Exercises on chapters 1 & 2

1. About the Gronwall lemma

Exercice 1.1. *Prove in full generality the following classical differential version of Gronwall lemma.*
Lemma. *We assume that $u \in C([0,T];\mathbb{R})$, $T \in (0,\infty)$, satisfies the differential inequality*

$$u' \le a(t)u + b(t) \quad on \quad (0,T), \tag{1.1}$$

in the distributional sense, for some $a, b \in L^1(0,T)$. Then, u satisfies the pointwise estimate

$$u(t) \le e^{A(t)}u(0) + \int_0^t b(s)e^{A(t)-A(s)}\,ds, \quad \forall\, t \in [0,T), \tag{1.2}$$

where we have defined the primitive function

$$A(t) := \int_0^t a(s)\,ds.$$

Exercice 1.2. *We assume that $u \in C([0,T];\mathbb{R})$, $T \in (0,\infty)$, satisfies the integral inequality*

$$u(t) \le B(t) + \int_0^t a(s)u(s)\,ds \quad on \quad [0,T), \tag{1.3}$$

for some $B \in C([0,T))$ and $0 \le a \in L^1(0,T)$. Prove that u satisfies the pointwise estimate

$$u(t) \le B(t) + \int_0^t a(s)B(s)e^{A(t)-A(s)}\,ds, \quad \forall\, t \in (0,T).$$

1) By considering the function

$$v(t) := \int_0^t a(s)u(s)\,ds.$$

2) By considering the function

$$v(t) := \int_0^t a(s)u(s)\,ds\, e^{-A(t)} - \int_0^t a(s)B(s)e^{-A(s)}\,ds.$$

Recover the fact that $u \in C([0,T];\mathbb{R})$, $T \in (0,\infty)$, satisfies the integral inequality

$$u(t) \le u_0 + \int_0^t a(s)u(s)\,ds + \int_0^t b(s)\,ds \quad on \quad [0,T),$$

for some $0 \le a \in L^1(0,T)$ and $b \in L^1(0,T)$, implies that u satisfies the pointwise estimate

$$u(t) \le u_0\, e^{A(t)} + \int_0^t b(s)e^{A(t)-A(s)}\,ds, \quad \forall\, t \in (0,T)$$

3) first in the case when $b = 0$;
4) next, in the general case.

2. About variational solutions

Exercice 2.1. *Consider $f \in L^1(\mathbb{R}^d)$ such that $\mathrm{div} f \in L^1(\mathbb{R}^d)$. Show that*

$$\int_{\mathbb{R}^d} \mathrm{div} f \, dx = 0.$$

Exercice 2.2. *We consider the nonlinear McKean-Vlasov equation*

$$\partial_t f = \Lambda[f] := \Delta f + div(F[f] f), \quad f(0) = f_0, \tag{2.1}$$

with

$$F[f] := a * f, \quad a \in W^{1,\infty}(\mathbb{R}^d)^d.$$

1) Prove the a priori estimates

$$\|f(t)\|_{L^1} = \|f_0\|_{L^1} \quad \forall\, t \geq 0, \quad \|f(t)\|_{L^2_k} \leq e^{Ct} \, \|f_0\|_{L^2_k} \quad \forall\, t \geq 0,$$

for any $k > 0$ and a constant $C := C(k, \|a\|_{W^{1,\infty}}, \|f_0\|_{L^1})$, where we define the weighted Lebesgue space L^2_k by its norm $\|f\|_{L^2_k} := \|f \langle x \rangle^k\|_{L^2}$, $\langle x \rangle := (1 + |x|^2)^{1/2}$.
2) We set $H := L^2_k$, $k > d/2$, and $V := H^1_k$, where we define the weighted Sobolev space H^1_k by its norm $\|f\|^2_{H^1_k} := \|f\|^2_{L^2_k} + \|\nabla f\|^2_{L^2_k}$. Observe that for any $f \in V$ the distribution $\Lambda[f]$ is well defined in V' thanks to the identity

$$\langle \Lambda[f], g \rangle := - \int_{\mathbb{R}^d} (\nabla f + (a * f) f) \cdot \nabla (g \langle x \rangle^{2k}) \, dx \quad \forall\, g \in V.$$

(Hint. Prove that $L^2_k \subset L^1$). Write the variational formulation associated to the nonlinear McKean-Vlasov equation. Establish that if moreover the variational solution to the nonlinear McKean-Vlasov equation is nonnegative then it is mass preserving, that is $\|f(t)\|_{L^1} = \|f_0\|_{L^1}$ for any $t \geq 0$. (Hint. Take $\chi_M \langle x \rangle^{-2k}$ as a test function in the variational formulation, with $\chi_M(x) := \chi(x/M)$, $\chi \in \mathcal{D}(\mathbb{R}^d)$, $\mathbf{1}_{B(0,1)} \leq \chi \leq \mathbf{1}_{B(0,2)}$).
3) Prove that for any $0 \leq f_0 \in H$ and $g \in C([0,T]; H)$ there exists a unique mass preserving variational solution $0 \leq f \in X_T$ to the linear McKean-Vlasov equation

$$\partial_t f = \Delta f + div(F[g] f), \quad f(0) = f_0.$$

Prove that the mapping $g \mapsto f$ is a contraction in $C([0,T]; H)$ for $T > 0$ small enough. Conclude to the existence and uniqueness of a global (in time) variational solution to the nonlinear McKean-Vlasov equation.

Exercice 2.3. *For $a, c \in L^\infty(\mathbb{R}^d)$, $f_0 \in L^p(\mathbb{R}^d)$, $1 \leq p \leq \infty$, we consider the linear parabolic equation*

$$\partial_t f = \Lambda f := \Delta f + a \cdot \nabla f + c f, \quad f(0) = f_0. \tag{2.2}$$

We introduce the usual notations $H := L^2$, $V := H^1$ and X_T the associated space for some given $T > 0$.
1) Prove that for $\gamma \in C^1(\mathbb{R})$, $\gamma(0) = 0$, $\gamma' \in L^\infty$, there holds $\gamma(f) \in H$ for any $f \in H$ and $\gamma(f) \in V$ for any $f \in V$.
2) Prove that $f \in X_T$ is a variational solution to (2.3) if and only if

$$\frac{d}{dt} f = \Lambda f \ \text{ in } V' \ \text{ a.e. on } (0,T).$$

3) On the other hand, prove that for any $f \in X_T$ and any function $\beta \in C^2(\mathbb{R})$, $\beta(0) = \beta'(0) = 0$, $\beta'' \in L^\infty$, there holds

$$\frac{d}{dt} \int_{\mathbb{R}^d} \beta(f) = \langle \frac{d}{dt} f, \beta'(f) \rangle_{V',V} \ \text{ a.e. on } (0,T).$$

(Hint. Consider $f_\varepsilon = f *_t \rho_\varepsilon \in C^1([0,T]; H^1)$ and pass to the limit $\varepsilon \to 0$).
4) Consider a convex function $\beta \in C^2(\mathbb{R})$ such that $\beta(0) = \beta'(0) = 0$ and $\beta'' \in L^\infty$. Prove that any variational solution $f \in X_T$ to the above linear parabolic equation satisfies

$$\int_{\mathbb{R}^d} \beta(f_t) \, dx \leq \int_{\mathbb{R}^d} \beta(f_0) \, dx + \int_0^t \int_{\mathbb{R}^d} \{ c \, f \, \beta'(f) - (div\, a) \, \beta(f) \} \, dx ds,$$

for any $t \geq 0$.

5) Assuming moreover that there exists a constant $K \in (0,\infty)$ *such that* $0 \leq s\,\beta'(s) \leq K\beta(s)$ *for any* $s \in \mathbb{R}$, *deduce that for some constant* $C := C(a,c,K)$, *there holds*

$$\int_{\mathbb{R}^d} \beta(f_t)\,dx \leq e^{Ct} \int_{\mathbb{R}^d} \beta(f_0)\,dx, \quad \forall\, t \geq 0.$$

6) Prove that for any $p \in [1,2]$, *for some constant* $C := C(a,c)$ *and for any* $f_0 \in L^2 \cap L^p$, *there holds*

$$\|f(t)\|_{L^p} \leq e^{Ct} \|f_0\|_{L^p}, \quad \forall\, t \geq 0.$$

(Hint. Define β *on* $\mathbb{R}_+$ *and extend it to* $\mathbb{R}$ *by symmetry. More precisely, define* $\beta''_\alpha(s) = 2\theta \mathbf{1}_{s\leq\alpha} + p(p-1)s^{p-2}\mathbf{1}_{s>\alpha}$, *with* $2\theta = p(p-1)\alpha^{p-2}$ *and then the primitives which vanish at the origin, which are thus defined by* $\beta'_\alpha(s) = 2\theta s \mathbf{1}_{s\leq\alpha} + (ps^{p-1} + p(p-2)\alpha^{p-1})\mathbf{1}_{s>\alpha}$, $\beta_\alpha(s) = \theta s^2 \mathbf{1}_{s\leq\alpha} + (s^p + p(p-2)\alpha^{p-1}s + A\alpha^p)\mathbf{1}_{s>\alpha}$, $A := p(p-1)/2 - 1 - p(p-2)$. *Observe that* $s\beta'_\alpha(s) \leq 2\beta_\alpha(s)$ *because* $s\beta''_\alpha(s) \leq \beta'_\alpha(s)$ *and* $\beta_\alpha(s) \leq \beta(s)$ *because* $\beta''_\alpha(s) \leq \beta''(s)$*).*

7) Prove that for any $p \in [2,\infty]$ *and for some constant* $C := C(a,c,p)$ *there holds*

$$\|f(t)\|_{L^p} \leq e^{Ct} \|f_0\|_{L^p}, \quad \forall\, t \geq 0.$$

(Hint. Define $\beta''_R(s) = p(p-1)s^{p-2}\mathbf{1}_{s\leq R} + 2\theta \mathbf{1}_{s>R}$, *with* $2\theta = p(p-1)R^{p-2}$, *and then the primitives which vanish in the origin and which are thus defined by* $\beta'_R(s) = ps^{p-1}\mathbf{1}_{s\leq R} + (pR^{p-1} + 2\theta(s-R))\mathbf{1}_{s>R}$, $\beta_R(s) = s^p \mathbf{1}_{s\leq R} + (R^p + pR^{p-1}(s-R) + \theta(s-R)^2)\mathbf{1}_{s>R}$. *Observe that* $s\beta'_R(s) \leq p\beta_R(s)$ *because* $s\beta''_R(s) \leq (p-1)\beta'_R(s)$ *and* $\beta_R(s) \leq \beta(s)$ *because* $\beta''_R(s) \leq \beta''(s)$. *Pass to the limit* $p \to \infty$ *in order to deal with the case* $p = \infty$*).*

8) Prove that for any $f_0 \in L^p(\mathbb{R}^d)$, $1 \leq p \leq \infty$, *there exists at least one weak (in the sense of distributions) solution to the linear parabolic equation* (2.3)*. (Hint: Consider* $f_{0,n} \in L^1 \cap L^\infty$ *such that* $f_{0,n} \to f_0$ *in* L^p, $1 \leq p < \infty$, *and prove that the associate variational solution* $f_n \in X_T$ *is a Cauchy sequence in* $C([0,T];L^p)$. *Conclude the proof by passing to the limit* $p \to \infty$*).*

9) Establsih the L^p *estimates with "optimal" constant* C *(that is the one given by the formal computations).*

10) Extend the above result to an equation with an integral term and/or a source term.

11) Prove the existence of a weak solution to the McKean-Vlasov equation (2.1) *for any initial datum* $f_0 \in L^1(\mathbb{R}^d)$.

12) Prove that $f_0 \geq 0$ *implies* $f(t) \geq 0$ *for any* $t \in (0,T)$*. (Hint. Choose* $\beta(s) := s_-$*).*