \|f\|_{L^3}^3 \quad \text{on} \quad (0,T).$$

Deduce that there exists some constant $A \in (0,\infty)$ such that

$$\frac{d}{dt}\|f\|_{L^2}^2 + \frac{1}{2}\|\nabla_x f\|_{L^2}^2 \leq A^2 M \quad \text{on} \quad (0,T).$$

Deduce next that there exists a constant $c_M > 0$ such that

$$\frac{d}{dt}\|f\|_{L^2}^2 + c_M\|f\|_{L^2}^4 \leq A^2 M \quad \text{on} \quad (0,T).$$

and finally that exists a constant K (which only depends on c_M, A^2M and T) such that

$$t\,\|f(t,.)\|_{L^2}^2 \leq K \quad \forall\, t \in (0,T).$$

Question 2. Using the splitting

$$\int f^2\,(\widetilde{\log}_+ f)^{-2} \leq \int_{f\leq R} f^2\,(\widetilde{\log}_+ f)^{-2} + \int_{f\geq R} f^2\,(\widetilde{\log}_+ f)^{-2}, \quad \forall\, R \in (0,\infty),$$

deduce from Question 1 that

$$t\int f^2\,(\widetilde{\log}_+ f)^{-2} \to 0 \quad \text{as} \quad t \to 0.$$

Question 3. Introducing the notation $\widetilde{\log}_+ g := 2 + (\log g)_+$ and using an Hölder inequality, prove that

$$\|g\|_{L^{4/3}} \leq C(\mathcal{H}(g), M_2(g)) \left(\int f^2 \, (\widetilde{\log}_+ g)^{-2} \right)^{1/4}.$$

Question 4. Establish that

$$t^{1/4} \|f(t,.)\|_{L^{4/3}} \to 0 \quad \text{as } t \to 0.$$

Briefly explain how to deduce the uniqueness of weak solutions to the Keller-Segel equation.

3 Problem III -

We consider the relaxation equation

$$\partial_t f = \mathcal{L}f := -v \cdot \nabla f + \rho_f M - f \quad \text{in} \quad (0,\infty) \times \mathbb{R}^{2d}, \tag{3.1}$$

on the unknown $f = f(t,x,v)$, $t \geq 0$, $x, v \in \mathbb{R}^d$, with

$$\rho_f(t,x) = \int_{\mathbb{R}^d} f(t,x,v)\, dv, \quad M(v) := \frac{1}{(2\pi)^{d/2}} \exp(-|v|^2/2).$$

We complement the equation with an initial condition

$$f(0,x,v) = f_0(x,v) \quad \text{in} \quad \mathbb{R}^{2d}. \tag{3.2}$$

Question 1. A priori estimates and associated semigroup. We denote by f a nice solution to the relaxation equation (3.1)–(3.2).
(a) Prove that f is mass conserving.
(b) Prove that

$$|\rho_g| \leq \|g\|_{L^2_v(M^{-1/2})}, \quad \forall\, g = g(v) \in L^2_v(M^{-1/2})$$

and deduce that

$$\|f(t,\cdot)\|_{L^2_{xv}(M^{-1/2})} \leq \|f_0\|_{L^2_{xv}(M^{-1/2})}.$$

(c) Consider $m = \langle v\rangle^k$, $k > d/2$. Prove that there exists a constant $C \in (0,\infty)$ such that

$$|\rho_g| \leq C\|g\|_{L^p_v(m)}, \quad \forall\, g = g(v) \in L^p_v(m), \quad p = 1, 2,$$

and deduce that

$$\|f(t,\cdot)\|_{L^p_{xv}(m)} \leq e^{\lambda t}\|f_0\|_{L^p_{xv}(m)},$$

for a constant $\lambda \in [0,\infty)$ that we will express in function of C.
(d) What strategy can be used in order to exhibit a semigroup $S(t)$ in $L^p_{xv}(m)$, $p = 2$, $p = 1$, which provides solutions to (3.1) for initial date in $L^p_{xv}(m)$? Is the semigroup positive? mass conservative? a contraction in some spaces?

The aim of the problem is to prove that the associated semigroup $S_{\mathcal{L}}$ to (3.1) is bounded in $L^p(m)$, $p = 1, 2$, without using the estimate proved in question (1b).

In the sequel we will not try to justify rigorously the a priori estimates we will establish, but we will carry on the proofs just as if there do exist nice (smooth and fast decaying) solutions.
We define

$$\mathcal{A}f := \rho_f M, \quad \mathcal{B}f = \mathcal{L}f - \mathcal{A}f.$$

Question 2. Prove that $S_{\mathcal{B}}$ satisfies a growth estimate $\mathcal{O}(e^{-t})$ in any $L^p_{xv}(m)$ space. Using the Duhamel formula

$$S_{\mathcal{L}} = S_{\mathcal{B}} + S_{\mathcal{B}}\mathcal{A} * S_{\mathcal{L}}$$

prove that $S_{\mathcal{L}}$ is bounded in $L^1_{xv}(m)$.

Question 3. Establish that $\mathcal{A} : L^1_{xv}(m) \to L^1_x L^\infty_v(m)$ where

$$\|g\|_{L^1_x L^p_v(m)} := \int_{\mathbb{R}^d} \|g(x,)\|_{L^p(m)}\, dx.$$

Prove that

$$\frac{d}{dt} \int \Big(\int f^p\, dx \Big)^{1/p} dv = \int \Big(\int (\partial_t f) f^{p-1}\, dx \Big) \Big(\int f^p\, dx \Big)^{1/p-1} dv.$$

Deduce that $S_\mathcal{B}$ satisfies a growth estimate $\mathcal{O}(e^{-t})$ in any $L^1_x L^p_v(m)$ space for $p \in (1,\infty)$, and then in $L^1_x L^\infty_v(m)$. Finally prove that $S_\mathcal{B}(t)\mathcal{A}$ is appropriately bounded in $\mathscr{B}(L^1, L^1_x L^\infty_v(m))$ and that $S_\mathcal{L}$ is bounded in $L^1_x L^\infty_v(m)$.

Question 4. We define $u(t) := \mathcal{A} S_\mathcal{B}(t)$. Establish that

$$(u(t) f_0)(x,v) = M(v) e^{-t} \int_{\mathbb{R}^d} f_0(x - v_* t, v_*)\, dv_*.$$

Deduce that

$$\|u(t) f_0\|_{L^\infty_{xv}(m)} \le C\, \frac{e^{-t}}{t^d}\, \|f_0\|_{L^1_x L^\infty_v(m)}.$$

Question 5. Establish that there exists some constants $n \ge 1$ and $C \in [1,\infty)$ such that

$$\|u^{(*n)}(t)\|_{L^1_{xv}(m) \to L^\infty_{xv}(m)} \le C\, e^{-t/2}.$$

Deduce that $S_\mathcal{L}$ is bounded in $L^\infty_{xv}(m)$.

Question 6. How to prove that $S_\mathcal{L}$ is bounded in $L^2_{xv}(m)$ in a similar way? How to shorten the proof of that last result by using question (1b)? Same question for the space $L^\infty_{xv}(m)$.