1 Problem I - Local in time estimates

Consider a smooth and fast decaying initial datum f_0, the associated solution $f = f(t, x)$, $t \ge 0$, $x \in \mathbb{R}^d$, to heat equation

$$\partial_t f = \frac{1}{2}\Delta f, \quad f(0, .) = f_0,$$

and for a given $\alpha \in \mathbb{R}^d$, define

$$g := f\, e^{\psi}, \quad \psi(x) := \alpha \cdot x.$$

(1) Establish that

$$\partial_t g = \frac{1}{2}\Delta g - \alpha \cdot \nabla g + \frac{1}{2}|\alpha|^2 g.$$

(2) Establish that $\|g(t, .)\|_{L^1} \le e^{\alpha^2 t/2}\, \|g_0\|_{L^1}$ for any $t \ge 0$.
(3) Establish that

$$\|g(t)\|^2_{L^2}\, e^{-\alpha^2 t} \le \frac{\|g_0\|^2_{L^1}}{(2/d\, C_N\, t)^{d/2}}, \quad \forall\, t > 0.$$

(4) Denoting by $T(t)$ the semigroup associated to the parabolic equation satisfies by g, prove successively that

$$T(t) : L^1 \to L^2, \quad L^2 \to L^\infty, \quad L^1 \to L^\infty,$$

for some constants $Ct^{-d/4}e^{\alpha^2 t/2}$, $Ct^{-d/4}e^{\alpha^2 t/2}$ and $Ct^{-d/2}e^{\alpha^2 t/2}$.
(5) Denoting by S the heat semigroup and by $F(t, x, y) := (S(t)\delta_x)(y)$ the fundamental solution associated to the heat equation when starting from the Dirac function in $x \in \mathbb{R}^d$, deduce

$$F(t, x, y) \le \frac{C}{t^{d/2}}\, e^{\alpha\cdot(x-y)+\alpha^2 t/2}, \quad \forall\, t > 0, \forall\, x, y, \alpha \in \mathbb{R}^d,$$

and then

$$F(t, x, y) \le \frac{C}{t^{d/2}}\, e^{-\frac{|x-y|^2}{2t}}, \quad \forall\, t > 0, \forall\, x, y \in \mathbb{R}^d.$$

(6) May we prove a similar result for the parabolic equation

$$\partial_t f = \mathrm{div}_x(A(x)\nabla_x f), \quad 0 < \nu \le A \in L^\infty\, ?$$

2 Problem II - Harris estimate

In a first part, we condider a Markov semigroup $S = S_{\mathcal{L}}$ on $L^1(\mathbb{R}^d)$ which fullfils

(H1) there exists some weight function $m : \mathbb{R}^d \to [1,\infty)$ which is increasing and satisfies $m(x) \to \infty$ as $x \to \infty$ and there exist some constants $\alpha > 0, b > 0$ such that

$$\mathcal{L}^* m \le -\alpha\, m + b;$$

(H2) there exists $T > 0$ and for any $R > 0$ there exists a positive and not zero measure $\nu = \nu_{T,R}$ such that

$$S_T f \ge \nu \int_{B_R} f, \quad \forall\, f \in L^1(\mathbb{R}^d),\ f \ge 0.$$

In the sequel, we fix $f_0 \in L^1(m)$ such that $\langle f_0 \rangle = 0$ and we denote $f_t := S_t f_0$.

(1) Prove

$$\|S_T f_0\|_{L^1} \le \|f_0\|_{L^1}.$$

(2) Prove

$$\frac{d}{dt}\|f_t\|_{L^1(m)} \le -\alpha \|f_t\|_{L^1(m)} + b\|f_t\|_{L^1},$$

and deduce

$$\|S_T f_0\|_{L^1(m)} \le \gamma \|f_0\|_{L^1(m)} + K\|f_0\|_{L^1},$$

with $\gamma \in (0,1)$ and $K > 0$.

We fix $R > 0$ large enough such that $K/A \le (1-\gamma)/2$ with $A := m(R)/4$, and we observe that the following altenative holds

$$\|f_0\|_{L^1(m)} > A\|f_0\|_{L^1} \tag{2.1}$$

or

$$\|f_0\|_{L^1(m)} \le A\|f_0\|_{L^1}. \tag{2.2}$$

(3) Prove that if (2.1) holds, then

$$\|S_T f_0\|_{L^1(m)} \le \gamma_1 \|f_0\|_{L^1(m)},$$

with $\gamma_1 \in (0,1)$. (For instance, $\gamma_1 := (1+\gamma)/2$ is suitable).

(4) We introduce

$$\|f\|_\beta := \|f\|_{L^1} + \beta\|f\|_{L^1(m)}, \quad \beta > 0.$$

Prove that if (2.1) holds, then

$$\|S_T f_0\|_\beta \le \gamma_2 \|f_0\|_\beta,$$

with $\gamma_2 \in (0,1)$ and for any $\beta > 0$. (For instance, $\gamma_2 := (\beta\gamma_1 + 1)/(\beta+1)$ is suitable).

(5) Prove that if (2.2) holds, then

$$\int_{B_R} f_{0\pm} \ge \frac{1}{4}\int |f_0|,$$

and next

$$\|S_T f_0\|_{L^1} \le (1 - \frac{\langle \nu \rangle}{2}) \|f_0\|_{L^1},$$

where $\nu = \nu_{T,R}$ is defined in hypothesis (H2).
(6) Prove that if (2.2) holds, then

$$\|S_T f_0\|_\beta \le \gamma_3 \|f_0\|_\beta,$$

with $\gamma_3 \in (0,1)$ and for $\beta > 0$ small enough.
(7) Conclude that there exist some constants $C \ge 1$ and $a < 0$ such that

$$\|S_t f_0\|_{L^1(m)} \le C\, e^{at} \|f_0\|_{L^1(m)}, \quad \forall\, t \ge 0,$$

for any $f_0 \in L^1(m)$, $\langle f \rangle = 0$.

We consider now the equation

$$\partial_t f = \mathcal{L} f := -\partial_x f - f + G \rho_f$$

on the function $f = f(t,x)$, $t > 0$, $x \in \mathbb{R}$, where we denote

$$G = \frac{e^{-x^2/2}}{(2\pi)^{1/2}}, \quad \rho_f := \int_{\mathbb{R}} f(y)\, dy.$$

(8) Prove that $\mathcal{L}$ is the generator of a semigroup in $L^p(m)$ for any exponent $p \in [1,\infty]$ and weight function $m : \mathbb{R} \to [1,\infty)$ such that $L^p(m) \subset L^1$.
(9) Prove that $S_{\mathcal{L}}$ satisfies (H1) with $m(x) = 1 + x^2$.
(10) We define $\mathcal{B} f := -\partial_x f - f$. Prove that $[S_{\mathcal{B}}(t) f_0](x) = f_0(x-t) e^{-t}$. Deduce that $S_{\mathcal{L}}$ satisfies (H2) for any $T > 0$ (and any $R > 0$!).

3 Problem III - the Keller-Segel equation with supercritical mass

We consider the Keller-Segel equation

$$\partial_t f = \Delta f + \mathrm{div}(\mathcal{K}_f f), \tag{3.1}$$

on the unknown $f = f(t,x)$, $t \ge 0$, $x \in \mathbb{R}^2$, with

$$\mathcal{K}_f := \mathcal{K} * f, \quad \mathcal{K} := \nabla \kappa = \frac{1}{2\pi} \frac{z}{|z|^2}.$$

We complement the equation with an initial condition

$$f(0,x) = f_0(x)$$

with $0 \le f_0 \in L^1_2(\mathbb{R}^2)$, $f_0 \log f_0 \in L^1(\mathbb{R}^2)$. We denote $M := \langle f_0 \rangle$ the initial mass.

(1) We define the relative entropy

$$S(f) := \int_{\mathbb{R}^2} (f \,\log(f/\mathscr{M}) - f + \mathscr{M})\, dx,$$

where $\mathscr{M} := M\,(2\pi)^{-1} \exp(-|x|^2/2)$, and the Fisher information

$$I(f) := \int_{\mathbb{R}^2} \frac{|\nabla f|^2}{f}\, dx.$$

Prove that any solution f to the Keller-Segel equation (3.1) formally satisfies

$$\frac{d}{dt} S(f(t)) \;\; \le \;\; -I(f(t)) + M \exp\{C_1 S(f(t))\} + C_2,$$

for some constants $C_1, C_2 > 0$ which only depend on the mass M.

(2) Deduce an a priori estimate

$$\sup_{(0,T)} S(f(t)) + \int_0^T I(f(t))\, dt \le C_T < \infty,$$

for some small enough final time $T > 0$, where C_T only depends on M, $S(f_0)$ and T, whatever is the value of $M \in \mathbb{R}_+^*$.

(3) Prove the existence of a positive and mass conservative solution f to the Keller-Segel equation (3.1) on the interval of time $(0, T)$.

(4) We define $T^* > 0$ as the maximal time of existence of a positive and mass conservative solution f to the Keller-Segel equation. Prove that when $M > 8\pi$, there hold

$$T^* < \infty \quad \text{and} \quad S(f(t)) \nearrow +\infty \text{ when } t \nearrow T^*.$$