On the vorticity equation

We consider the 2 dimensional vorticity equation

$$\partial_t \omega = \Delta \omega - u \cdot \nabla \omega, \quad \omega(0) = \omega_0, \tag{0.1}$$

on the vorticity function $\omega : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$, $(t,x) \mapsto \omega_t(x) = \omega(t,x)$, where the velocity vector field $u : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}^2$ is defined through the Biot-Savart formula

$$u := K * \omega, \quad K(x) := \frac{1}{2\pi} \frac{x^\perp}{|x|^2}, \quad x^\perp := (-x_2, x_1). \tag{0.2}$$

Parts I and II are mostly independent. I believe that part A and D (except question (17)) are the simplest ones.

Part I - Existence

A. Local in time existence

In this part, we aim to establish the existence (and uniqueness) of a (mild) solution to the vorticity equation (0.1) for any $\omega_0 \in L^1(\mathbb{R}^2)$ on a small interval of time $[0,T]$. We write the vorticity equation into its mild formulation

$$\omega_t = e^{\Delta t} \omega_0 + \int_0^t e^{\Delta(t-s)} (-u_s \cdot \nabla \omega_s)\, ds, \tag{0.2}$$

where $e^{\Delta t}$ stands for the heat semigroup.

(1) We recall the Young inequality for convolution product

$$\|g * f\|_{L^c} \le \|g\|_{L^b} \|f\|_{L^a}, \quad \frac{1}{c} = \frac{1}{b} + \frac{1}{a} - 1,$$

for any $f \in L^a$, $g \in L^b$, $1/b + 1/a > 1$. Establish the estimates

$$\|e^{\Delta t} f\|_{L^p(\mathbb{R}^2)} \le \frac{C}{t^{\frac{1}{q} - \frac{1}{p}}} \|f\|_{L^q(\mathbb{R}^2)}$$

and

$$\|\nabla e^{\Delta t} f\|_{L^p(\mathbb{R}^2)} \le \frac{C}{t^{\frac{1}{2} + \frac{1}{q} - \frac{1}{p}}} \|f\|_{L^q(\mathbb{R}^2)},$$

for any $f \in L^q(\mathbb{R}^2)$ and $1 \le q \le p \le \infty$. Deduce that for any $p \in (1,\infty]$ and $\omega_0 \in L^q$, $p > q \ge 1$, there holds

$$t^{1-\frac{1}{p}} \|e^{\Delta t} \omega_0\|_{L^p(\mathbb{R}^2)} \to 0, \quad \text{as} \quad t \to 0,$$

with an estimate which only depends on $\|\omega_0\|_{L^q}$ when $q > 1$. (Hint. First consider the case $q > 1$. For the case $q = 1$, introduce the splitting $\omega_0 = \omega_0 \mathbf{1}_{|\omega_0| \le n} + \omega_0 \mathbf{1}_{|\omega_0| > n}$).

We define

$$\|f\|_{p,T} := \sup_{s \in (0,T)} s^{1-\frac{1}{p}} \|f_t\|_{L^p}$$

and the Banach space

$$X_{p,T} := \{f \in L^\infty_{\rm loc}((0,T]; L^p(\mathbb{R}^2));\ \|f\|_{p,T} < \infty\}.$$

(2) We define

$$\mathcal{Q}_t(f,g) := \int_0^t e^{\Delta(t-s)}(-K * f_s \cdot \nabla g_s)\, ds, \quad \mathcal{Q}_t(\omega) := \mathcal{Q}_t(\omega, \omega).$$

Prove that $\operatorname{div} K = 0$ and then

$$\mathcal{Q}_t(\omega) = -\int_0^t \operatorname{div}[e^{\Delta(t-s)}(K * \omega_s\, \omega_s)]\, ds.$$

We recall that for any $q \in (1,2)$, the following (HLS) inequality holds true

$$\|\frac{1}{|x|} * g\|_{L^\beta(\mathbb{R}^2)} \le C\, \|g\|_{L^q(\mathbb{R}^2)}, \quad \frac{1}{\beta} = \frac{1}{q} - \frac{1}{2}, \quad C = C(q). \tag{0.3}$$

Establish that for any $p \in (1,2)$, there exists a constant $C_p \ge 1$ such that

$$\forall\, T > 0,\ \forall\, \omega \in X_{p,T}, \quad \|\mathcal{Q}(\omega)\|_{p,T} \le C_p \|\omega\|^2_{p,T}.$$

(3) For $f \in X_{p,T}$, we define

$$g_t := e^{\Delta t}\omega_0 + \mathcal{Q}_t(f),$$

and $F : f \to g$. We fix $B > 0$ small enough such that $\alpha := 2C_p B < 1$. Prove that for $T > 0$ small enough, $F : \mathcal{B} \to \mathcal{B}$, where

$$\mathcal{B} := \{f \in X_{p,T};\ \|f\|_{p,T} \le B\}.$$

Deduce that F is a contraction of constant α. Conclude that there exists a unique $\omega \in X_{p,T}$ solution to the mild formulation (0.2) to the vorticity equation. Observe that we can take $T = T(\|\omega_0\|_{L^p})$ if $p > 1$.

B. More estimates

(4) For $\omega_0 \in L^1$, denote $\omega_0^n := \omega_0 \mathbf{1}_{|\omega_0| \le n}$ and next $\omega^n \in X_{4/3,T}$ the associated mild solution construct in question (3) (observe that T can chosen independent of n). Prove that

$$\omega_t - \omega_t^n = e^{\Delta t}(\omega_0 - \omega_0^n) + \mathcal{Q}_t(\omega, \omega - \omega^n) + \mathcal{Q}_t(\omega - \omega^n, \omega^n)$$

and deduce

$$(1-\alpha)\|\omega - \omega^n\|_{4/3,T} \le \|e^{\Delta t}(\omega_0 - \omega_0^n)\|_{4/3,T} \to 0, \quad \text{as} \quad n \to \infty.$$

(5) Prove that

$$\|\mathcal{Q}_t(f,g)\|_{L^1} \le C\|f\|_{4/3,t}\|g\|_{4/3,t}, \quad \forall\, t>0.$$

Deduce that $\omega \in X_{1,T}$ and next $\|\omega - \omega^n\|_{1,T} \to 0$ as $n\to\infty$.

(6) Prove that

$$\sup_{t\in[0,T]} \|\mathcal{Q}_t(f,g)\|_{L^p} \le C T^{1-1/p}\|f\|_{L^\infty(0,T;L^p)}\|g\|_{L^\infty(0,T;L^p)}.$$

Deduce that $\omega \in L^\infty(0,T;L^p)$ if $\omega_0\in L^p$, $1<p<\infty$.

(7) Prove that

$$t\mapsto \mathcal{Q}_t(\omega)\in C([0,T];L^1)$$

when $\omega\in L^\infty(0,T;L^{4/3})$. With the help of (6) deduce first that $\omega\in C([0,T];L^1)$ when $\omega_0\in L^1\cap L^{4/3}$. With the help of (5) deduce next that $\omega\in C([0,T];L^1)$ when $\omega_0\in L^1$.

(8) Prove that ω is a weak solution to the vorticity equation (0.1). Reciprocally, prove that any weak solution $\omega\in C([0,T);L^1)$ satisfying $\|\omega\|_{4/3,T}<\infty$ is a mild solution.

(9) We introduce the splitting $\mathcal{Q}_t := \mathcal{Q}^1_t + \mathcal{Q}^2_t$ with

$$\mathcal{Q}^1_t(\omega) := -\int_0^{t/2} \mathrm{div}[e^{\Delta(t-s)}(K*\omega_s\,\omega_s)]\,ds, \quad \mathcal{Q}^2_t(\omega) := -\int_{t/2}^{t} \mathrm{div}[e^{\Delta(t-s)}(K*\omega_s\,\omega_s)]\,ds.$$

We fix $q\in(1,2)$ and we denote by C some constants which only depend on q. Prove that

$$\|\mathcal{Q}^1(\omega)\|_{\infty,t} \le C\|\omega\|_{q,t}.$$

Prove that

$$\|K*f\|_{L^\infty} \le C\|f\|_{L^q}^{q/2}\|f\|_{L^\infty}^{1-q/2}.$$

(Hint. Introduce the splitting $K(x) = K(x)\mathbf{1}_{|x|\le A} + K(x)\mathbf{1}_{|x|>A}$*)* and deduce that

$$\|\mathcal{Q}^2(\omega)\|_{\infty,t} \le C\|\omega\|_{q,t}^{1+q/2}\|\omega\|_{\infty,t}^{1-q/2}.$$

Finally prove that $\omega\in X_{p,T}$ for any $1\le p\le\infty$. *(Hint. First consider the case* $p=\infty$*).*

C. Global existence

(10) Accepting and using the Calderón-Zygmond inequality

$$\|\nabla(K*f)\|_{L^p} \le C_p\|f\|_{L^p}, \quad 1<p<\infty,$$

observe that $\mathcal{K} := K*\omega \in L^\infty(\tau,T;W^{1,p})$ for any $1<p<\infty$, $0<\tau<T$, and prove that

$$r_\varepsilon := (\mathcal{K}\cdot\nabla\omega)*\rho_\varepsilon - \mathcal{K}\cdot\nabla\omega^\varepsilon \to 0 \ \text{ in } \ L^1_{loc}, \quad \omega^\varepsilon := \omega*\rho_\varepsilon,$$

for any given mollifier (ρ_ε) on $\mathbb{R}^2$. Deduce that for any smooth nonnegative convex function β such that $\beta(0)=\beta'(0)=0$ and $\beta''\in C_c(\mathbb{R})$ there holds

$$\partial_t\beta(\omega^\varepsilon) = \Delta\beta(\omega^\varepsilon) - |\nabla\omega^\varepsilon|^2\beta''(\omega^\varepsilon) - \mathcal{K}\cdot\nabla\beta(\omega^\varepsilon) + r_\varepsilon\beta'(\omega^\varepsilon)$$

and next

$$\int \beta(\omega_{t_1})\varphi\, dx \le \int \beta(\omega_{t_0})\varphi\, dx + \int_{t_0}^{t_1}\int \beta(\omega_s)\left\{\Delta\varphi + \mathcal{K}\cdot\nabla\varphi\right\} dxds,$$

for any $\varphi \in C_c^2(\mathbb{R}^2)$, $\varphi \ge 0$, and any $t_0, t_1 \in (0,T)$, $t_1 > t_0$. Conclude that

$$\|\omega_{t_1}\|_{L^p} \le \|\omega_{t_0}\|_{L^p}.$$

(11) Establish that for any $\omega_0 \in L^1(\mathbb{R}^2)$ there exists a unique weak and mild solution

$$\omega \in C([0,\infty); L^1(\mathbb{R}^2)) \cap C((0,\infty); L^\infty(\mathbb{R}^2))$$

to the vorticity equation (0.1) such that for any $p \in [1,\infty]$, $0 \le t_0 < t_1 < T$, there hold

$$\|\omega\|_{p,T} < \infty \quad \text{and} \quad \|\omega_{t_1}\|_{L^p} \le \|\omega_{t_0}\|_{L^p}.$$

Prove furthermore $\omega(t,\cdot) \ge 0$ for any $t \ge 0$ if $\omega_0 \ge 0$.

Part II - Long time behaviour.

In this part, identities and estimates have only to be *formally established*.

(12) Prove that if ω is a classical solution to the vorticity equation (0.1) then the rescaled function

$$w(t,x) := e^{2t}\omega\left(\frac{e^{2t}-1}{2}, e^t x\right)$$

satisfies the rescaled vorticity equation

$$\partial_t w = \Delta w + \operatorname{div}(xw) - v\cdot\nabla w, \quad w(0) = \omega_0, \tag{0.4}$$

where v is defined through the Biot-Savart formula (0.2), namely $v := K * w$.

(13) Prove that

$$\begin{aligned}
\frac{d}{dt}\int w(t,x)\, dx &= 0,\\
\frac{d}{dt}\int w(t,x)\, x\, dx &= -\int w(t,x)\, x\, dx,\\
\frac{d}{dt}\int w(t,x)\, |x|^2\, dx &= 4\int w(t,x)\, dx - 2\int w(t,x)\, |x|^2\, dx,
\end{aligned}$$

by using that $K(-z) = -K(z)$ in the second line and also $K(z)\cdot z = 0$ in the third line. Defining

$$X := \left\{ f \in L^1_2;\ \int f = 1,\ \int fx = 0,\ \int f|x|^2 = 2 \right\},$$

deduce that $w(t,\cdot) \in X$ for any $t \ge 0$ if $\omega_0 \in X$. Define the gaussian function

$$G(x) := \frac{1}{2\pi} e^{-|x|^2/2}.$$

Observe/admit that the solution Φ to the Poisson equation $-\Delta\Phi = G$ is radially symmetric, that is $\Phi(x) = \phi(|x|)$ with $(r\phi')' = rG(r)$ and find $\phi'(r) = (1-e^{-r^2/2})/(2\pi r)$. Conclude that

$$K * G(x) = \frac{x^\perp}{|x|^2}\frac{1-e^{-|x|^2/2}}{2\pi}.$$

Deduce that

$$0 \le G \in X, \quad 0 = \Delta G + \operatorname{div}(xG) - K * G \cdot \nabla G.$$

(14) Assume furthermore that $\omega_0 \ge 0$ and thus $w(t,\cdot) \ge 0$ for any $t \ge 0$. Prove that

$$\frac{d}{dt}H(w|G) = -I(w|G),$$

with $H(w|G) := H(w) - H(G)$, $I(w|G) := I(w) - I(G)$,

$$H(f) = \int_{\mathbb{R}^2} f \log f \, dx, \quad I(f) := \int_{\mathbb{R}^2} \frac{|\nabla f|^2}{f}\, dx.$$

Deduce that

$$\|w(t) - G\|_{L^1} \le C(\omega_0)e^{-t}, \quad \forall\, t \ge 0.$$

In the sequel, we do not assume $\omega_0 \ge 0$ anymore.

(15) Consider the linear operator $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$ with

$$\mathcal{L}_0 f = \Delta f + \operatorname{div}(xf), \quad \mathcal{L}_1 f = -K * f \cdot \nabla G - K * G \cdot \nabla f.$$

Prove that

$$\int (\mathcal{L}f) f G^{-1} = \int (\mathcal{L}_0 f) f G^{-1}.$$

Deduce that the semigroup $S_\mathcal{L}$ satisfies the decay estimate

$$\|S_\mathcal{L}(t) f\|_H \le e^{-t}\|f\|_H, \quad \forall\, t \ge 0,$$

for any $f \in H := L^2(G^{-1})$ such that $\int f dx = 0$.

We define $L^p_k := \{f \in L^p;\ \|f\|_{L^p_k} := \|f\langle x\rangle^k\|_{L^p} < \infty\}$ and $\mathcal{H} := L^2_k$.

(16) **In all the sequel, we fix $k > 2$.** Prove that

$$\|f\|_{L^1} \le C\|f\|_\mathcal{H} \quad \text{and} \quad \|f\|_{L^{4/3}} \le C_k\|f\|_\mathcal{H}, \tag{0.5}$$

We thus may define $\mathcal{H}_0 := \{f \in \mathcal{H};\ \int f = 0\}$.

(17) Establish that

$$\int (\mathcal{L}f) f \langle x\rangle^{2k} = -\int |\nabla f|^2 \langle x\rangle^{2k} - \int f^2 \langle x\rangle^{2k}\Big[(k-1) - \frac{k}{\langle x\rangle^2} + \frac{\Delta\langle x\rangle^{2k}}{2\langle x\rangle^{2k}}\Big] - \int f(K*f)\cdot\nabla G\langle x\rangle^{2k}.$$

Using (0.5) and the HLS inequality (0.3), deduce that for any $a \in (1-k, 0)$, there exists C_a such that

$$\int (\mathcal{L}f) f \langle x\rangle^{2k} \le a \int f^2 \langle x\rangle^{2k} + C_a \int f^2.$$

Defining

$$\mathcal{B}f := \mathcal{L}f - M\chi_R f,$$

with $\chi_R(x) = \chi(x/R)$, $\chi \in \mathcal{D}(\mathbb{R}^2)$, $\mathbf{1}_{B(0,1)} \le \chi \le \mathbf{1}_{B(0,2)}$, prove that for any $a \in (k-1, 0)$, there exist $R, M > 0$ large enough such that

$$\|S_{\mathcal{B}}(t) f\|_{\mathcal{H}} \le e^{at} \|f\|_{\mathcal{H}}, \quad \forall\, t \ge 0,\ \forall\, f \in \mathcal{H}.$$

In the sequel, we fix $a \in (1-k, -1)$ and the above associated constants $R, M > 0$ and operator $\mathcal{B}$. Establish that the semigroup $S_{\mathcal{L}}$ satisfies the (same) decay estimate (as in H)

$$\|S_{\mathcal{L}}(t) f\|_{\mathcal{H}} \le C e^{-t} \|f\|_{\mathcal{H}}, \quad \forall\, t \ge 0,$$

for any $f \in \mathcal{H}_0$.

(18) We come back to the rescaled vorticity equation (0.4) and we introduce the variation $f := w - G$ around the steady state G for an initial datum $f_0 \in \mathcal{H}_0$. Establish that f satisfies the mild equation

$$f_t = S_{\mathcal{L}}(t) f_0 + \int_0^t \mathrm{div}[S_{\mathcal{L}}(t-s)[(K * f_s) f_s]]\, ds.$$

Prove that

$$\|\nabla S_{\mathcal{L}}(\tau) f\|_{\mathcal{H}} \le \frac{C e^{a\tau}}{\tau^{2/3}} \|f\|_{L^{3/2}_k}, \quad \forall\, \tau > 0,\ \forall\, f \in L^{3/2}_k,\ \int f = 0,$$

and deduce by duality (do not really try !) that

$$\|\Pi^\perp S_{\mathcal{L}}(\tau) \nabla f\|_{\mathcal{H}} \le \frac{C e^{a\tau}}{\tau^{2/3}} \|f\|_{L^{3/2}_k}, \quad \forall\, \tau > 0,\ \forall\, f \in L^{3/2}_k,$$

where $\Pi^\perp f := f - G \int f$.

(19) Deduce that $u(t) := \|f_t\|_{\mathcal{H}}$ satisfies the integral inequality

$$u(t) \le C_1 e^{-t} u(0) + C_2 \int_0^t \frac{e^{-(t-s)}}{(t-s)^{2/3}} u(s)^2\, ds, \quad \forall\, t > 0.$$

Conclude that for $\|w_0 - G\|_{\mathcal{H}}$ small enough, there holds

$$\|w(t) - G\|_{\mathcal{H}} \le C e^{-t}, \quad \forall\, t \ge 0.$$