**Université Paris-Dauphine** **2019-2020**
An introduction to evolution PDEs December 2019

**On the Landau equation**

We aim to estbalish the existence of solutions to the Landau equation

$$\partial_t f(t,v) = Q(f,f)(t,v), \quad f(0,v) = f_0(v), \tag{0.1}$$

on the density function $f = f(t,v) \ge 0$, $t \ge 0$, $v \in \mathbb{R}^d$, $d \ge 2$, where the Landau kernel is defined by the formula

$$Q(f,f)(v) := \frac{\partial}{\partial v_i}\Big\{\int_{\mathbb{R}^d} a_{ij}(v-v_*)\Big(f(v_*)\frac{\partial f}{\partial v_j}(v) - f(v)\frac{\partial f}{\partial v_j}(v_*)\Big)\,dv_*\Big\}.$$

Here and the sequel we use Einstein's convention of sommation of repeated indices. The matrix $a = (a_{ij})$ is defined by

$$a(z) = |z|^2\Pi(z), \quad \Pi_{ij}(z) := \delta_{ij} - \hat z_i \hat z_j, \quad \hat z_k := \frac{z_k}{|z|},$$

so that $\Pi$ is the is the orthogonal projection on the hyperplan $z^\perp := \{y \in \mathbb{R}^d;\ y\cdot z = 0\}$.

# 1 Part I - Physical properties and a priori estimates.

(1) Observe that $a(z)z = 0$ for any $z \in \mathbb{R}^d$ and $a(z)\xi\xi \ge 0$ for any $z,\xi \in \mathbb{R}^d$. Here and below, we use the bilinear form notation $auv = {}^t vau = v\cdot au$. In particular, the symetric matrix $a$ is positive but not strictly positive.

(2) For any nice functions $f,\varphi : \mathbb{R}^d \to \mathbb{R}$, $f \ge 0$, prove that

$$\int Q(f,f)\varphi\,dv = \frac12 \iint a(v-v_*)(f\nabla_* f_* - f_*\nabla f)(\nabla\varphi - \nabla_*\varphi_*)\,dvdv_*,$$

where $f_* = f(v_*)$, $\nabla_*\psi_* = (\nabla\psi)(v_*)$. Deduce that

$$\int Q(f,f)\varphi\,dv = 0, \quad \text{for } \varphi = 1, v_i, |v|^2,$$

and

$$-D(f) := \int Q(f,f)\log f\,dv \le 0.$$

Establish then

$$\Big|\int Q(f,f)\varphi\,dv\Big| \le D(f)^{1/2}\Big(\frac12\iint ff_* a(v-v_*)(\nabla\varphi - \nabla_*\varphi_*)(\nabla\varphi-\nabla_*\varphi_*)\,dvdv_*\Big)^{1/2}.$$

(3) For $H_0 \in \mathbb{R}$, we define $\mathcal{E}_{H_0}$ the set of functions

$$\begin{aligned} \mathcal{E}_{H_0} \;&:=\; \Big\{ f \in L^1_2(\mathbb{R}^d);\ f \ge 0,\ \int f\,dv = 1,\ \int f\,v\,dv = 0, \\ &\qquad \int f\,|v|^2\,dv \le d,\ H(f) := \int f\,\log f\,dv \le H_0 \Big\}. \end{aligned}$$

Prove that there exists a constant $C_0$ such that

$$H_-(f) := \int f(\log f)_- dv \le C_0, \quad \forall\, f \in \mathcal{E}_{H_0},$$

and define $D_0 := H_0 + C_0$. Deduce that for any nice positive solution $f$ to the Landau equation such that $f_0 \in \mathcal{E}_{H_0}$, there holds

$$f \in \mathcal{F}_T := \Big\{ g \in C([0,T]; L^1_2);\ g(t) \in \mathcal{E}_{H_0},\ \forall\, t \in (0,T),\ \int_0^T D(g(t))\,dt \le D_0 \Big\}.$$

We say that $f \in C([0,T); L^1)$ is a weak solution to the Landau equation if $f \in \mathcal{F}_T$ and (0.1) holds in the distributional sense. Why the definition is meaningful?

(4) Prove that

$$Q(f,f) = \partial_i(\bar a_{ij}\partial_j f - \bar b_i f) = \partial^2_{ij}(\bar a_{ij} f) - 2\partial_i(\bar b_i f) = \bar a_{ij}\partial^2_{ij} f - \bar c f,$$

with

$$\bar a_{ij} = \bar a^f_{ij} := a_{ij} * f, \quad \bar b_i = \bar b^f_i := b_i * f, \quad \bar c = \bar c^f := c * f, \tag{1.2}$$

and

$$b_i := \sum_{j=1}^d \partial_j a_{ij} = -(d-1) z_i, \quad c := \sum_{i=1}^d \partial_i b_i = -(d-1)d.$$

Prove that there existe $C \in (0,\infty)$ such that

$$|\bar a_{ij}| \le C(1+|v|^2), \quad |\bar b_i| \le C(1+|v|),$$

## 2 Part II - On the ellipticity of $\bar a$.

We fix $H_0 \in \mathbb{R}$ and $f \in \mathcal{E}_{H_0}$.

(5a) Show that there exists a function $\eta \ge 0$ (only depending of $D_0$) such that

$$\forall\, A \subset \mathbb{R}^d, \quad \int_A f\,dv \le \eta(|A|)$$

and $\eta(r) \to 0$ when $r \to 0$. Here $|A|$ denotes the Lebesgue measure of $A$. Deduce that

$$\forall\, R, \varepsilon > 0, \quad \int f\,\mathbf{1}_{|v|\le R}\,\mathbf{1}_{|v_i|\le\varepsilon}\,dv \le \eta_R(\varepsilon)$$

and $\eta_R(r) \to 0$ when $r \to 0$.
(5b) Show that

$$\int f \mathbf{1}_{|v|\le R} \ \ge 1 - \frac{d}{R^2}.$$

(5c) Deduce from the two previous questions that

$$\forall\, i = 1, \dots, d, \quad T_i := \int f v_i^2 dv \ge \lambda,$$

for some constant $\lambda > 0$ which only depends on $D_0$. Generalize the last estimate into

$$\forall\, \xi \in \mathbb{R}^d, \quad T(\xi) := \int f |v \cdot \xi|^2 dv \ge \lambda |\xi|^2.$$

(6) Deduce that

$$\forall\, v, \xi \in \mathbb{R}^d, \quad \bar{a}(v)\xi\xi := \sum_{i,j=1}^{d} \bar{a}_{ij}(v)\xi_i\xi_j \ge (d-1)\lambda\, |\xi|^2.$$

Prove that any weak solution formally satisfies

$$\frac{d}{dt}\, H(f) = -\int \bar{a}_{ij}\, \frac{\partial_i f \partial_j f}{f} - \int \bar{c} f,$$

and thus the following bound on the Fisher information

$$I(f) := \int \frac{|\nabla f|^2}{f} \in L^1(0,T)$$

## 3 Part III - Weak stability.

We consider here a sequence of weak solutions $(f_n)$ to the Landau equation such that $f_n \in \mathcal{F}_T$ for any $n \ge 1$.
(7) Prove that

$$\int_0^T \int |\nabla_v f_n|\, dv dt \le C_T$$

and that

$$\frac{d}{dt} \int f_n \varphi\, dv \text{ is bounded in } L^\infty(0,T), \quad \forall\, \varphi \in C_c^2(\mathbb{R}^d).$$

Deduce that $(f_n)$ belongs to a compact set of $L^1((0,T) \times \mathbb{R}^d)$. Up to the extraction of a subsequence, we then have

$$f_n \to f \text{ strongly in } L^1((0,T) \times \mathbb{R}^d).$$

Deduce that

$$Q(f_n, f_n) \rightharpoonup Q(f,f) \text{ weakly in } \mathcal{D}((0,T) \times \mathbb{R}^d)$$

and that $f$ is a weak solution to the Landau equation.
(8) (Difficult, here $d = 3$) Take $f \in \mathcal{E}_{H_0}$ with energy equals to $d$. Establish that $D(f) = 0$ if, and only if,

$$\frac{\nabla f}{f} - \frac{\nabla f_*}{f_*} = \lambda(v, v_*)(v - v_*), \quad \forall\, v, v_* \in \mathbb{R}^d,$$

for some scalar function $(v, v_*) \mapsto \lambda(v, v_*)$. Establish then that the last equation is equivalent to

$$\log f = \lambda_1 |v|^2/2 + \lambda_2 v + \lambda_3, \quad \forall\, v \in \mathbb{R}^d,$$

for some constants $\lambda_1 \in \mathbb{R}$, $\lambda_2 \in \mathbb{R}^d$, $\lambda_3 \in \mathbb{R}$. Conclude that

$$D(f) = 0 \text{ if, and only if, } f = M(v) := (2\pi)^{-3/2} \exp(-|v|^2/2).$$

(9) (very difficult, because needs many steps) Prove that for any global weak solution $f$ associated to $f_0 \in L^1_3 \cap \mathcal{E}_{H_0}$ with energy equals $d$, there holds $f(t) \rightharpoonup M$ when $t \to \infty$. (Hint. Accept that the energy $M_2(f(t)) = d$ and prove that the third moment $M_3(f(t))$ is uniformly bounded).

# 4 Part IV - Existence.

(10) We fix $k = d + 4$. Show that $\mathcal{H} := L^2_k \subset L^1_3$ and that $H_0 := H(f_0) \in \mathbb{R}$ if $0 \le f_0 \in L^2_k$. In the sequel, we first assume that $f_0 \in \mathcal{E}_{H_0} \cap \mathcal{H}$.
(11) For $f \in C([0, T]; \mathcal{E}_{H_0})$, we define $\bar a$, $\bar b$ and $\bar c$ thanks to (1.2) and then

$$\tilde a_{ij} := \bar a_{ij} + \varepsilon |v|^2 \delta_{ij}, \quad \tilde b_i := \bar b_i - \varepsilon \frac{d+2}{2} v_i, \quad \varepsilon \in (0, \lambda).$$

We define $\mathcal{V} := H^1_{k+2}$ and then

$$\forall\, g \in \mathcal{V}, \quad Lg := \partial_i(\tilde a_{ij} \partial_j g - \tilde b_i g) \in \mathcal{V}'.$$

Show that for some constant $C_i \in (0, \infty)$, there hold

$$(Lg, g)_{\mathcal{H}} \le -\varepsilon \|g\|^2_{\mathcal{V}} + C_1 \|g\|^2_{\mathcal{H}}, \quad |(Lg, h)_{\mathcal{H}}| \le C_2 \|g\|_{\mathcal{V}} \|h\|_{\mathcal{V}}, \quad \forall\, g, h \in \mathcal{V}.$$

Deduce that there exists a unique variational solution

$$g \in \mathcal{X}_T := C([0, T]; \mathcal{H}) \cap L^2(0, T; \mathcal{V}) \cap H^1(0, T; \mathcal{V}')$$

to the parabolic equation

$$\partial_t g = Lg, \quad g(0) = f_0.$$

Prove furthermore that $g \in \mathcal{F}_T$.
(12) Prove that there exists a unique fonction

$$f_\varepsilon \in C([0, T]; L^2_k) \cap L^2(0, T; H^1_k) \cap \mathcal{F}_T$$

solution to the nonlinear parabolic equation

$$\partial_t f_\varepsilon = \partial_i(\tilde{a}_{ij}^{f_\varepsilon}\partial_j f_\varepsilon + \tilde{b}_i^{f_\varepsilon} f_\varepsilon), \quad f_\varepsilon(0) = f_0,$$

where $\tilde{a}_{ij}^{f_\varepsilon}$ denotes the
(13) For $f_0 \in \mathcal{E}_{H_0}$ and $T > 0$, prove that there exists at least one weak solution $f \in \mathcal{F}_T$ to the Landau equation.