1 Problem I - Local in time estimates

Consider a weak solution $0 \le f \in C([0,T]; L^1(\mathbb{R}^2))$ to the Keller-Segel equation, that is

$$\partial_t f := \Delta f + \mathrm{div}(\bar{\mathcal{K}} f),$$

with

$$\bar{\mathcal{K}} := \mathcal{K} * f, \quad \mathcal{K} := \nabla \kappa, \quad \kappa := \frac{1}{2\pi} \log |z|,$$

such that

$$\mathcal{A}_T := \sup_{[0,T]} \int_{\mathbb{R}^2} f(1 + |x|^2 + (\log f)_+)\, dx + \int_0^T I(f)\, dt < \infty.$$

(1) Establish that $f^2, |\nabla_x \bar{\mathcal{K}}|\, f \in L^1((0,T) \times \mathbb{R}^2)$ and that for any mollifier sequence (ρ_n), the function $f^n := f *_x \rho_n$ satisfies

$$\partial_t f^n - \bar{\mathcal{K}} \cdot \nabla_x f^n - \Delta_x f^n = r^n \to f^2 \quad \text{in} \quad L^1(0,T; L^1_{loc}(\mathbb{R}^2)).$$

(2) Deduce that

$$\int_{\mathbb{R}^2} \beta(f^n_{t_1})\, \chi\, dx + \int_{t_0}^{t_1} \int_{\mathbb{R}^2} \beta''(f^n_s)\, |\nabla_x f^n_s|^2\, \chi\, dxds = \int_{\mathbb{R}^2} \beta(f^n_{t_0})\, \chi\, dx$$
$$+ \int_{t_0}^{t_1} \int_{\mathbb{R}^2} \Big\{ \beta'(f^n_s)\, r^n\, \chi + \beta(f^n_s)\, \Delta\chi - \beta(f^n_s)\, \mathrm{div}_x(\bar{\mathcal{K}}\chi) \Big\}\, dxds,$$

for all $0 \le t_0 < t_1 \le T$, $0 \le \chi \in C^2_c(\mathbb{R}^2)$ and $\beta \in C^1(\mathbb{R}) \cap W^{2,\infty}_{loc}(\mathbb{R})$ such that β'' is non-negative and vanishes outside of a compact set, and next that

$$\int_{\mathbb{R}^2} \beta(f_{t_1})\, dx + \int_{t_0}^{t_1} \int_{\mathbb{R}^2} \beta''(f_s)\, |\nabla f_s|^2\, dxds \tag{1.1}$$
$$\le \int_{\mathbb{R}^2} \beta(f_{t_0})\, dx + \int_{t_0}^{t_1} \int_{\mathbb{R}^2} (\beta'(f_s)\, f_s^2 - \beta(f_s)\, f_s)_+\, dxds.$$

(3) Deduce that the same inequality holds for any renormalizing function $\beta : \mathbb{R} \to \mathbb{R}$ which is convex, piecewise of class C^1 and such that

$$|\beta(u)| \le C\,(1 + u\,(\log u)_+), \quad (\beta'(u)\, u^2 - \beta(u)\, u)_+ \le C\,(1 + u^2) \quad \forall\, u \in \mathbb{R}.$$

(4) We define the renormalizing function $\beta_K : \mathbb{R}_+ \to \mathbb{R}_+$, $K \ge e^2$, by

$$\beta_K(u) := u\,(\widetilde{\log}\,u)^2 \ \text{ if } \ u \le K, \qquad \beta_K(u) := (2+\log K)\,u\log u - 2K\log K \ \text{ if } \ u \ge K,$$

and the function $\widetilde{\log}_K$ by

$$\widetilde{\log}_K\,u := \mathbf{1}_{u\le e} + (\log u)\mathbf{1}_{e<u\le K} + (\log K)\mathbf{1}_{u>K}.$$

Deduce from (1.1) that

$$\begin{aligned}\int_{\mathbb{R}^2} \beta_K(f_{t_1})\,dx + \int_{t_0}^{t_1}\int_{\mathbb{R}^2} \frac{|\nabla f|^2}{f}\,(\widetilde{\log}_K f)\,\mathbf{1}_{f\ge e}\,dxds \\ \le \int_{\mathbb{R}^2} \beta_K(f_{t_0})\,dx + 4\int_{t_0}^{t_1}\int_{\mathbb{R}^2} f^2\,\widetilde{\log}_K f\,dxds.\end{aligned} \tag{1.2}$$

Establish that for any $A \in (e,K)$

$$\int_{\mathbb{R}^2} f^2\,\widetilde{\log}_K f\,dx \quad\le\quad (A\log A)\,M + \Big(\frac{\mathcal{H}^+(f)}{\log A}\Big)^{1/2}\Big(\int_{\mathbb{R}^2}(f\,\widetilde{\log}_K f)^3\,dx\Big)^{1/2},$$

as well as

$$\Big(\int_{\mathbb{R}^2}(f\,\widetilde{\log}_K f)^3\,dx\Big)^{1/2} \quad\le\quad C\,\Big(M+\mathcal{H}^+(f)\Big)^{1/2}\Big(\int_{\mathbb{R}^2}\frac{|\nabla f|^2}{f}\,(\widetilde{\log}_K f)\,\mathbf{1}_{f\ge e}\,dx + I(f)\Big).$$

By choosing $A > 0$ large enough, conclude that there exists $C := C(\mathcal{A}_T)$ such that

$$\mathcal{H}_2(f(t_1)) \le \mathcal{H}_2(f(t_0)) + C, \tag{1.3}$$

with

$$\mathcal{H}_2(g) := \int_{\mathbb{R}^2} f\,(\widetilde{\log} g)^2\,dx, \quad \widetilde{\log}\,u := \mathbf{1}_{u\le e} + (\log u)\mathbf{1}_{u>e}.$$

(5) We define the new renormalizing function $\beta_K : \mathbb{R}_+ \to \mathbb{R}_+$, $K \ge 2$, by

$$\beta_K(u) := \frac{u^p}{p} \ \text{ if } \ u \le K, \quad \beta_K(u) := \frac{K^{p-1}}{\log K}(u\,\log u - u) - \frac{1}{p'}K^p + \frac{K^p}{\log K} \ \text{ if } \ u \ge K.$$

Prove that

$$\begin{aligned}\int_{\mathbb{R}^2} \beta_K(f_{t_1})\,dx + \frac{4}{p'p}\int_{t_0}^{t_1}\int_{\mathbb{R}^2} |\nabla(f^{p/2})|^2\,\mathbf{1}_{f\le K}\,dxds + \frac{K^{p-1}}{\log K}\int_{t_0}^{t_1}\int_{\mathbb{R}^2}\frac{|\nabla f|^2}{f}\,\mathbf{1}_{f\ge K}\,dxds \\ \le \int_{\mathbb{R}^2}\beta_K(f_{t_0})\,dx + \frac{1}{p'}\int_{t_0}^{t_1}\mathcal{T}_1\,ds + 2\,K^{p-1}\int_{t_0}^{t_1}\mathcal{T}_2\,ds,\end{aligned}$$

with

$$\mathcal{T}_1 := \int_{\mathbb{R}^2} f^{p+1}\,\mathbf{1}_{f\le K}\,dx \ \text{ and } \ \mathcal{T}_2 := \int_{\mathbb{R}^2} f^2\,\mathbf{1}_{f\ge K}\,dx.$$

On the one hand, prove that

$$\frac{1}{p'}\int_{t_0}^{t_1} \mathcal{T}_1\, ds \le \frac{2^p}{p'}\, A^p\, M\, T + \frac{1}{p'p}\int_{t_0}^{t_1}\int_{\mathbb{R}^2} |\nabla_x (f^{p/2})|^2\, \mathbf{1}_{f\le K}\, dxds,$$

for $A = A(p, \mathcal{A}_T) > 1$ large enough.
On the other hand, prove that

$$\mathcal{T}_2 \le 8\, K^{p-1} \Big(\int_{\mathbb{R}^2} |\nabla f|\, \mathbf{1}_{f\ge K/2}\, dx\Big)^2$$

and next that

$$\mathcal{T}_2 \le 8\, K^{p-1} \Big\{ \frac{4}{p^2}\int_{\mathbb{R}^2} |\nabla (f^{p/2})|^2\, (\frac{2}{K})^{p-1}\, \mathbf{1}_{f\le K} + \int_{\mathbb{R}^2} \frac{|\nabla f|^2}{f}\, \mathbf{1}_{f\ge K}\Big\} \frac{\mathcal{H}_2(f)}{(\log(K/2))^2}.$$

Finally deduce from (1.3) that

$$\int_{t_0}^{t_1} \mathcal{T}_2\, ds \;\; \le \;\; \frac{1}{p'p}\int_{t_0}^{t_1}\int_{\mathbb{R}^2} |\nabla (f^{p/2})|^2\, \mathbf{1}_{f\le K}\, dxds + \frac{K^{p-1}}{\log K}\int_{t_0}^{t_1}\int_{\mathbb{R}^2} \frac{|\nabla f|^2}{f}\, \mathbf{1}_{f\ge K}\, dxds,$$

for any $K \ge K^* = K^*(p, \mathcal{A}_T) > \max(A, 4)$ large enough.
(6) Deduce that any exponent $p \in (1,\infty)$ and for any time $t_0 \in (0,T)$, f satisfies

$$f \in L^\infty(t_0, T; L^p(\mathbb{R}^2)) \quad \text{and} \quad \nabla_x f \in L^2((t_0,T)\times\mathbb{R}^2).$$

2 Exercise II - Decay estimates

(1) Prove that a function $u \in C^1(\mathbb{R})$ which satisfies the differential inequality

$$u' \le -\lambda\, u, \quad \lambda > 0,$$

also satisfies the integral inequality

$$\forall t \ge 0, \quad \lambda \int_t^\infty u(s)\, ds \le u(t). \tag{2.1}$$

Prove that if $u : \mathbb{R}_+ \to \mathbb{R}_+$ is decreasing and satisfies (2.1), there exists $C = C(\lambda, u(0)) > 0$ such that

$$\forall t \ge 0, \qquad u(t) \le C\, e^{-\lambda t}.$$

(Hint. Introduce the function $v(t) := \int_t^\infty u(s)\, ds$).
(2) Prove that a function $u \in C^1(\mathbb{R})$ which satisfies the differential inequality

$$u' \le -K\, u^{1+\alpha}, \quad \alpha > 0,\ K > 0,$$

also satisfies the integral inequality

$$\forall t \ge 0, \quad K \int_t^\infty u^{1+\alpha}(s)\, ds \le u(t). \tag{2.2}$$

Prove that if $u : \mathbb{R}_+ \to \mathbb{R}_+$ is decreasing and satisfies (2.2), there exists $C = C(\alpha, u(0)) > 0$ such that

$$\forall t \ge 0, \qquad u(t) \le C\, \frac{1}{t^{1/\alpha}}.$$

3 Problem III - Subgeometric Harris estimate

In this part, we consider a Markov semigroup $S = S_{\mathcal{L}}$ on $L^1(\mathbb{R}^d)$ which fulfills
(H1) there exist some weight functions $m_i : \mathbb{R}^d \to [1, \infty)$ satisfying $m_1 \geq m_0$, $m_0(x) \to \infty$ as $x \to \infty$ and there exists constant $b > 0$ such that

$$\mathcal{L}^* m_1 \leq -m_0 + b;$$

(H2) there exists a constant $T > 0$ and for any $R \geq R_0 \geq 0$ there exists a positive and not zero measure ν such that

$$S_T f \geq \nu \int_{B_R} f, \quad \forall\, f \in L^1,\ f \geq 0;$$

(H3) there exists $m_2 \geq m_1$ such that and for any $\lambda > 0$ there exists ξ_λ such that

$$m_1 \leq \lambda m_0 + \xi_\lambda m_2, \quad \xi_\lambda \to 0 \ \text{ as } \lambda \to \infty.$$

(H4) We also assume that

$$\sup_{t \geq 0} \|S_t f\|_{L^1(m_i)} \leq M_i \|f\|_{L^1(m_i)}, \quad M_i \geq 1,\ i = 1, 2.$$

(1) Prove

$$\|S_T f_0\|_{L^1} \leq \|f_0\|_{L^1}, \quad \forall\, T > 0,\ \forall\, f_0 \in L^1.$$

In the sequel, we fix $f_0 \in L^1(m_2)$ such that $\langle f_0 \rangle = 0$ and we denote $f_t := S_t f_0$.
(2) Prove that

$$\frac{d}{dt} \|f_t\|_{L^1(m_1)} \leq -\|f_t\|_{L^1(m_0)} + b \|f_t\|_{L^1},$$

and deduce that

$$\|S_T f_0\|_{L^1(m_1)} + \frac{T}{M_0} \|S_T f_0\|_{L^1(m_0)} \leq \|f_0\|_{L^1(m_1)} + K \|f_0\|_{L^1}.$$

We define

$$\|f\|_\beta := \|f\|_{L^1} + \beta \|f\|_{L^1(m_1)}, \quad \beta > 0.$$

We fix $R \geq R_0$ large enough such that $A := m(R)/4 \geq 3M_0/T$, and we observe that the following alternative holds

$$\|f_0\|_{L^1(m_0)} \leq A \|f_0\|_{L^1} \tag{3.1}$$

or

$$\|f_0\|_{L^1(m_0)} > A \|f_0\|_{L^1}. \tag{3.2}$$

(3) We assume that condition (3.1) holds. Prove that

$$\|S_T f_0\|_{L^1} \leq \gamma_1 \|f_0\|_{L^1},$$

with $\gamma_1 \in (0,1)$. Deduce that

$$\|S_T f_0\|_\beta \le \gamma_1 \|f_0\|_{L^1} - \frac{\beta T}{M_0}\|S_T f_0\|_{L^1(m_0)} + \beta \|f_0\|_{L^1(m_1)} + \beta K \|f_0\|_{L^1}$$

and next

$$\|S_T f_0\|_\beta + \frac{\beta T}{M_0}\|S_T f_0\|_{L^1(m_0)} \le \|f_0\|_\beta,$$

for $\beta > 0$ small enough.

(4) We assume that condition (3.2) holds. Prove that

$$\|S_T f_0\|_{L^1(m_1)} + \frac{T}{M_0}\|S_T f_0\|_{L^1(m_0)} \le \|f_0\|_{L^1(m_1)} + \frac{T}{3M_0}\|f_0\|_{L^1(m_0)},$$

and deduce

$$\|S_T f_0\|_\beta + \frac{\beta T}{M_0}\|S_T f_0\|_{L^1(m_0)} \le \|f_0\|_\beta + \frac{\beta T}{3M_0}\|f_0\|_{L^1(m_0)}.$$

(5) Observe that in both cases (3.1) and (3.2), there holds

$$\|S_T f_0\|_\beta + 3\alpha \|S_T f_0\|_{L^1(m_0)} \le \|f_0\|_\beta + \alpha \|f_0\|_{L^1(m_0)},$$

where from now β and α are fixed constants. Deduce that

$$Z(u_1 + \alpha v_1) \le u_0 + \alpha v_0 + \frac{\xi_\lambda}{\lambda}\alpha w_1,$$

with

$$u_n := \|S_{nT} f_0\|_\beta, \quad v_n := \|S_{nT} f_0\|_{L^1(m_0)}, \quad w_n := \|S_{nT} f_0\|_{L^1(m_2)}$$

and for $\lambda \ge \lambda_0 \ge 1$ large enough

$$Z := 1 + \frac{\delta}{\lambda} \le 2, \quad \delta := \frac{\alpha}{1+\beta}.$$

Deduce that for any $n \ge 1$, there holds

$$u_n \le Z^{-n}(u_0 + \alpha v_0) + \frac{Z}{Z-1}\frac{\xi_\lambda \alpha}{\lambda}\sup_{i \ge 1} w_i,$$

and next

$$\|S_{nT} f_0\|_\beta \le \Big(e^{-\frac{nT}{\lambda}\frac{\delta}{2T}} + \xi_\lambda\Big) C \|f_0\|_{L^1(m_2)}, \quad \forall\, \lambda \ge \lambda_0.$$

(6) Prove that

$$\|S_t f_0\|_{L^1} \le \Theta(t) \|f_0\|_{L^1(m_2)}, \quad \forall\, t \ge 0,\ \forall\, f_0 \in L^1(m_2),\ \langle f \rangle = 0,$$

for the function Θ given by

$$\Theta(t) := C \inf_{\lambda > 0}\{e^{-\kappa t/\lambda} + \xi_\lambda\}.$$

What is the value of Θ when $m_0 = 1$, $m_1 = \langle x \rangle$, $m_2 = \langle x \rangle^2$?

4 Problem IV - An application to the Fokker-Planck equation

In all the problem, we consider the Fokker-Planck equation

$$\partial_t f = \mathcal{L} f := \Delta_x f + \mathrm{div}_x(f\, E) \quad \text{in } (0,\infty)\times\mathbb{R}^d \tag{4.1}$$

for the confinement potential $E := \nabla\phi$, $\phi := \langle x\rangle^\gamma/\gamma$, $\langle x\rangle^2 := 1 + |x|^2$, that we complement with an initial condition

$$f(0,x) = f_0(x) \quad \text{in } \mathbb{R}^d. \tag{4.2}$$

Question 1

Give a strategy in order to build solutions to (4.1) when $f_0 \in L^p_k(\mathbb{R}^d)$, $p \in [1,\infty]$, $k \geq 0$.

We assume from now on that $f_0 \in L^1_k(\mathbb{R}^d)$, $k > 0$, and that we are able to build a unique weak (and renormalized) solution $f \in C([0,\infty); L^1_k)$ to equation (4.1)-(4.2).
We also assume that $\gamma \geq 2$.

Question 2

Prove

$$\langle f(t)\rangle = \langle f_0\rangle \quad \text{and} \quad f(t,.) \geq 0 \text{ if } f_0 \geq 0.$$

Question 3

Prove that there exist $\alpha > 0$ and $K \geq 0$ such that

$$\mathcal{L}^* \langle x\rangle^k \leq -\alpha \langle x\rangle^k + K,$$

and deduce that

$$\sup_{t\geq 0} \|f(t,\cdot)\|_{L^1_k} \leq C_1 \|f_0\|_{L^1_k}.$$

(Hint. A possible constant is $C_1 := \max(1, K/\alpha)$).

Question 4

Prove that

$$\sup_{t\geq 0} \|f(t,\cdot)\|_{L^2_k} \leq C_2 \|f_0\|_{L^2_k},$$

at least for $k > 0$ large enough.

Question 5

Prove that

$$\sup_{t\geq 0} \|f(t,\cdot)\|_{H^1_k} \leq C_3 \|f_0\|_{H^1_k},$$

at least for $k > 0$ large enough.

Question 6

Prove that

$$\|f(t,\cdot)\|_{H^1} \leq \frac{C_4}{t^\alpha}\|f_0\|_{L^1_k},$$

at least for $k > 0$ large enough and for some constant $\alpha > 0$ to be specified.
(Hint. Consider the functional $\mathcal{F}(t) := \|f(t)\|_{L^1_k} + t^\alpha \|\nabla_x f(t)\|^2_{L^2}$).

We assume from now on that $d = 1$, so that $C^{0,1/2} \subset H^1$.

Question 7

We fix $f_0 \in L^1$ such that $f_0 \geq 0$ and $\operatorname{supp} f_0 \subset B_R$, $R > 0$. Using question 3, prove that

$$\int_{B_\rho} f(t) \geq \frac{1}{2}\int_{B_R} f_0,$$

for any $t \geq 0$ by choosing $\rho > 0$ large enough. Using question 6, prove that there exist $r, \kappa > 0$ and for any $t > 0$ there exists $x_0 \in B_R$ such that

$$f(t) \geq \kappa, \quad \forall\, x \in B(x_0, r).$$

We accept the spreading of the positivity property, namely that for ant $r_0, r_1 > 0$, $x_0 \in \mathbb{R}^d$, there exist $t_1, \kappa_1 > 0$ such that

$$f_0 \geq \mathbf{1}_{B(x_0,r_0)} \quad \Rightarrow \quad f(t_1,\cdot) \geq \kappa_1 \mathbf{1}_{B(x_0,r_1)}.$$

Deduce that there exist $\theta > 0$ and $T > 0$ such that

$$f(T,\cdot) \geq \theta \mathbf{1}_{B(0,R)} \int_{B_R} f_0 \, dx.$$

Question 8

Prove that for any $k > 0$, there exists $C, \lambda > 0$ such that $f_0 \in L^1_k$ satisfying $\langle f_0 \rangle = 0$, there holds

$$\forall\, t > 0, \qquad \|f(t,.)\|_{L^1_k} \leq C e^{-\lambda t}\|f_0\|_{L^1_k}.$$

Question 9

We assume now $\gamma \in (0, 2)$. We define

$$\mathcal{B} f := \mathcal{L} f - M \chi_R f$$

with $\chi_R(x) := \chi(x/R)$, $\chi \in \mathcal{D}(\mathbb{R}^d)$, $0 \leq \chi \leq 1$, $\chi(x) = 1$ for any $|x| \leq 1$, and with $M, R > 0$ to be fixed.
We denote by $f_{\mathcal{B}}(t) = S_{\mathcal{B}}(t) f_0$ the solution associated to the evolution PDE corresponding to the operator $\mathcal{B}$ and the initial condition f_0.
(1) Why such a solution is well defined (no more than one sentence of explanation)?

(2) Prove that there exists $M, R > 0$ such that for any $k \geq 0$ there holds

$$\frac{d}{dt} \int_{\mathbb{R}^d} f_{\mathcal{B}}(t) \langle x \rangle^k \, dx \leq -c_k \int_{\mathbb{R}^d} f_{\mathcal{B}}(t) \langle x \rangle^{k+\gamma-2} \leq 0,$$

for some constant $c_k \geq 0$, $c_k > 0$ if $k > 0$, and

$$\|S_{\mathcal{B}}(t)\|_{L^1_k \to L^1_k} \leq 1.$$

(3) Prove that for any $k_1 < k < k_2$ there exists $\theta \in (0, 1)$ such that

$$\forall \, f \geq 0 \quad M_k \leq M_{k_1}^{\theta} \, M_{k_2}^{1-\theta}, \quad M_\ell := \int_{\mathbb{R}^d} f(x) \, \langle x \rangle^\ell \, dx$$

and write θ as a function of k_1, k and k_2.
(4) Prove that if $\ell > k > 0$ there exists $\alpha > 0$ such that

$$\|S_{\mathcal{B}}(t)\|_{L^1_\ell \to L^1} \leq \|S_{\mathcal{B}}(t)\|_{L^1_\ell \to L^1_k} \leq C / \langle t \rangle^\alpha,$$

and that $\alpha > 1$ if ℓ is large enough (to be specified).
(6) Prove that

$$S_{\mathcal{L}} = S_{\mathcal{B}} + S_{\mathcal{B}} * (\mathcal{A} S_{\mathcal{L}}),$$

and deduce that for k large enough (to be specified)

$$\|S_{\mathcal{L}}\|_{L^1_k \to L^1_k} \leq C.$$

Question 10

Still in the case $\gamma \in (0, 2)$, what can we say about the decay of

$$\|f(t, .)\|_{L^1}$$

when $f_0 \in L^1_k$, $k > 0$, satisfies $\langle f_0 \rangle = 0$?