Exam, January 12, 2023

The exam is made of two independent problems. One may obtain the maximal score without dealing with the *"more difficult questions"* that one finds at the end of each problem. The answers may be written in english, french or spanish (choose only one language!).

1 Problem 1 - The kinetic Fokker-Planck equation.

In this problem, we consider the kinetic Fokker-Planck (or Kolmogorov) equation

$$\begin{cases} \partial_t f = -v\partial_x f + \partial^2_{vv} f \quad \text{in} \quad (0,\infty)\times\mathbb{R}\times\mathbb{R}, \\ f(0,\cdot) = f_0 \quad \text{on} \quad \mathbb{R}\times\mathbb{R}, \end{cases} \tag{1.1}$$

on the function $f = f(t,x,v)$, $t \geq 0$, $x \in \mathbb{R}$, $v \in \mathbb{R}$.

The aim of this problem is to prove the existence and uniqueness of the solution to the Fokker-Planck equation as well as an ultracontractivity property.

Question 1

For $f_0 \in L^2(\mathbb{R})$ and $T > 0$, establish that there exists a weak solution $f \in L^2((0,T)\times\mathbb{R}; H^1_v(\mathbb{R}))$ to equation (1.1) in the sense

$$\int_{\mathcal{U}} \{f(-\partial_t\varphi - v\partial_x\varphi) + \partial_v f\partial_v\varphi\} dtdxdv = \int_{\mathbb{R}^2} f_0\varphi(0,\cdot)dxdv \tag{1.2}$$

for any $\varphi \in C^1_c([0,T)\times\mathbb{R}^2)$, $\mathcal{U} := (0,T)\times\mathbb{R}^2$.

In the next two questions, we accept furthermore that

$$f \in C(\mathbb{R}_+; L^2(\mathbb{R}^2)). \tag{1.3}$$

Question 2

Consider a mollifer (ρ_ε) in $\mathbb{R}^2$ and define $f_\varepsilon := f *_{x,v} \rho_\varepsilon$. Prove that

$$\partial_t f_\varepsilon + v\partial_x f_\varepsilon - \partial^2_{vv} f_\varepsilon = r_\varepsilon,$$

in $\mathcal{D}'([0,T)\times\mathbb{R}^2)$, with $r_\varepsilon \to 0$ in $L^2_{\text{loc}}((0,T)\times\mathbb{R}^2)$. Deduce that for any $\beta \in C^2$, $\beta'' \in L^\infty(\mathbb{R})$, there holds

$$\partial_t \beta(f) = -v\partial_x \beta(f) + \partial^2_{vv}\beta(f) - \beta''(f)|\partial_v f|^2, \tag{1.4}$$

in $\mathcal{D}'([0,T)\times\mathbb{R}^2)$, and more precisely

$$\int_{\mathcal{U}} \{\beta(f)(-\partial_t\varphi - v\partial_x\varphi) + \partial_v\beta(f)\partial_v\varphi + \beta''(f)|\partial_v f|^2\varphi\} dtdxdv = \int_{\mathbb{R}^2} \beta(f_0)\varphi(0,\cdot)dxdv,$$

for any $\varphi \in C^1_c([0,T)\times\mathbb{R}^2)$.

Question 3

Prove that f is the unique function satisfying both (1.2) and (1.3).

In the seven next questions, we assume that the involved functions are nice (smooth, fast decaying) in order to justify the computations.

Question 4

We denote $L^p = L^p(\mathbb{R}^2)$ and we assume $f_0 \in L^p$, $1 \le p \le \infty$. For a solution f to (1.1) establish that

$$\sup_{t\ge 0} \|f(t,\cdot)\|_{L^p} \le \|f_0\|_{L^p}.$$

Question 5

We denote $L^2 = L^2(\mathbb{R}^2)$ and $(\cdot,\cdot)$ the associated scalar product. Prove that a solution f to (1.1) satisfies

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|f\|^2_{L^2} &= -\|\partial_v f\|^2_{L^2}\\
\frac{1}{2}\frac{d}{dt}\|\partial_x f\|^2_{L^2} &= -\|\partial^2_{xv} f\|^2_{L^2}\\
\frac{1}{2}\frac{d}{dt}\|\partial_v f\|^2_{L^2} &= -(\partial_x f, \partial_v f) - \|\partial^2_{vv} f\|^2_{L^2}\\
\frac{d}{dt}(\partial_x f, \partial_v f) &= -\|\partial_x f\|^2_{L^2} - 2(\partial^2_{xv} f, \partial^2_{vv} f).
\end{aligned}$$

Question 6

We denote $H^1 = H^1(\mathbb{R}^2)$ and, for $f \in H^1$, we define

$$\|f\|_{H^1}^2 := \|f\|_{L^2}^2 + \|\partial_x f\|_{L^2}^2 + \|\partial_v f\|_{L^2}^2, \quad \mathcal{F} = \mathcal{F}(f) := \|f\|_{H^1}^2 + (\partial_x f, \partial_v f).$$

Establish that

$$\frac{1}{2}\|f\|_{H^1}^2 \le \mathcal{F} \le \frac{3}{2}\|f\|_{H^1}^2, \quad \forall\, f \in H^1.$$

For a solution f to (1.1), we denote $\mathcal{F}_t := \mathcal{F}(f(t,\cdot))$. Establish that $\mathcal{F}_t' \le 0$ and deduce

$$\|f(t,\cdot)\|_{H^1} \le \sqrt{3}\,\|f_0\|_{H^1}, \quad \forall\, t \ge 0.$$

Question 7

We define

$$\mathcal{G} = \mathcal{G}(f) := \varepsilon^{-1}\|f\|_{L^2}^2 + \varepsilon^3 t^3 \|\partial_x f\|_{L^2}^2 + \varepsilon t \|\partial_v f\|_{L^2}^2 + \varepsilon^2 t^2 (\partial_x f, \partial_v f).$$

For a solution f to (1.1), we denote $\mathcal{G}_t := \mathcal{G}(f(t,\cdot))$. Establish that for some fixed $\varepsilon \in (0,1)$ small enough, there holds $\mathcal{G}_t' \le 0$ for any $t > 0$. *(Hint. At some point, use the Young inequalities*

$$2\|\partial_v f\|_{L^2}\|\partial_x f\|_{L^2} \le \varepsilon^{-2} t^{-1} \|\partial_v f\|_{L^2}^2 + \varepsilon^2 t \|\partial_x f\|_{L^2}^2,$$
$$2\|\partial_{vv}^2 f\|_{L^2}\|\partial_{xv}^2 f\|_{L^2} \le \varepsilon^{-1} t^{-1} \|\partial_{vv} f\|_{L^2}^2 + \varepsilon t \|\partial_{vx} f\|_{L^2}^2).$$

Deduce that $f(t) \in H^1$ for any $t > 0$ if $f_0 \in L^2$, and more precisely

$$t^3 \|f(t,\cdot)\|_{H^1}^2 \le \frac{2}{\varepsilon^4}\|f_0\|_{L^2}^2, \quad \forall\, t \in (0,1).$$

Question 8

Deduce that for any $q \in (2,\infty)$ and $T \in (0,1)$, there exists $C(q)$ such that

$$\|f(t,\cdot)\|_{L^q} \le \frac{C(q)}{t^{3/2}}\|f_0\|_{L^2}, \quad \forall\, t \in (0,T).$$

Together with question 4, establish that there exists $C > 0$ such that

$$\|f(t,\cdot)\|_{L^3} \le \frac{C}{(t-s)^{2/3}}\|f(s,\cdot)\|_{L^2}, \quad \forall\, s,t,\ 0 < s < t < T.$$

Question 9

Denote $\rho_a := \|f(t,\cdot)\|_{L^a}$. For $\varphi \geq 0$, establish that

$$\int_0^T \varphi(t)\rho_3(t)dt \leq C \int_0^T \Phi(s)\rho_2(s)ds,$$

with

$$\Phi(s) := \int_s^T \frac{\varphi(t)}{t} \frac{dt}{(t-s)^{2/3}}.$$

Together with question 4, deduce that

$$A(T)\rho_3(T) \lesssim B(T)\rho_{3/2}(0), \quad \forall T \in (0,1),$$

with

$$A(T) := \int_0^T \varphi(t)dt, \quad B(T) := \int_0^T \Phi^2(s)\varphi^{-1}(s)ds.$$

Choosing finally $\varphi(t) := \psi(t/T)$ and ψ adequately, establish that

$$\rho_3(T) \lesssim T^{-4/3}\rho_{3/2}(0), \quad \forall T \in (0,1).$$

Question 10 (more difficult)

Taking into account the previous two questions and repeating the arguments, establish successively

$$\begin{aligned} \|f(t,\cdot)\|_{L^2} &\leq \frac{C_1}{(t-s)^{2/3}}\|f(s,\cdot)\|_{L^{3/2}}, \qquad \forall s,t,\ 0 < s < t < 1, \\ \|f(t,\cdot)\|_{L^2} &\leq \frac{C_2}{t^\bullet}\|f_0\|_{L^1}, \quad \forall t \in (0,1), \end{aligned}$$

for some $C_i > 0$ and $\bullet > 0$ to be made explicit. Establish next

$$\|f(t,\cdot)\|_{L^\infty} \leq \frac{C_3}{t^{2\bullet}}\|f_0\|_{L^1}, \quad \forall t \in (0,1).$$

The aim of the next questions is to prove (1.3).

Question 11 (more difficult)

For $f_0 \in L^2(\mathbb{R})$, we consider a function $f \in L^2((0,T) \times \mathbb{R}; H^1_v(\mathbb{R}))$ which satisfies (1.2) as established in Question 1. We denote by (f_ε) and (r_ε) the sequences of functions defined in Question 2.

For $\varepsilon_n \to 0$, define $g_{n,m} := f_{\varepsilon_m} - f_{\varepsilon_n}$, $R_{n,m} := r_{\varepsilon_m} - r_{\varepsilon_n}$, and establish that

$$\begin{aligned} \Big| \int_{\mathbb{R}^2} \beta(g_{n,m})(t_1)\phi dxdv - \int_{\mathbb{R}^2} \beta(g_{n,m})(t_0)\phi dxdv \Big| \\ \leq \int_{t_0}^{t_1} \int_{\mathbb{R}^2} \big\{ |g_{n,m}||v\partial_x \phi + \partial^2_{vv}\phi| + |R_{n,m}|\phi \big\} dvdxdt, \end{aligned}$$

for any $0 \leq t_0 \leq t_1 \leq T$, any $n, m \geq 1$, any $\phi \in C^2_c(\mathbb{R}^2)$, first for $\beta \in C^2$, $0 \leq \beta'' \in L^\infty(\mathbb{R})$, $0 \leq \beta(s) \leq |s|$, next for $\beta(s) = |s|$.
Observing that there exists $t_1 \in (0, T)$ such that $f_{\varepsilon_n}(t_1, \cdot) \to f(t_1, \cdot)$ in L^2 as $n \to \infty$, establish that $(f_{\varepsilon_n}\phi)$ is a Cauhy sequence in $C([0, T]; L^1)$ for any $0 \leq \phi \in C^2_c(\mathbb{R}^2)$ and any $T > 0$. Deduce that there exists a function $\widetilde{f} \in C([0, T]; L^1(B_R))$ for any $T, R > 0$ such that $\widetilde{f} = f$ a.e. on $\mathbb{R}_+ \times \mathbb{R}^2$. In the sequel, we adopt the notation f for $\widetilde{f}$.

Question 12 (more difficult)

Repeating the proof of Question 2, establish that (1.4) holds for any $\beta \in C^2$, $\beta', \beta'' \in L^\infty(\mathbb{R})$. Deduce that $f \in L^\infty(0, T; L^2)$ and next that (1.4) holds for any $\beta \in C^2$, $\beta'' \in L^\infty(\mathbb{R})$.

Question 13 (more difficult)

Establish that $f \in C(\mathbb{R}_+; L^2_{weak})$ in the sense that

$$t \mapsto \int_{\mathbb{R}^2} f(t, \cdot)\phi dx \in C(\mathbb{R}_+)$$

for any $\phi \in L^2(\mathbb{R}^2)$ and that $t \mapsto \|f(t, \cdot)\|_{L^2} \in C(\mathbb{R}_+)$. Deduce that (1.3) holds.

2 Problem 2 - The Vlasov-Poisson equation.

In this problem, we consider the (nonlinear) Vlasov-Poisson equation

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + E_f \cdot \nabla_v f = 0 \ \text{ in } \ (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d, \\ f(0,\cdot) = f_0 \ \text{ on } \ \mathbb{R}^d \times \mathbb{R}^d, \end{cases} \tag{2.1}$$

on the function $f = f(t,x,v)$, $t \geq 0$, $x \in \mathbb{R}^d$, $v \in \mathbb{R}^d$, **in dimension** $d = 2$, where

$$E_f = -\nabla_x \Phi_f, \quad -\Delta_x \Phi_f = \rho_f \ \text{ in } \ (0,\infty) \times \mathbb{R}^d,$$

and

$$\rho_f(t,x) := \int_{\mathbb{R}^d} f(t,x,v)dv.$$

We assume that

$$0 \leq f_0 \in (L^1 \cap L^\infty)(\mathbb{R}^{2d}), \quad f_0|v|^2 \in L^1(\mathbb{R}^{2d}), \quad E_{f_0} \in L^2(\mathbb{R}^d). \tag{2.2}$$

The **aim of this problem** is to establsih the existence of a weak solution to the Vlasov-Poisson equation (2.1).

Question 21

Prove formally that

$$\|f(t,\cdot)\|_{L^p(\mathbb{R}^{2d})} = \|f_0\|_{L^p(\mathbb{R}^{2d})}, \quad \forall\, t \geq 0, \ \forall\, p \in [1,\infty],$$

and $f(t,\cdot) \geq 0$ on $\mathbb{R}^{2d}$ for any $t \geq 0$.

Question 22

Prove formally that

$$\frac{d}{dt} \int_{\mathbb{R}^{2d}} f|v|^2 dxdv = 2 \int_{\mathbb{R}^d} j_f \cdot E_f dx,$$

with

$$j_f(t,x) := \int_{\mathbb{R}^d} f(t,x,v)v dv.$$

Prove formally that

$$\partial_t \rho_f = -\mathrm{div}_x j_f$$

and establish next that

$$2 \int_{\mathbb{R}^d} j_f \cdot E_f dx = -2 \int_{\mathbb{R}^d} \partial_t \rho_f \Phi_f = -\frac{d}{dt} \int_{\mathbb{R}^d} |\nabla_x \Phi_f|^2.$$

Deduce that

$$\int_{\mathbb{R}^{2d}} f|v|^2 dxdv + \int_{\mathbb{R}^d} |\nabla_x \Phi_f|^2 = \int_{\mathbb{R}^{2d}} f_0|v|^2 dxdv + \int_{\mathbb{R}^d} |\nabla_x \Phi_{f_0}|^2, \quad \forall\, t \geq 0.$$

Question 23

Using the splitting $\mathbb{R}^2 = B_R \cup B_R^c$ for any $R > 0$, establish that

$$\int g dv \leq C \|g\|_{L^\infty}^{1/2} \||v|^2 g\|_{L^1}^{1/2}, \quad \forall\, g = g(v) \geq 0,$$

for a constant $C > 0$. Deduce that

$$\|\rho_f\|_{L^2} \lesssim \|f\|_{L^\infty_{xv}}^{1/2} \||v|^2 f\|_{L^1_{xv}}^{1/2}, \quad \forall\, f = f(x,v) \geq 0. \tag{2.3}$$

Question 24

Establish that at least formally

$$\|D_x E_f\|_{L^2((0,T)\times\mathbb{R}^d)} = \|\rho_f\|_{L^2((0,T)\times\mathbb{R}^d)}. \tag{2.4}$$

Question 25

We recall the Plancherel identity

$$\int_{\mathbb{R}^2} E\psi = \int_{\mathbb{R}^2} \widehat{E}\check{\psi},$$

for $E, \psi : \mathbb{R}^2 \to \mathbb{R}$, where the Fourier transforms are defined by

$$\widehat{E}(\xi) := \frac{1}{2\pi}\int_{\mathbb{R}^2} E(x) e^{-ix\cdot\xi} dx, \quad \check{\psi}(\xi) := \frac{1}{2\pi}\int_{\mathbb{R}^2} \psi(x) e^{ix\cdot\xi} dx.$$

Establish that if f is a nice solution to the Vlasov-Poisson equation and $\psi = \psi(x)$ is a nice function then

$$\frac{d}{dt}\int_{\mathbb{R}^d} E_f \psi = F, \quad F_i := \int_{\mathbb{R}^2} \frac{\xi_i}{|\xi|^2} \xi \cdot \widehat{\jmath}_f \check{\psi},$$

and next

$$\|F(t,\cdot)\|_{L^\infty} \leq \|j_f\|_{L^1} \|\check{\psi}\|_{L^1}.$$

We recall that for $\psi \in W^{3,1}(\mathbb{R}^2)$, we have $\check{\psi} \in L^1$ and $\|\check{\psi}\|_{L^1} \leq C\|\psi\|_{W^{3,1}}$. Deduce from the above relations that

$$\|E_f *_x \eta\|_{H^1_{tx}((0,T)\times\mathbb{R}^2)} \leq C(\eta, T)(\|fv\|_{L^\infty_t L^1_{xv}} + \|E_f\|_{L^2_{tx}((0,T)\times\mathbb{R}^2)}), \tag{2.5}$$

for any $\eta \in W^{3,1}(\mathbb{R}^2) \cap W^{1,\infty}(\mathbb{R}^2)$.

Question 26

If not proved, we may accept (2.3), (2.4) and (2.5). Consider a sequence (f_n) of solutions to the Vlasov-Poisson equation (2.1) in the distributional sense of $\mathcal{D}'([0,T)\times\mathbb{R}^{2d})$ and assume that

$$\|f_n(t,\cdot)\|_{L^1\cap L^\infty} + \int_{\mathbb{R}^{2d}} f_n(t,\cdot)|v|^2 dxdv + \int_{\mathbb{R}^d} |E_{f_n(t,\cdot)}|^2 \leq C,$$

for any $t \in (0, T)$ and $n \geq 1$. Prove that there exists a subsequence (f_{n_k}) of (f_n) and a function f such that $f_{n_k} \rightharpoonup f$ as $k \to \infty$ and f is a solution to the **nonlinear** Vlasov-Poisson equation (2.1) in the distributional sense of $\mathcal{D}'([0, T) \times \mathbb{R}^{2d})$.

Question 27 (more difficult)

Suggest a strategy for proving the existence of a solution to the Vlasov-Poisson equation (2.1) when the initial datum f_0 satisfies the requirement (2.2).