Exam, January 18, 2023

Problem 1 - The heat equation in the half-line.

In this problem, we consider the heat equation in the half-line

$$\begin{cases} \partial_t f = \partial^2_{xx} f \quad \text{in} \quad (0,\infty)\times\mathbb{R}_+, \\ f(0) = f_0 \quad \text{on} \quad \mathbb{R}_+, \end{cases} \tag{0.1}$$

with Neumann condition

$$\partial_x f(t,0) = 0 \quad \text{on} \quad (0,\infty), \tag{0.2}$$

or Dirichlet condition

$$f(t,0) = 0 \quad \text{on} \quad (0,\infty), \tag{0.3}$$

and we investigate the long time behaviour of the solutions. The aim of the problem is to recover the classical decay estimate in the case of the Neumann problem (0.1)-(0.2) and to show that this estimate may be improved in the case of the Dirichlet problem (0.1)-(0.3).

Question 1

For $f \in L^2(\mathbb{R}_+)$, we denote $\bar f(x) = f(x)$ if $x > 0$, $\bar f(x) = f(-x)$ if $x < 0$. Observe that for $f \in C^1(\mathbb{R}_+)$ there is equivalence between $f'(0) = 0$ and $\bar f \in C^1(\mathbb{R})$. Show that if $f \in C([0,T); L^2(\mathbb{R}_+)) \cap L^2(0,T; H^1(\mathbb{R}_+))$ is a solution to (0.1)-(0.2) in the sense

$$\frac{d}{dt}\int_0^\infty f\varphi = -\int_0^\infty \partial_x f\,\partial_x\varphi, \quad \forall\,\varphi\in H^1(\mathbb{R}_+),$$

then $\bar f \in C([0,T); L^2(\mathbb{R})) \cap L^2(0,T; H^1(\mathbb{R}))$ is a solution to the heat equation on the real line in the sense

$$\frac{d}{dt}\int_{\mathbb{R}} \bar f\varphi = -\int_{\mathbb{R}} \partial_x \bar f\,\partial_x\varphi, \quad \forall\,\varphi\in H^1(\mathbb{R}).$$

Deduce that if $f_0 \in L^1(\mathbb{R}_+)$, there exists at least one solution f to (0.1)-(0.2) and this one satisfies

$$\|f(t,\cdot)\|^2_{L^2} \le \frac{C}{t^{1/2}}\|f_0\|^2_{L^1}, \quad \forall\, t > 0.$$

Question 2

Prove that a solution f to the Dirichlet problem (0.1)-(0.3) satisfies (at least formally)

$$\frac{d}{dt}\int_0^\infty |f|dx \le 0,$$
$$\frac{d}{dt}\int_0^\infty fxdx = 0, \qquad \frac{d}{dt}\int_0^\infty |f|xdx \le 0,$$
$$\frac{d}{dt}\int_0^\infty f^2dx = -\int_0^\infty (\partial_x f)^2,$$
$$\frac{d}{dt}\int_0^\infty f_-^2dx = -\int_0^\infty (\partial_x f_-)^2.$$

Again formally, establish that if $f_0 \ge 0$, the associated solution satisfies $f \ge 0$.

Question 3

For $f \in C(\mathbb{R}_+)$, we denote $\widetilde{f}(x) = f(x)$ if $x > 0$, $\widetilde{f}(x) = -f(-x)$ if $x < 0$. Show that if $f \in C([0,T]; L^2(\mathbb{R}_+)) \cap L^2(0,T; H_0^1(\mathbb{R}_+))$ is a solution to (0.1)-(0.3) in the sense

$$\frac{d}{dt}\int_0^\infty f\varphi = -\int_0^\infty \partial_x f \partial_x \varphi, \quad \forall \varphi \in H_0^1(\mathbb{R}_+),$$

then $\widetilde{f} \in C([0,T]; L^2(\mathbb{R})) \cap L^2(0,T; H^1(\mathbb{R}))$ is a solution to the heat equation on the real line in the sense

$$\frac{d}{dt}\int_{\mathbb{R}} \widetilde{f}\varphi = -\int_{\mathbb{R}} \partial_x \widetilde{f} \partial_x \varphi, \quad \forall \varphi \in H^1(\mathbb{R}). \tag{0.4}$$

Reciprocally, deduce that if $\widetilde{f} \in C([0,T]; L^2(\mathbb{R})) \cap L^2(0,T; H^1(\mathbb{R}))$ is an **odd** solution to the heat equation on the real line in the sense that it satisfies (0.4) and $\widetilde{f}(t,x) = \widetilde{f}(t,-x)$ for any $t \in (0,T)$ and a.e. $x \in \mathbb{R}_+$, then the restriction $f := \widetilde{f}_{|\mathbb{R}+}$ is a solution to the Dirichlet problem (0.1)-(0.3). Deduce that if $f_0 \in L^2(\mathbb{R}_+)$, there exists at least one solution f to (0.1)-(0.3).

Question 4 (improved Nash inequality)

We denote by L^1_1 the Lebesgue space endowed with the norm $\|g\|_{L^1_1} := \|(1+|x|)g\|_{L^1}$. Show that there exists a constant $C_{iN} > 0$ such that for any $g \in L^1_1(\mathbb{R}) \cap H^1(\mathbb{R})$ with **zero mean**, there holds

$$\|g\|_{L^2}^{5/3} \le C_{iN} \|g\|_{L^1_1}^{2/3} \|g'\|_{L^2}.$$

(Hint. Use first Parseval's identity and adapt the classical Nash inequality).
Deduce that there exists a constant $C > 0$ such that for any $h \in L^1_1(\mathbb{R}_+) \cap H^1(\mathbb{R}_+)$ with $h(0) = 0$, there holds

$$\|h\|_{L^2}^{5/3} \le C \|h\|_{L^1_1}^{2/3} \|h'\|_{L^2}.$$

(Hint. Use the previous inequality for a well chosen g).

Question 5

Consider $f_0 \in L^1_1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$. Establish (at least formally) that the associated solution satisfies the improved decay

$$\|f(t,\cdot)\|^2_{L^2} \leq \frac{C}{t^{3/2}}\|f_0\|^2_{L^1_1}, \quad \forall\, t > 0.$$

Question 6

Explain briefly how to make the above result completely rigorous.

Question 7

Recover the same result for both problems (0.1)-(0.3) for any solution associated to an initial datum $f_0 \in L^1_1(\mathbb{R}_+)$ with **zero mean** by using the Fokker-Planck equation and the optimal Poincaré estimate.

Problem 2 - Krein-Rutman theorem in a weakly dissipative framework

The aim of this problem is to extend the Krein-Rutman theorem to a more general framework and to apply it to an example of parabolic equation.

1 Part I - An abstract Krein-Rutman theorem.

In this part X denotes a Banach lattice (e.g. $X := L^p_m(E, \mathscr{A}, \mu)$ associated to the measurable space $(E, \mathscr{A}, \mu) := (\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d), \lambda)$ and the weight function $m : E \to [1, \infty)$), Y denotes a dual Banach lattice (e.g. $Y := L^{p'}_\nu$, $\nu := m^{-1}$) and $S = S_\mathcal{L}$ is a positive continuous semigroup of bounded operators on X.
We recall that X (and Y) is a Banach space equipped with an order $\geq$ such that any vector $f \in X$ splits as $f = f_+ - f_-$ with $f_\pm \in X_+ := \{g \in X;\ g \geq 0\}$ and we may associate its absolute value $|f| := f_+ + f_-$. We finally recall that $S(t)f \geq 0$ for any $t \geq 0$ if $f \geq 0$. We denote the dual bracket between X and Y by

$$\langle f, \psi \rangle := \int_E f \psi d\mu.$$

We further assume that

(i) there exists $\psi_0 \in Y_+ \backslash \{0\}$ such that $[S_t f]_0 \geq [f]_0$ for any $f \in X_+$ and $t \geq 0$, we denote by $\mathcal{X}$ the vector space X endowed with the (semi)norm $\|f\|_\mathcal{X} = [f]_0 := \langle |f|, \psi_0 \rangle$;

(ii) there exist $v \in L^\infty(\mathbb{R}_+; \mathscr{B}(X))$ and $0 \leq w \in L^1(\mathbb{R}_+; \mathscr{B}(\mathcal{X}, X))$ such that

$$S = v + w * S, \tag{1.1}$$

and we set

$$M := \sup_{t \geq 0} \|v(t)\|_{\mathscr{B}(X)} < \infty, \quad \Theta(t) := \|w(t)\|_{\mathscr{B}(\mathcal{X}, X)} \in L^1(\mathbb{R}_+). \tag{1.2}$$

(iii) B_X is weakly $\sigma(X, Y)$ sequentially compact (so that Y is separable).

We define the positive number

$$R := \max(2\|\Theta\|_{L^1}, \|g_0\|),$$

for some $g_0 \in X_+$ such that $[g_0]_0 = 1$, the set

$$\mathcal{C} := \{f \in X_+;\ [f]_0 = 1,\ \|f\| \leq R\},$$

as well as the function λ and its supremum λ^* by

$$\lambda(t) := \inf_{f \in \mathcal{C}} [S(t) f]_0, \quad \forall\, t \geq 0, \qquad \lambda^* := \sup_{t \geq 0} \lambda(t).$$

Question 8

Show that $\mathcal{C}$ is nonempty convex and weakly compact. Show that

$$[S_t f]_0 \geq [S_s f]_0, \quad \forall\, t \geq s \geq 0,\ \forall\, f \geq 0,$$

and that λ is an increasing function.

Question 9

We assume $\lambda^* > 2M$. Choosing $T_0 > 0$ such that

$$\forall\, f \in \mathcal{C}, \quad [S_{T_0} f]_0 \geq 2M$$

(why is it possible?), we define

$$\Phi_{T_0} f := \frac{S_{T_0} f}{[S_{T_0} f]_0}, \quad \forall\, f \in \mathcal{C}.$$

Show that $\Phi_{T_0} : \mathcal{C} \to \mathcal{C}$ and next that there exists $f_{T_0} \in X$ such that

$$f_{T_0} \geq 0, \quad [f_{T_0}]_0 = 1, \quad S_{T_0} f_{T_0} = e^{\lambda_1 T_0} f_{T_0}.$$

Conclude that there exists $(\lambda_1, f_1) \in \mathbb{R} \times X$ such that

$$\lambda_1 > 0, \quad f_1 \in X_+ \backslash \{0\}, \quad \mathcal{L} f_1 = \lambda_1 f_1.$$

From now on in this part, we assume $\lambda^* \leq 2M$.

Question 10

Prove that there exists $f_0 \in \mathcal{C}$ such that

$$\forall\, t \geq 0, \quad [S(t) f_0]_0 \leq 2M. \tag{1.3}$$

(Hint. Start observing that for any $n \geq 1$, there exists $f_n \in \mathcal{C}$ such that $[S(n) f_n]_0 \leq 2M$).

Taking advantage of the splitting structure (1.1) of the semigroup S, for any $T > 0$, we introduce the associated Cesàro means

$$U_T := \frac{1}{T} \int_0^T S(t)\, dt, \quad V_T := \frac{1}{T} \int_0^T v(t)\, dt, \quad W_T := \frac{1}{T} \int_0^T (w * S)(t)\, dt.$$

Question 11

Establish that

$$\|V_T\|_{\mathscr{B}(X)} \leq M$$

and

$$\|W_T f_0\| \leq \|\Theta\|_{L^1}\, [U_T f_0]_0.$$

(Hint. For the second inequality, use the Fubini theorem). Deduce that

$$1 \leq [S_T f_0]_0 \leq 2M \quad \text{and} \quad \|U_T f_0\| \leq M \|f_0\| + 2M \|\Theta\|_{L^1}.$$

Question 12

Deduce that there exist $T_k \to +\infty$ and $f_1 \in X$ such that $U_{T_k} f \rightharpoonup f_1$ and f_1 satisfies

$$f_1 \in X_+ \backslash \{0\}, \quad \mathcal{L} f_1 = \lambda_1 f_1, \quad \text{with} \quad \lambda_1 = 0.$$

Summarize the result established under assumptions **(i)**, **(ii)** and **(iii)**.

2 Part II - The Krein-Rutman theorem for a parabolic equation.

We consider the parabolic equation

$$\partial_t f = \mathcal{L} f := \Delta f + \operatorname{div}(bf) + cf \quad \text{in} \quad (0,\infty) \times \mathbb{R}^d, \tag{2.1}$$

with

$$b = \frac{1}{\gamma} \nabla \langle x \rangle^\gamma, \gamma \in (0,1), \quad 0 \le \langle x \rangle c \in L^\infty.$$

We define the multiplication operator $\mathcal{A}$ and the elliptic operator $\mathcal{B}$ by

$$\mathcal{A} := M \chi_R, \quad \mathcal{B} := \mathcal{L} - \mathcal{A},$$

for $M, R > 0$ and $\chi_R(x) := \chi(x/R)$ with $\chi \in \mathcal{D}(\mathbb{R}^d)$, $\mathbf{1}_{B_1} \le \chi \le \mathbf{1}_{B_2}$.

Question 13

Recall why the parabolic equation (2.1) has a solution and generates a positive semigroup in L^2_m for any weight function m, we choose m increasing enough in such a way that $L^2_m \subset L^1$. Exhibit $\psi_0 \in L^\infty \cap C^2$ such that

$$\mathcal{L}^* \psi_0 \ge 0.$$

Deduce that S satisfies property **(i)** stated at the beginning of Part I.

Question 14

We consider the family of weight functions $m := \langle x \rangle^k$, for some $k > 0$ large enough to be specified. Prove that we may find $M, R > 0$ large enough such that

$$\int (\mathcal{B} f) \langle x \rangle^k dx \le -c_k \int f \langle x \rangle^{k+\gamma-2} dx$$

for some $c_k > 0$ and for any $k > 0$ and any nice function $f \ge 0$. Prove similarly that

$$\int (\mathcal{B} f) f \langle x \rangle^{2k} dx \le -c_k \int |\nabla f|^2 \langle x \rangle^{2k} dx - c_k \int f^2 \langle x \rangle^{2k+\gamma-2}\, dx$$

for any nice function f and any $k > 0$ large enough (at least $k > d/2$). Deduce that

(a) $S_\mathcal{B} \in L^\infty_t(\mathscr{B}(L^2_m))$ and $S_\mathcal{B} \in L^\infty_t(\mathscr{B}(L^1_m))$.

Question 15

Establish that if $0 \le u \in C^1(\mathbb{R}_+)$ satisfies

$$u' \le -cu^{1+1/\alpha}, \quad c, \alpha > 0,$$

there exists $C := C(c, \alpha) > 0$ such that

$$u(t) \le C/t^\alpha, \quad \forall\, t > 0.$$

Prove successively that if $\ell > k > 0$ there exists $\alpha = \alpha(k, \ell) > 0$ such that

$$\|S_\mathcal{B}(t)\|_{L^1_\ell \to L^1_k} \le C/\langle t\rangle^\alpha,$$
$$\|S_\mathcal{B}(t)\|_{L^2_\ell \to L^2_k} \le C/\langle t\rangle^\alpha,$$
$$\|S_\mathcal{B}(t)\|_{L^1_\ell \to L^2_k} \le C/t^{d/4}.$$

Question 16

For $N := [d/4] + 1$ and m chosen conveniently, deduce that

(b) $S_\mathcal{B}\mathcal{A} \in L^1_t(\mathscr{B}(L^2_m))$;

(c) $(S_\mathcal{B}\mathcal{A})^{(*N)} \in L^1_t(\mathscr{B}(L^1, L^2_m))$.

Question 17

Conclude that there exists $f_1 \in H^1_m$ and $\lambda_1 \ge 0$ such that

$$\mathcal{L} f_1 = \lambda_1 f_1, \quad f_1 \ge 0, \quad f_1 \not\equiv 0.$$