CHAPTER 7
THE NAVIER-STOKES EQUATION

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We present some mathematical results on the Navier-Stokes equation about incompressible fluids in dimension $d = 2, 3$.

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1. Introduction, a priori estimate, vortex formulation

We consider the Navier-Stokes equation

$$\left\{ \begin{array}{l} \partial_t u - \nu \Delta u + u \cdot \nabla u + \nabla p = 0 \\ \operatorname{div} u = 0, \end{array} \right. \tag{1.1}$$

on the vector (velocity) field $u : (0, T) \times \Omega \to \mathbb{R}^d$, with $\Omega \subset \mathbb{R}^d$. For the sake of simplicity, we will only consider the case $\Omega = \mathbb{R}^d$ with $d = 2, 3$, and a viscosity coefficient $\nu = 1$. The first equation is vectorial, and it is thus equivalent to

$$\partial_t u_i - \Delta u_i + u \cdot \nabla u_i + \partial_i p = 0, \quad \forall\, i = 1, \dots, d.$$

• **The pressure.** The term ∇p is linked to the vanishing divergence condition and can be interpreted as a "Lagrange multiplicator" associated to this constraint. More precisely, computing the divergence of each term involved in the first vectorial equation, we get

$$-\Delta p = \operatorname{div}(u \cdot \nabla u) = \sum_{ij} \partial^2_{ij} (u_i u_j),$$

where we have used that $\operatorname{div} u = 0$ in the last equality. After having properly defined the inverse of the Laplacian operator, we may thus write

$$p := (-\Delta)^{-1} \Big(\sum_{ij} \partial^2_{ij} (u_i u_j) \Big).$$

We will come back on that fundamental (but technical) point later.

• **The energy identity.** We compute

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 &= \int \partial_t u \cdot u \\ &= -\int (u\cdot\nabla u)\cdot u + \int \Delta u \cdot u - \int \nabla p \cdot u.\end{aligned}$$

Using integrations by part, we compute each term separately. We have

$$\begin{aligned}-\int \nabla p \cdot u &= \int p\,(\operatorname{div} u) = 0, \\ \int \Delta u \cdot u &= -\int |\nabla u|^2 := -\sum_{ij} \int (\partial_i u_j)^2,\end{aligned}$$

and also (with Einstein's convention of summation of repeated indices)

$$\int (u\cdot\nabla u)\cdot u = \int (u_j \partial_j u_i) u_i = \frac{1}{2}\int u_j \partial_j |u|^2 = -\frac{1}{2}\int (\operatorname{div} u)|u|^2 = 0.$$

All together, we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 = -\|\nabla u\|_{L^2}^2,$$

and finally after time integration

$$\|u(t)\|_{L^2}^2 + 2\int_0^t \|\nabla u(s)\|_{L^2}^2\,ds = \|u_0\|_{L^2}^2, \quad \forall\, t \ge 0. \tag{1.2}$$

• **The vortex and the dimension** $d = 2$**.** In dimension $d = 3$, we define the vortex vector field

$$\Omega := \operatorname{curl} u := \nabla \wedge u = \begin{pmatrix} \partial_2 u_3 - \partial_3 u_2 \\ \partial_3 u_1 - \partial_1 u_3 \\ \partial_1 u_2 - \partial_2 u_1 \end{pmatrix}.$$

In dimension $d = 2$, we may associated the $3d$ vector field $\tilde u : \mathbb{R}^3 \to \mathbb{R}^3$, $\tilde u(x_1,x_2,x_3) = (u_1(x_1,x_2), u_2(x_1,x_2), 0)$, so that

$$\widetilde\Omega = \operatorname{curl} \tilde u = \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix}, \quad \omega := \partial_1 u_2 - \partial_2 u_1,$$

and the vortex is thus a scalar. In any dimension, we may verify (that is left as an exercise) that the vortex satisfies the evolution equation

$$\partial_t \Omega + u\cdot\nabla\Omega - \Omega\cdot\nabla u - \Delta\Omega = 0,$$

where we have just used that $\operatorname{curl} \nabla p = (\nabla\wedge\nabla)p = 0$. We deduce from this that in dimension $d = 2$, we have

$$\partial_t \omega + u\cdot\nabla\omega - \Delta\omega = 0,$$

where we have used

$$\operatorname{div} \widetilde\Omega = \nabla\cdot(\nabla\wedge\tilde u) = 0, \quad \text{and then } \widetilde\Omega\cdot\nabla\tilde u = \nabla\cdot(\widetilde\Omega\otimes\tilde u) = 0.$$

We may rather compute directly the evolution equation satisfied by the vortex function ω. We indeed have

$$\begin{aligned}\partial_t\omega &= \partial_1(\Delta u_2 - u\cdot\nabla u_2 - \partial_2 p) - \partial_2(\Delta u_1 - u\cdot\nabla u_1 - \partial_1 p) \\ &= \Delta\omega - u\cdot\nabla\omega + Q,\end{aligned}$$

with

$$\begin{aligned} Q &:= -\partial_1 u \cdot \nabla u_2 + \partial_2 u \cdot \nabla u_1 \\ &= -\partial_1 u_1 \partial_1 u_2 - \partial_1 u_2 \partial_2 u_2 + \partial_2 u_1 \partial_1 u_1 + \partial_2 u_2 \partial_2 u_1 \\ &= \partial_1 u_1 (\partial_2 u_1 - \partial_1 u_2) + \partial_2 u_2 (\partial_2 u_1 - \partial_1 u_2) \\ &= (\operatorname{div} u)\, \omega = 0. \end{aligned}$$

We end with a last observation. We compute

$$\operatorname{curl} \tilde\Omega = \operatorname{curl} \operatorname{curl} \tilde u = (\nabla \operatorname{div} - \Delta)\tilde u = -\Delta \tilde u,$$

so that

$$\tilde u = -\Delta^{-1} \operatorname{curl} \tilde\Omega = -\operatorname{curl} \Delta^{-1} \tilde\Omega = -\nabla \wedge \begin{pmatrix} 0 \\ 0 \\ \Delta^{-1}\omega \end{pmatrix} = \begin{pmatrix} -\partial_2(\Delta^{-1}\omega) \\ \partial_1(\Delta^{-1}\omega) \\ 0 \end{pmatrix}.$$

Recalling that

$$\Delta E = \delta$$

with $E(x) = (2\pi)^{-1} \log |x|$ in dimension $d = 2$, we get

$$u = \begin{pmatrix} \tilde u_1 \\ \tilde u_2 \end{pmatrix} = K * \omega \tag{1.3}$$

with

$$K(x) := \nabla^\perp E(x) = \frac{1}{2\pi} \frac{x^\perp}{|x|^2}, \quad x^\perp = (-x_2, x_1).$$

As a conslusion, in dimension $d = 2$ we obtain the following scalar equation on the vortex

$$\partial_t \omega + (K * \omega) \cdot \nabla \omega - \Delta \omega = 0,$$

from what we may reconstruct the velocy filed thanks to the Biot-Savart equation (1.3).

2. Around the presure issue

Theorem 2.1 (Hodge decomposition and pressure)**.** *For any $T : \Omega \to \mathbb{R}^d$, there exist $\varphi : \Omega \to \mathbb{R}$ and $\psi : \Omega \to \mathbb{R}^d$ such that*

$$T = \operatorname{curl} \psi + \nabla \varphi,$$

with $\psi = 0$ and $T = \nabla\varphi$ if $\operatorname{curl} T = 0$ and $\varphi = 0$ et $T = \operatorname{curl} \psi$ if $\operatorname{div} T = 0$. As a consequence, when $T \perp \{S,\ \operatorname{div} S = 0\}$ there exists $p : \Omega \to \mathbb{R}$ such that $T = \nabla p$.

Idea of the proof. We make the following observations

$$\operatorname{curl} \nabla = 0, \quad \operatorname{div} \operatorname{curl} = 0, \quad \Delta := \nabla \operatorname{div} - \operatorname{curl} \operatorname{curl}$$

We set

$$w := \Delta^{-1} T, \quad \varphi := \operatorname{div} w, \quad \psi := -\operatorname{curl} w,$$

so that

$$T = \Delta w = (\nabla \operatorname{div} - \operatorname{curl} \operatorname{curl})w = \nabla \varphi + \operatorname{curl} \psi.$$

On the other hand, the operator Δ^{-1} commutes with translations and thus with both operators div and curl. To see this in another way, we may write

$$\Delta^{-1} \operatorname{div} = \Delta^{-1} \operatorname{div} (\nabla \operatorname{div} - \operatorname{curl} \operatorname{curl}) \Delta^{-1} = \Delta^{-1} \operatorname{div} \nabla \operatorname{div} \Delta^{-1} = \operatorname{div} \Delta^{-1}$$

and

$$\begin{aligned}\Delta^{-1}\mathrm{curl} &= \Delta^{-1}\mathrm{curl}\,(\nabla\mathrm{div}\, - \mathrm{curl}\,\mathrm{curl}\,)\Delta^{-1} = -\Delta^{-1}\mathrm{curl}\,\mathrm{curl}\,\mathrm{curl}\,\Delta^{-1} \\ &= -\Delta^{-1}(\nabla\mathrm{div}\, - \Delta)\,\mathrm{curl}\,\Delta^{-1} = \Delta^{-1}\Delta\,\mathrm{curl}\,\Delta^{-1} = \mathrm{curl}\,\Delta^{-1}.\end{aligned}$$

Be careful with the fact that we have used $\Delta\Delta^{-1} = I$ what is always true, but also $\Delta^{-1}\Delta = I$ which is less clear, but also true in the case when $\Omega = \mathbb{R}^d$. As a consequence, we have

$$\psi = \mathrm{curl}\,(\Delta^{-1}T) = \Delta^{-1}(\mathrm{curl}\,T) = 0 \quad \text{when} \quad \mathrm{curl}\,T = 0$$

and

$$\varphi = \mathrm{div}\,(\Delta^{-1}T) = \Delta^{-1}(\mathrm{div}\,T) = 0 \quad \text{when} \quad \mathrm{div}\,T = 0.$$

Finally, if $T \perp \{S,\ \mathrm{div}S = 0\}$, we have in particular for any $\phi \in \mathcal{D}(\mathbb{R}^d)$,

$$\langle \mathrm{curl}\,T, \phi\rangle = \langle T, -\mathrm{curl}\,\phi\rangle = 0,$$

because $\mathrm{div}\,\mathrm{curl}\,\phi = 0$. We deduce that $\mathrm{curl}\,T = 0$, and thus $T = \nabla p$. □

We introduce the close sets $\mathcal{V}$ and $\mathcal{R}$ of the Hilbert space $L^2(\mathbb{R}^d)$ of vector field, by defining

$$\begin{aligned}\mathcal{V} &:= \{u \in L^2(\mathbb{R}^d);\ \mathrm{div}\,u = 0\}, \\ \mathcal{R} &:= \{u \in L^2(\mathbb{R}^d);\ \mathrm{curl}\,u = 0\}.\end{aligned}$$

Theorem 2.2 (Hodge decomposition and pressure in $L^2(\mathbb{R}^d)$)**.** *For any vector field $u \in L^2(\mathbb{R}^d)$, there exist a scalar function $\varphi \in \dot H^1(\mathbb{R}^d)$ and a vector field $\psi \in \dot H^1(\mathbb{R}^d)$ such that*

$$u = \mathrm{curl}\,\psi + \nabla\varphi, \quad \|\nabla\varphi\| + \|D\psi\| \lesssim \|u\|,$$

with $\psi = 0$ and thus $u = \nabla\varphi$ if $u \in \mathcal{R}$ and $\varphi = 0$ and thus $u = \mathrm{curl}\,\psi$ if $u \in \mathcal{V}$. As a consequence, when $u \perp \mathcal{V}$ there exists $p \in \dot H^1$ such that $u = \nabla p$.

Idea of the proof. In the Fourier side, everything becomes easier. □

Theorem 2.3 (pressure again)**.** *We define the projection*

$$\mathbf{P} = I - \nabla\Delta^{-1}\mathrm{div}\,.$$

There is equivalence between

$$\begin{aligned}&(1) \quad T = \nabla p, \\ &(2) \quad \mathbf{P}T = 0, \\ &(3) \quad T \perp \mathcal{V}.\end{aligned}$$

Idea of the proof. (1) ⇒ (2) For $T = \nabla p$, we have

$$\mathbf{P}T = T - \nabla\Delta^{-1}\mathrm{div}\,\nabla p = T - \nabla p = 0.$$

(2) ⇒ (3) Assume $\mathbf{P}T = 0$. For any $v \in \mathcal{V}$, there holds $\mathbf{P}v = v$, and therefore

$$(T, v) = (T, \mathbf{P}v) = (\mathbf{P}T, v) = 0.$$

That means $T \perp \mathcal{V}$.
(3) ⇒ (1) That follows immediately from the Hodge decomposition. □

3. Leray solutions to the Navier-Stokes equation

We consider the Navier-Stokes equation

$$\left\{\begin{array}{l} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = 0 \\ \operatorname{div} u = 0, \end{array}\right. \tag{3.1}$$

on the vector fields $u : (0,T) \times \Omega \to \mathbb{R}^d$, with $d = 2, 3$.

Theorem 3.1. *For any $u_0 \in \mathcal{V}$, there exists at least one weak global solution*

$$u \in X_T := L^\infty(0,T;\mathcal{V}) \cap L^2(0,T;H^1), \quad \forall T > 0,$$

in the sense that

$$\int u_t \cdot \psi_t \, dx + \int_0^t \int \nabla u : \nabla \psi \, dxds =$$
$$= \int u_0 \cdot \psi_0 \, dx + \int_0^t \int u \otimes u : \nabla \psi \, dxds - \int_0^t \int u \cdot \partial_s \psi \, dxds,$$

for any $t \in (0,T)$ and $\psi \in C^1$, $\operatorname{div} \psi = 0$. Moreover, the following energy inequality holds

$$\|u(t)\|_{L^2}^2 + 2 \int_0^t \|\nabla u(s)\|_{L^2}^2 \, ds \le \|u_0\|_{L^2}^2, \quad \forall t \ge 0. \tag{3.2}$$

We accept that the above Navier-Stokes equation is equivalent to the two following formulations

$$\partial_t u - \Delta u - \mathbf{P}(u \cdot \nabla u) = 0 \tag{3.3}$$

and

$$\partial_t u - \Delta u - u \cdot \nabla u \perp \mathcal{D} \otimes \mathcal{V}. \tag{3.4}$$

We observe that (3.1) $\Rightarrow$ (3.3) is just a consequence of the fact that $\mathbf{P}u = u$ because $\operatorname{div} u = 0$ and the fact that $\mathbf{P}\nabla p = 0$.

There exist two classical strategies in order to establish the Leray Theorem 3.1 about existence of solutions.

A first way consists in considering a Friedrich discretization scheme

$$\partial_t u_n - S_n[\Delta u_n - \mathbf{P}\operatorname{div}(S_n u_n \otimes S_n u_n)] = 0,$$

where S_n is a finite dimensional range operator for which existence of solutions is given by the Cauchy-Lipschitz on ODE, and then to pass to the limit $n \to \infty$.

A second way consists in considering the regularized equation

$$\partial_t u_\varepsilon - \Delta u_\varepsilon + (\rho_\varepsilon * u_\varepsilon) \cdot \nabla u_\varepsilon + \nabla p_\varepsilon = 0, \tag{3.5}$$

for a sequence of mollifiers (ρ_ε) and then to pass to the limit $\varepsilon \to 0$.

We will follow that second way but we rather start by proving a stability principle.

Theorem 3.2. *Consider a sequence $(u_{0,n})$ of $\mathcal{V}$ such that $u_{0,n} \to u_0$ in L^2 and a sequence of associated Leray weak global solutions (u_n). Then, there exists $u \in X_T$ and a subsequence $(u_{n'})$ such that $u_{n'} \to u$ and u is a Leray weak global solution associated to u_0.*

In the proof we use in a crucial way the following classical compactness lemma.

Lemma 3.3 (Aubin-Lions)**.** *Consider a sequence (u_n) which satisfies*

$$\begin{aligned}&(i) \quad (u_n) \ \text{ is bounded in } \ L^2_{tx},\\ &(ii) \quad (\partial_t u_n) \ \text{ is bounded in } \ L^2_t(H^{-s}_x), \ s \in \mathbb{R}_+,\\ &(iii) \quad (\nabla_x u_n) \ \text{ is bounded in } \ L^2_{tx}.\end{aligned}$$

Then, there exists $u \in L^2_{tx}$ and a subsequence $(u_{n'})$ such that $u_{n'} \to u$ strongly in $L^2((0,T) \times B_R)$ as $n \to \infty$ for any $R > 0$.

Idea of the proof. Step 1. We may write $\partial_t u_n = D^s g_n$ with (g_n) bounded in L^2_{tx}. We introduce a sequence of mollifiers (ρ_ε), that is $\rho_\varepsilon(x) := \varepsilon^{-d}\rho(\varepsilon^{-1}x)$ with $0 \le \rho \in \mathcal{D}(\mathbb{R}^d)$, $\langle \rho \rangle = 1$. We observe that

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^d} u_n(t,y)\,\rho_\varepsilon(x-y)\,dx = \int_{\mathbb{R}^d} g_n(t,y)\,D^s\rho_\varepsilon(x-y)\,dy,$$

where the RHS term is bounded in $L^2((0,T)\times\mathbb{R}^d)$ uniformly in n for any fixed $\varepsilon > 0$. We also clearly have

$$\nabla_x \int_{\mathbb{R}^d} u_n(t,y)\,\rho_\varepsilon(x-y)\,dx = -\int_{\mathbb{R}^d} u_n \nabla_y \rho_\varepsilon(x-y)\,dy,$$

where again the RHS term is bounded in $L^2((0,T)\times\mathbb{R}^d)$ uniformly in n for any fixed $\varepsilon > 0$. In other words, $u_n * \rho_\varepsilon$ is bounded in $H^1((0,T)\times\mathbb{R}^d)$. Thanks to the Rellich-Kondrachov Theorem, we get that (up to the extraction of a subsequence) $(u_n * \rho_\varepsilon)_n$ is strongly convergent in $L^2((0,T)\times B_R)$, for any $R > 0$. On the other hand, from (i) and the Banach-Alaoglu weak compacteness theorem, we know that there exists $u \in L^2_{tx}$ and a subsequence $(u_{n'})$ such that $u_{n'} \rightharpoonup u$ weakly in L^2_{tx}. All together, for any fixed $\varepsilon > 0$, we then get

$$u_n * \rho_\varepsilon \to u * \rho_\varepsilon \text{ strongly in } L^2((0,T)\times B_R) \text{ as } n \to \infty.$$

Step 2. We now observe that

$$\begin{aligned}\int_{(0,T)\times\mathbb{R}^d} |w - w*\rho_\varepsilon|^2\,dxdt &= \int_{(0,T)\times\mathbb{R}^d}\Big|\int_{\mathbb{R}^d}(w(t,x)-w(t,x-y))\rho_\varepsilon(y)\,dy\Big|^2 dxdt\\ &= \int_{(0,T)\times\mathbb{R}^d}\Big|\int_{\mathbb{R}^d}\int_0^1 \nabla_x w(t,z_s)\cdot y\rho_\varepsilon(y)\,dsdy\Big|^2 dxdt,\end{aligned}$$

with $z_s := x + sy$. From the Jensen (or Cauchy-Schwarz) inequality, we deduce

$$\begin{aligned}\int_{(0,T)\times\mathbb{R}^d} |w - w*\rho_\varepsilon|^2\,dxdt &\le \varepsilon^2\int_{(0,T)\times\mathbb{R}^d}\int_{\mathbb{R}^d}\int_0^1 |\nabla_x w(t,z_s)|^2\frac{1}{\varepsilon^d}\frac{|y|^2}{\varepsilon^2}\rho\big(\frac{y}{\varepsilon}\big)\,dsdydxdt\\ &\le \varepsilon^2\int_{(0,T)\times\mathbb{R}^d}|\nabla_x w(t,z)|^2 dtdz\int_{\mathbb{R}^d}|z|\rho(z)\,dy\\ &\le \varepsilon^2 C_\rho\|\nabla_x w\|^2_{L^2_{tx}}.\end{aligned}$$

We conclude that $u_n \to u$ in $L^2((0,T)\times B_R)$ by writing

$$u_n - u = (u_n - u_n*\rho_\varepsilon) + (u_n*\rho - u*\rho) + (u*\rho_\varepsilon - u)$$

and using the previous convergence and estimates. □

Idea of the proof of Theorem 3.2. From an interoplation theorem and Sobolev embedding, in dimension $d = 2$, we have

$$L^\infty_t(L^2)\cap L^2_t(\dot H^1) \subset L^4_t(\dot H^{1/2}) \subset L^4_{tx},$$

because for $s = 1/2$, the relation $1/s^* = 1/2 - s/d$ gives $s^* = 4$. We deduce that

$$u_n \otimes u_n \text{ is bounded in } L^2((0,T) \times \mathbb{R}^2).$$

Similarly, in dimension $d = 3$, we have

$$L^\infty_t(L^2) \cap L^2_t(\dot H^1) \subset L^4_t(\dot H^{1/2}) \subset L^4_t(L^3),$$

because for $s = 1/2$, the relation $1/s^* = 1/2 - s/d$ gives $s^* = 3$. Because of the same Sobolev embedding, we have $L^{3/2} \subset \dot H^{-1/2}$, and then

$$u_n \otimes u_n \text{ is bounded in } L^2_t(L^{3/2}(\mathbb{R}^3)) \subset L^2_t(\dot H^{-1/2}(\mathbb{R}^3)).$$

We recall the definition of the projector

$$\mathbf{P} = I - \nabla \Delta^{-1} \mathrm{div}\,,$$

which in the Fourier side also writes

$$\hat{\mathbf{P}} = I - \frac{\xi \otimes \xi}{|\xi|^2}.$$

Arguing in the Fourier side, we see that

$$\mathbf{P}\mathrm{div}\,(u_n \otimes u_n) \text{ is bounded in } L^2_t(H^s(\mathbb{R}^d)),$$

with $s = -1$ if $d = 2$ and $s = -3/2$ if $d = 3$. As a consequence, we find

$$\partial_t u_n = \Delta u_n - \mathbf{P}\mathrm{div}\,(u_n \otimes u_n),$$

which is bounded in $L^2(0,T; H^s(\mathbb{R}^d))$, for the same values of s as above when $d = 2, 3$. From, Aubin-Lions' lemma, we deduce that $u_n \to u$ strongly L^2_{loc} and then $u_n \otimes u_n \to u \otimes u$ strongly L^1_{loc}. As a consequence, we may pass to the limit in the weak formulation in the sense that

$$\begin{aligned}\int u_n(t) \cdot \psi(t)\, dx &= -\int_0^t \int \nabla u_n : \nabla \psi \, dxds + \int_0^t \int u_n \otimes u_n : \nabla \psi \, dxds \\ &\quad + \int u_{0,n} \cdot \psi_0 \, dx + \int_0^t \int u_n \cdot \partial_t \psi \, dxds,\end{aligned}$$

for any $\psi \in C^1_c(\mathbb{R}^{d+1})$, $\mathrm{div}_x \psi = 0$, and we get that u also satisfies the same weak formulation of the Navier-Stokes equation. □

Idea of the proof of Theorem 3.1. We now consider the equation (3.5), and we start with considering the linear mapping which to a vector field $u : (0,T) \times \mathbb{R}^d \to \mathbb{R}^d$ associates the solution $v : (0,T) \times \mathbb{R}^d \to \mathbb{R}^d$ to the linear equation

$$\left\{ \begin{array}{l} \partial_t v - \Delta v + (u * \rho_\varepsilon) \cdot \nabla v + \nabla p = 0 \\ \mathrm{div}\, v = 0, \quad v(0) = u_0. \end{array} \right. \tag{3.6}$$

Let us make the problem more precise. Fix $u_0 \in \mathcal{V}$ and (ρ_ε) a mollifier. We define

$$\mathcal{X}_T := C([0,T]; \mathcal{V}) \cap L^2(0,T; H^1(\mathbb{R}^d)) \cap H^1(0,T; H^{-1}(\mathbb{R}^d)),$$

and for $u, v, \psi \in \mathcal{X}_T$, we may multiply (3.6) and integrate the resulting equation in order to get (at least formally)

$$\begin{aligned}\int v_t \cdot \psi_t \, dx &= -\int_0^t \int \nabla v : \nabla \psi \, dxds + \int_0^t \int (u * \rho_\varepsilon) \otimes v : \nabla \psi \, dxds \\ &\quad + \int u_0 \cdot \psi_0 \, dx + \int_0^t \int v \cdot \partial_t \psi \, dxds.\end{aligned}$$

It is worth emphasizing that because of $\operatorname{div} u = 0$, we also have $\operatorname{div}(u * \rho_\varepsilon) = 0$, and then $(u*\rho_\varepsilon)\cdot\nabla v = \operatorname{div}((u*\rho_\varepsilon)\otimes v)$. That is nothing but the J.-L. Lions' variational formulation of (3.6) when making of spaces and operator

$$H := \mathcal{V}, \quad V := H^1, \quad \Lambda_t v := \Delta v - (u_t * \rho_\varepsilon)\cdot\nabla v - \nabla p,$$

where p stands for a Lagrange multiplicator. We have to show that $-\Lambda_t$ is bounded and satisfies *Gårding's inequality*. We indeed observe that $\mathcal{V}$ endowed with the L^2 scalar product is an Hilbert space (because it is closed in L^2) and next for any $v, \psi \in \mathcal{V}\cap H^1$, we have

$$|(\Lambda_t v, \psi)_{L^2}| = \Big|\int ((u_t*\rho_\varepsilon)\otimes v + \nabla v) : \nabla\psi\Big| \le C_\varepsilon \|v\|_{H^1}\|\psi\|_{H^1},$$

using that $\|u(t)*\rho_\varepsilon\|_\infty \le \|u(t)\|_{L^2}\|\rho_\varepsilon\|_{L^2} \le C_\varepsilon$, and

$$(\Lambda_t v, v)_{L^2} = -\int ((u_t*\rho_\varepsilon)\otimes v + \nabla v) : \nabla v \le -\frac{1}{2}\|v\|^2_{H^1} + C_\varepsilon\|v\|^2_{L^2}.$$

From J.-L. Lions theorem, we therefore establish the existence and uniqueness of a variational solution $v \in \mathcal{X}_T$ to the linear equation (3.6) and that one satisfies the energy identity

$$\|v(t)\|^2_{L^2} + 2\int_0^t \|\nabla v(s)\|^2_{L^2}\,ds = \|u_0\|^2_{L^2}, \quad \forall\, t\in(0,T),$$

by just repeating the arguments leading to (1.2).

Denoting $v := \Phi(u)$, we have shown $\Phi : \mathcal{X}_T \to \mathcal{X}_T$. Considering $u_1, u_2 \in \mathcal{X}_T$ with $\|u_i(t)\|_2 \le \|u_0\|_2$ for any $t \in (0,T)$ and denoting $v_i := \Phi(u_i)$, $v = v_2 - v_1$, $u = u_2 - u_1$, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int |v|^2 &= \int (\Delta v_2 - (u_2*\rho_\varepsilon)\cdot\nabla v_2 - \nabla p_2)\cdot v \\
&\quad - \int (\Delta v_1 - (u_1*\rho_\varepsilon)\cdot\nabla v_1 - \nabla p_1)\cdot v \\
&= \int (\nabla v_2 + (u_2*\rho_\varepsilon)\otimes v_2) : \nabla v - \int (\nabla v_1 + (u_1*\rho_\varepsilon)\otimes v_1) : \nabla v \\
&= \int |\nabla v|^2 + \int [(u_2*\rho_\varepsilon)\otimes v) + (u*\rho_\varepsilon)\otimes v_1)] : \nabla v.
\end{aligned}$$

As a consequence, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|v\|_2^2 &\le \int |(u_2*\rho_\varepsilon)\otimes v|^2 + \int |(u*\rho_\varepsilon)\otimes v_1)|^2 \\
&\le \|u_2*\rho_\varepsilon\|^2_\infty \int |v|^2 + \|u*\rho_\varepsilon\|^2_\infty \int |v_1|^2 \\
&\le C_\varepsilon \|u_0\|_2^2\,\|v\|_2^2 + C_\varepsilon\|u\|_2^2\,\|u_0\|_2^2,
\end{aligned}$$

and from the Gronwall lemma, we deduce

$$\|v_t\|_2^2 \le \int_0^t C e^{C(t-s)}\|u_s\|_2^2\,ds, \quad C := C_\varepsilon\,\|u_0\|_2^2,$$

and finally

$$\sup_{[0,T^*]}\|v_t\|_2^2 \le [e^{CT^*} - 1]\,\sup_{[0,T^*]}\|u_t\|_2^2.$$

As a conclusion, when $T^* > 0$ is small enough, Φ is a contraction in $\mathcal{X}_{T^*}$, so that Φ admits a unique fixed point which is thus the unique solution to the regularized Navier-Stokes equation (3.6).
Let us come back to the above computation in order to make it completely rigorous when we have to deal with variational solutions. With the same notations $v_i := \Phi(u_i)$, $v = v_2 - v_1$, $u = u_2 - u_1$ and $\psi = v$, we write first

$$\begin{aligned}
\frac{1}{2}\|v(t)\|^2 &= \|v(t)\|^2 - \int_0^t \langle v'(s), v(s)\rangle \, ds \\
&= (v_2(t), \psi_t) - (u_0, \psi_0) - \int_0^t \langle \psi_s', v_1(s)\rangle \, ds \\
&\qquad -(v_1(t), \psi_t) + (u_0, \psi_0) + \int_0^t \langle \psi_s', v_1(s)\rangle \, ds \\
&= -\int_0^t \int \nabla v_2 : \nabla\psi \, dxds + \int_0^t \int (u_2 * \rho_\varepsilon) \otimes v_2 : \nabla\psi \, dxds \\
&\qquad \int_0^t \int \nabla v_1 : \nabla\psi \, dxds - \int_0^t \int (u_1 * \rho_\varepsilon) \otimes v_1 : \nabla\psi \, dxds \\
&= -\int_0^t \int [\nabla v + (u_2 * \rho_\varepsilon) \otimes v_2 - (u_1 * \rho_\varepsilon) \otimes v_1] : \nabla v \, dxds,
\end{aligned}$$

and we may then estimate the RHS term similarly as we did above.

4. The Navier-Stokes equation in vortex formulation in 2D

In this section, we consider the Navier-Stokes equation in dimension $d = 2$ in its vortex formulation, that we will simply call from now on the *vortex equation*, namely

$$\partial_t \omega + u \cdot \nabla\omega - \Delta\omega = 0, \tag{4.1}$$

where u is given that the Biot-Savart equation (1.3).

$$u := K * \omega, \quad K(x) := \frac{1}{2\pi} \frac{x^\perp}{|x|^2}, \quad x^\perp = (-x_2, x_1).$$

Theorem 4.1. *For any $\omega_0 \in L^1(\mathbb{R}^2)$, there exists a unique global solution to the vortex equation*

$$\omega \in C([0,\infty); L^1(\mathbb{R}^2)) \cap C((0,\infty); L^\infty(\mathbb{R}^2))$$

such that

$$\|\omega(t)\|_{L^1} \le \|\omega_0\|_{L^1}, \quad t^{1-\frac{1}{p}} \|\omega(t)\|_{L^p} \le C\|\omega_0\|_{L^1},$$

for any $t \ge 0$ and any $p \in (1,\infty]$ for a universal constant $C > 0$. The equation has to be understood in the mild sense

$$\omega_t = e^{\Delta t}\omega_0 - \int_0^t e^{(t-s)\Delta}(u_s \cdot \nabla\omega_s) \, ds,$$

where $e^{\Delta t}$ stands for the heat semigroup.

We will not give the proof in the full generality, but rather explain how to proceed in the particular case $\omega_0 \in L^1 \cap L^\rho$, $\rho > 1$. That case yet contains the main difficulties and the proof we will see thus contains the main tools in order to overcome them.

• We start with some a priori estimates. For $p \in [1,\infty)$, we compute

$$\begin{aligned}\frac{d}{dt}\int |\omega|^p\,dx &= p\int \omega|\omega|^{p-2}\Delta\omega\,dx - p\int \omega|\omega|^{p-2}\,u\cdot\nabla\omega\,dx\\ &= -p(p-1)\int |\omega|^{p-2}|\nabla\omega|^2\,dx - \int u\cdot\nabla|\omega|^p\,dx\\ &= -4\,\frac{p-1}{p}\int |\nabla(\omega|\omega|^{p/2-1})|^2\,dx + \int(\operatorname{div} u)|\omega|^p\,dx\\ &= -4\,\frac{p-1}{p}\int |\nabla(\omega|\omega|^{p/2-1})|^2\,dx,\end{aligned}$$

by performing two integrations by parts. We deduce

$$\|\omega_t\|_{L^p} \le \|\omega_0\|_{L^p}, \quad \forall\, t\ge 0,\ \forall\, p\in[1,\infty], \tag{4.2}$$

as well as

$$\int_0^\infty \|\nabla(\omega_s|\omega_s|^{p/2-1})\|_{L^2}\,ds \lesssim \|\omega_0\|_{L^p}, \qquad \forall\, p\in(1,\infty). \tag{4.3}$$

• We now briefly explain a possible strategy in order to obtain the existence of solution which consists to follow the same line of arguments as in the previous section when we have considered the NS equation in dimension $d = 2, 3$. We introduce the kernel $K_\varepsilon := K * \rho_\varepsilon$ for a sequence of mollifiers (ρ_ε) and we first consider the regularized equation

$$\partial_t\omega - \Delta\omega + K_\varepsilon * \omega\cdot\nabla\omega = 0, \quad \omega(0) = \omega_0. \tag{4.4}$$

That equation may be tackled thanks to the classical way. We define the mapping $g \mapsto f$, where f is the solution to the linear equation

$$\partial_t f - \Delta f + K_\varepsilon * g\cdot\nabla f = 0, \quad f(0) = \omega_0.$$

When furthemore, $\omega_0 \in L^2$, we prove that this mapping has a fixed point in the J.-L.-Lions space X_T for $T > 0$ small enough, and we get then a global solution $\omega_\varepsilon \in X_T$, $\forall T > 0$, to the regularized equation (4.4) by repeating the argument (valid on a small intervalle $(0,T)$, with $T := T(\|\,\omega_0\|_{L^2})$). When $p = 2$, we may pass to the limit $\varepsilon \to 0$ exacltly as in the previous section and in that way we obtain a solution

$$\omega \in C(\mathbb{R}_+; L^2_w) \cap L^\infty(\mathbb{R}_+; L^2) \cap L^2(\mathbb{R}_+, \dot H^1)$$

to the vortex equation (4.1). For the more general case $p > 1$, we may also proceed similarly by adapting Aubin-Lions' lemma to a L^p framework, we however do not developp further that issue.

• We finally concentrate on the uniqueness issue for solutions which belong to $L^\infty(\mathbb{R}_+; L^\rho(\mathbb{R}^2))$, $\rho > 1$. We start with several intermediate results of independant interest.

Lemma 4.2. *For any $1 \le r \le p \le \infty$ and for any $g \in L^q(\mathbb{R}^2)$, there holds*

$$\|e^{t\Delta} g\|_{L^p(\mathbb{R}^2)} \lesssim \frac{1}{t^{\frac1r - \frac1p}}\,\|g\|_{L^r(\mathbb{R}^2)}$$

and

$$\|\nabla e^{t\Delta} g\|_{L^p(\mathbb{R}^2)} \lesssim \frac{1}{t^{\frac12 + \frac1r - \frac1p}}\,\|g\|_{L^r(\mathbb{R}^2)}.$$

The proof is straightforward by writing the kernel representation of the heat semigroup.

We also recall the following formulation of the Holder inequality

$$\|f\,g\|_{L^r} \le \|f\|_{L^\beta}\|g\|_{L^p}, \quad \frac{1}{r} = \frac{1}{\beta} + \frac{1}{p}, \tag{4.5}$$

for $\beta, p \ge 1$ such that $1/\beta + 1/p \le 1$.

We finally accept the following consequence of the Hardy-Littlewood-Sobolev (HLS) inequality

$$\|K * g\|_{L^\beta(\mathbb{R}^2)} \le C_q\, \|g\|_{L^q(\mathbb{R}^2)}, \quad \frac{1}{\beta} = \frac{1}{q} - \frac{1}{2}, \tag{4.6}$$

which holds true for any $q \in (1,2)$. Recalling the Young inequality about convolution product stipulating that

$$\|f * g\|_{L^\beta} \le \|f\|_{L^p}\|g\|_{L^q}, \quad \forall\, f \in L^p,\ g \in L^q, \tag{4.7}$$

for any $p, q \in [1,+\infty]$ such that $1/p + 1/q < 1$ and with

$$\frac{1}{\beta} := \frac{1}{q} + \frac{1}{p} - 1,$$

the HLS inequality can be seen as a critical case of the Young inequality in the case when $f = K \notin L^2(\mathbb{R}^2)$.

Let us start the uniqueness proof and thus consider two solutions ω^i, $i = 1,2$, to the vortex equation that we write in mild form

$$\omega^i_t = e^{t\Delta}\omega_0 - \int_0^t e^{(t-s)\Delta}(\mathrm{div}\,(u_s\omega_s))\,ds.$$

Introducing the difference $\omega := \omega^2 - \omega^1$ and $u = u^2 - u^1$, we have

$$\omega_t = \int_0^t e^{(t-s)\Delta}(\mathrm{div}\,(u_s\omega^2_s))\,ds + \int_0^t e^{(t-s)\Delta}(\mathrm{div}\,(u^1_s\omega_s))\,ds =: \mathcal{Q}^2_t + \mathcal{Q}^1_t.$$

Let us fix $p \in (1, \min(\rho, 2))$ and observe that

$$\|\omega^i\|_{p,t} \le t^{1-1/p}\|\omega_0\|_{L^p} \to 0, \quad \text{as } t \to 0,$$

where we have set

$$\|f\|_{p,t} := \sup_{s\in(0,t)} [s^{1-1/p}\|f_s\|_{L^p}].$$

We estimate

$$\begin{aligned}
\|\mathcal{Q}^1_t\|_{L^p} \quad &\le \quad \int_0^t \|e^{(t-s)\Delta}(\mathrm{div}\,(u^1_s\omega_s))\|_{L^p}\ \ ds \\
&\lesssim \quad \int_0^t \frac{1}{(t-s)^{\frac12+\frac1r-\frac1p}}\|u^1_s\omega_s\|_{L^r}\ \ ds \\
&\lesssim \quad \int_0^t \frac{1}{(t-s)^{\frac1p}}\|u^1_s\|_{L^\beta}\|\omega_s\|_{L^p}\ \ ds \\
&\lesssim \quad \int_0^t \frac{1}{(t-s)^{\frac1p}}\|\omega^1_s\|_{L^q}\|\omega_s\|_{L^p}\ \ ds,
\end{aligned}$$

where we have successively used the second estimate on the heat semigroup as stated in Lemma 4.2 with r defined by

$$\frac{1}{r} = \frac{2}{p} - \frac{1}{2},$$

next the Holder inequality (4.5) with

$$\frac{1}{\beta} = \frac{1}{r} - \frac{1}{p},$$

and finally the HLS inequality with

$$\frac{1}{\beta} = \frac{1}{q} - \frac{1}{2}.$$

From the three above definitions, we have

$$\frac{1}{q} = \frac{1}{\beta} + \frac{1}{2} = \frac{1}{r} - \frac{1}{p} + \frac{1}{2} = \frac{1}{p}.$$

As a consequence $q = p$, and we may write

$$\begin{aligned} \|\mathcal{Q}^1_t\|_{L^p} &\lesssim \int_0^t \frac{ds}{(t-s)^{\frac{1}{p}} s^{2(1-\frac{1}{p})}} \|\omega^1\|_{p,t} \|\omega\|_{p,t} \\ &\lesssim C \frac{1}{t^{1-\frac{1}{p}}} \|\omega^1\|_{p,t} \|\omega\|_{p,t} \end{aligned}$$

with the constant

$$C := \int_0^1 \frac{du}{(1-u)^{\frac{1}{p}} u^{2(1-\frac{1}{p})}}$$

is finite because of the choice of $p \in (1,2)$. The same computations lead to As a consequence $q = p$, and we may write

$$\|\mathcal{Q}^2_t\|_{L^p} \lesssim C \frac{1}{t^{1-\frac{1}{p}}} \|\omega^2\|_{p,t} \|\omega\|_{p,t}$$

Coming back to the estimate on the mild formulation, we deduc e

$$\begin{aligned} t^{1-\frac{1}{p}} \|\omega_t\|_{L^p} &\leq C[\|\omega^1\|_{p,t} + \|\omega^1\|_{p,t}] \; \|\omega\|_{p,t} \\ &\leq C[t^{1-1/p} \|\omega_0\|_{L^p}] \; \|\omega\|_{p,t}, \end{aligned}$$

so that

$$\|\omega\|_{p,t} \leq \frac{1}{2} \|\omega\|_{p,t},$$

for $t > 0$ small enough, and that ends the proof of the uniqueness.

5. Interpolation inequalities

About interpolation inequalities :

- $L^p \cap L^q \subset L^r$ for any $p \leq r \leq q$ and

$$\|u\|_{L^r} \leq \|u\|^{\theta}_{L^p} \|u\|^{1-\theta}_{L^q}, \quad \frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}, \quad \forall\, u \in L^p \cap L^q.$$

We use the Holder inequality and we then write

$$\begin{aligned}\|u\|_{L^r} &= \Big(\int |u|^{r\theta}|u|^{r(1-\theta)}\Big)^{1/r}\\ &\le \Big(\int |u|^{r\theta s}\Big)^{1/(rs)}\Big(\int |u|^{s'r(1-\theta)}\Big)^{1/(rs')},\end{aligned}$$

with $1/s+1/s'=1$ choosen in such a way that $r\theta s=p$. As a consequence, $1/(rs)=\theta/p$ and

$$\frac{1}{rs'}=\frac{1}{r}\big(1-\frac{1}{s}\big)=\frac{1}{r}\big(1-\frac{\theta r}{p}\big)=\frac{1}{r}-\frac{\theta}{p}=\frac{1-\theta}{q},$$

and that ends the proof of the first interpolation inequality.

- $L^{p_1}(L^{p_2})\cap L^{q_1}(L^{q_2})\subset L^{r_1}(L^{r_2})$, and more precisely

$$\|u\|_{L^{r_1}(L^{r_2})}\le \|u\|^{\theta}_{L^{p_1}(L^{p_2})}\|u\|^{1-\theta}_{L^{q_1}(L^{q_2})},\qquad \frac{1}{r_i}=\frac{\theta}{p_i}+\frac{1-\theta}{q_i}$$

for any $u\in L^{p_1}(L^{p_2})\cap L^{q_1}(L^{q_2})$ for <u>a same</u> $\theta\in(0,1)$.
We proceed similarly as above. Thanks to the first interpolation inequality and the Holder inequality, we then write

$$\begin{aligned}\|u\|_{L^{r_1}(L^{r_2})} &= \Big(\int |\|u\|^{r_1}_{L^{r_2}}\Big)^{1/r_1}\\ &\le \Big(\int \|u\|^{r_1\theta}_{L^{p_2}}\|u\|^{r_1(1-\theta)}_{L^{q_2}}\Big)^{1/r_1}\\ &\le \Big(\int \|u\|^{r_1\theta s}_{L^{p_2}}\Big)^{1/(r_1 s)}\Big(\int \|u\|^{r_1(1-\theta)s'}_{L^{q_2}}\Big)^{1/(r_1 s')},\end{aligned}$$

with $1/s+1/s'=1$ choosen in such a way that $r_1\theta s=p_1$. As a consequence, $1/(r_1 s)=\theta/p_1$ and

$$\frac{1}{r_1 s'}=\frac{1}{r_1}\big(1-\frac{1}{s}\big)=\frac{1}{r_1}\big(1-\frac{\theta r_1}{p_1}\big)=\frac{1}{r}-\frac{\theta}{p_1}=\frac{1-\theta}{q_1},$$

and that ends the proof of the second interpolation inequality.

- $\dot H^{s_1}\cap\dot H^{s_2}\subset\dot H^{s}$ for any $s_1\le s\le s_2$ and

$$\|u\|_{\dot H^s}\le\|u\|^{\theta}_{\dot H^{s_1}}\|u\|^{1-\theta}_{\dot H^{s_2}},\quad s=\theta s_1+(1-\theta)s_2,\quad \forall\, u\in\dot H^{s_1}\cap\dot H^{s_2}.$$

On the Fourier side, we have

$$\begin{aligned}\|u\|^2_{\dot H^s} &= \int |\hat u|^{2\theta}|\xi|^{2\theta s_1}|\hat u|^{2(1-\theta)}|\xi|^{2(1-\theta)s_2}\,d\xi\\ &\le \Big(\int |\hat u|^2|\xi|^{2s_1}\,d\xi\Big)^{\theta}\Big(\int |\hat u|^2|\xi|^{2s_2}\,d\xi\Big)^{1-\theta},\end{aligned}$$

by using the Holder inequality with exponent $p=1/\theta$ and $p'=1/(1-\theta)$.

- $L^p(\dot H^a)\cap L^q(\dot H^b)\subset L^r(\dot H^c)$, and more precisely

$$\|u\|_{L^r(\dot H^c)}\le\|u\|^{\theta}_{L^p(\dot H^a)}\|u\|^{1-\theta}_{L^q(\dot H^b)},\quad \frac{1}{r}=\frac{\theta}{p}+\frac{1-\theta}{q},\ c=\theta a+(1-\theta)b$$

for any $u\in L^p(\dot H^a)\cap L^q(\dot H^b)$ for <u>a same</u> $\theta\in(0,1)$.
We write indeed

$$\|u\|_{L^r(\dot H^c)}\le\|\|u\|^{\theta}_{\dot H^a}\|u\|^{1-\theta}_{\dot H^b}\|_{L^r}$$

and we argue exactly as in the proof of the second interpolation inequality.

• Sobolev embedding. For any $s \in (0, d/2)$, there holds

$$\dot H^s(\mathbb{R}^d) \subset L^p(\mathbb{R}^d), \quad \text{with} \quad \frac{1}{p} = \frac{1}{2} - \frac{s}{d}. \tag{5.1}$$

In particular, $\dot H^{1/2}(\mathbb{R}^2) \subset L^4(\mathbb{R}^2)$ and $\dot H^{1/2}(\mathbb{R}^3) \subset L^3(\mathbb{R}^2)$. The proof of (5.1) is a bit tricky and it is presented below. We start with the Cavalieri's principle

$$\|f\|^p_{L^p} = p \int_0^\infty t^{p-1} |\{|f(x)| > t\}| \, dt \tag{5.2}$$

because

$$\int_{\mathbb{R}^d} |f(x)|^p \, dx = \int_{\mathbb{R}^d} \int_0^\infty \mathbf{1}_{t<|f(x)|} p t^{p-1} \, dt \, dx = p \int_0^\infty \Big\{ \int_{\mathbb{R}^d} \mathbf{1}_{|f(x)|>t} \, dx \Big\} t^{p-1} \, dt,$$

by the Fubini theorem. We then introduce the splitting

$$f = f_A + f^A,$$

with

$$f_A := \mathcal{F}^{-1}(\mathbf{1}_{|\xi|\le A} \hat f), \quad f^A := \mathcal{F}^{-1}(\mathbf{1}_{|\xi|\ge A} \hat f),$$

where for a function $g : \mathbb{R}^d \to \mathbb{C}$ we define its Fourier transform $\hat g = \mathcal{F} g$ and its inverse Fourier transform $\mathcal{F}^{-1} g$ by

$$\mathcal{F} g(\xi) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} g(x) \, e^{-ix\cdot\xi} \, dx, \quad \mathcal{F}^{-1} g(x) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} g(\xi) \, e^{-ix\cdot\xi} \, d\xi.$$

We observe that

$$\begin{aligned} \|f_A\|_\infty &\le (2\pi)^{-d/2} \|\hat f_A\|_1 = (2\pi)^{-d/2} \int \mathbf{1}_{|\xi|\le A} |\hat f| \, d\xi \\ &\le (2\pi)^{-d/2} \Big(\int \mathbf{1}_{|\xi|\le A} |\xi|^{-2s} \, d\xi \Big)^{1/2} \Big(\int |\xi|^{2s} |\hat f|^2 \, d\xi \Big)^{1/2} \\ &\le (C_1/2) A^{d/2-s} \|f\|_{\dot H^s}, \end{aligned}$$

where we have used the Cauchy-Schwartz inequality and we denote here and below by C_i some constant which only depend on p, s and d. We deduce that

$$\{|f_A| > t/2\} = \emptyset \quad \text{if} \quad t \ge t_A := C_1 A^{d/2-s} \|f\|_{\dot H^s}.$$

Together with the inclusion

$$\{|f(x)| > t\} \subset \{|f_A(x)| > t/2\} \cup \{|f^A(x)| > t/2\},$$

we get

$$\{|f(x)| > t\} \subset \{|f^{A_t}(x)| > t/2\},$$

where we have defined

$$A_t := \big(C_1 t / \|f\|_{\dot H^s} \big)^{1/(d/2-s)}.$$

Coming back to (5.2), we then have

$$\|f\|^p_{L^p} \le p \int_0^\infty t^{p-1} |\{|f^{A_t}(x)| > t/2\}| \, dt$$

We recall the Tchebychev inequality

$$|\{|g| > \lambda\}| \le \int \frac{|g|^2}{\lambda^2} \mathbf{1}_{\{|g|>\lambda\}} \, dx \le \frac{1}{\lambda^2} \|g\|_{L^2}$$

and the Plancherel idenitity

$$\|g\|_{L^2} = \|\hat g\|_{L^2}.$$

The three last inequalities together, we have

$$\|f\|^p_{L^p} \le p \int_0^\infty t^{p-1} \frac{4}{t^2} \|\hat f^{A_t}\|^2_{L^2} dt.$$

We then write

$$\begin{aligned} \int_0^\infty t^{p-3} \|\hat f^{A_t}\|^2_{L^2} dt &= \int_0^\infty t^{p-3} \int_{\mathbb{R}^d} |\hat f|^2 \mathbf{1}_{|\xi|\ge A_t}\, d\xi dt \\ &= \int_{\mathbb{R}^d} |\hat f|^2 \Big(\int_0^{t_{|\xi|}} t^{p-3}\, dt \Big) d\xi \\ &= \frac{1}{p-2} \int_{\mathbb{R}^d} |\hat f|^2 \Big(C_1 |\xi|^{d/2-s} \|f\|_{\dot H^s} \Big)^{p-2} d\xi, \end{aligned}$$

where we have used the Fubini theorem at the second line. From the very definition $p := 2d/(d-2s)$, we have

$$\big(\frac{d}{2} - s\big)(p-2) = (d-2s)\big(\frac{d}{d-2s} - 1\big) = d - (d-2s) = 2s.$$

Putting together the three last estimates and identities, we thus obtain

$$\|f\|^p_{L^p} \le C_2 \int_{\mathbb{R}^d} |\hat f|^2\, |\xi|^{2s}\, d\xi \|f\|^{p-2}_{\dot H^s} = C_2 \|f\|^p_{\dot H^s},$$

which is nothing but the announced estimate. □

• Young inequality for convolution. For any $p, q \in [1, +\infty]$ such that $1/p + 1/q < 1$, there holds

$$(5.3) \qquad \|f * g\|_{L^r} \le \|f\|_{L^p} \|g\|_{L^q}, \quad \forall\, f \in L^p,\ g \in L^q,$$

where $r \in (1, \infty)$ is defined by

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}.$$

We fix $0 \le f \in L^p$, $0 \le g \in L^q$, $0 \le h \in L^{r'}$, and observing that

$$\frac{1}{r} + \frac{1}{q'} = \frac{1}{p}, \quad \frac{1}{r} + \frac{1}{p'} = \frac{1}{q}, \quad \frac{1}{p'} + \frac{1}{q'} = \frac{1}{r'},$$

we may write

$$\int (f*g) h\, dx \;=\; \iint (f(x-y)^{\frac{p}{r}} g(y)^{\frac{q}{r}})(g(y)^{\frac{q}{p'}} h(y)^{\frac{r'}{p'}})(f(x-y)^{\frac{p}{q'}} h(x)^{\frac{r'}{q'}})\, dxdy.$$

Because

$$\frac{1}{p'} + \frac{1}{q'} + \frac{1}{r} = 1,$$

we may use the generalized Holder inequality and we get

$$\begin{aligned} \int (f*g) h\, dx &\le \Big(\iint f^p g^q dxdy \Big)^{1/r} \Big(\iint g^q h^{r'} dxdy \Big)^{1/p'} \Big(\iint f^p h^{r'}\, dxdy \Big)^{1/q'} \\ &= \|f\|^{p/r}_{L^p} \|g\|^{q/r}_{L^q} \|g\|^{q/p'}_{L^q} \|h\|^{r'/p'}_{L^{r'}} \|f\|^{p/q'}_{L^p} \|h\|^{r'/q'}_{L^{r'}} = \|f\|_{L^p} \|g\|_{L^q} \|h\|_{L^{r'}}. \end{aligned}$$

Together with the duality estimate

$$\|f*g\|_{L^r} = \sup_{\|h\|_{L^{r'}} \le 1} \int (f*g) h\, dx,$$

we immedialtely conclude that (6.11) for positive functions, and then for any functions. □

6. The Hardy-Littlewood-Sobolev inequality

We establish now a slight but fundamental improvement of the Young inequality. The proof uses smart and accurate arguments of *real harmonic analysis*.

6.1. **The maximal function.**

Lemma 6.1 (Covering lemma)**.** *Let $E \subset \mathbb{R}^d$ measurable which is covered by a familly of balls $\{B_j\}$ of volume bounded by a same constant. Then we can select a (disjoint finite or denombrable) subsequence denoted by $\mathcal{B}_1, ..., \mathcal{B}_k, ...$ so that*

$$\sum_k \lambda(\mathcal{B}_k) \geq K^{-1}\,\lambda(E), \tag{6.1}$$

where λ stands for the Lebesgue measure and $K := 5^d$ is a convenient choice.

Proof of Lemma 6.1 . We note J the set of indices for the set of distinct balls $\{B_j\}$? As a first step, we choose $\mathcal{B}_1$ in the family $\{B_j\}_{j\in J}$ which satisfies

$$\text{diam}\mathcal{B}_1 \geq \frac{1}{2}\sup\{\text{diam}B_j;\, j \in J\}.$$

We proceed recursively by choosing $\mathcal{B}_k$, $k \geq 2$, such that

$$\text{diam}\mathcal{B}_k \geq \frac{1}{2}\sup\{\text{diam}B_j,\ j \in J \text{ and } B_j \cap \mathcal{B}_i = \emptyset\ \forall\, i = 1, ..., k-1),$$

and build in that way a finite or denombrable sequence $\mathcal{B}_1, ..., \mathcal{B}_k, ...$. We can assume

$$\sum_k \lambda(\mathcal{B}_k) < \infty, \tag{6.2}$$

otherwise inequality (6.1) is trivial. We define $\mathcal{B}^*_k$ as the ball having same center as $\mathcal{B}_k$ but whose radius is five times larger and we claim that

$$E \subset \bigcup_k \mathcal{B}^*_k,$$

from what we immediately deduce (6.1) with $K = 5^d$. It is thus enough to prove that for any B_j in the familly $\{B_j\}$ which is not in the family $\{\mathcal{B}_k\}$, we have

$$B_j \subset \bigcup_k \mathcal{B}^*_k.$$

We fix such a ball B_j. From (6.2), we clearly have $\lambda(\mathcal{B}_k) \to 0$ when $k \to \infty$, and we then may choose k as the first integer such that

$$\text{diam}\,\mathcal{B}_{k+1} < \frac{1}{2}\text{diam}\,B_j.$$

From the definition of $(\mathcal{B}_\ell)$ we must have $B_j \cap \mathcal{B}_i \neq \emptyset$ for some $i \in \{1, ..., k\}$ (otherwise we have $B_j = \mathcal{B}_{k+1}$ because of the diameter condition and that contradict our choice of B_j which precisely not belongs to that family). We conclude that $B_j \subset \mathcal{B}^*_i$, and that ends the proof. □

For a function $f \in L^1_{loc}(\mathbb{R}^d)$, we define the maximal function

$$Mf(x) := \sup_{r>0} \frac{1}{\lambda(B(x,r))}\int_{B(x,r)} |f(y)|\,dy.$$

The maximal function Mf is measurable (take $r \in \mathbb{Q}^+_*$) and positive whenever $f \not\equiv 0$.

Theorem 6.2 (Maximal function)**.** *For any $p \in (1,\infty)$ and any $f \in L^p(\mathbb{R}^d)$, we have*

$$\|Mf\|_{L^p} \le C\,\|f\|_{L^p}, \tag{6.3}$$

for some positive constant C which only depends on p and d.

Proof of Theorem 6.2 . *Step 1.* We prove that for $g \in L^1(\mathbb{R}^d)$, there holds

$$\lambda(\{x \in \mathbb{R}^d; (Mg)(x) > \alpha\}) \le \frac{K}{\alpha}\,\|g\|_{L^1}, \quad \forall\,\alpha > 0, \tag{6.4}$$

for the same constant K as in the covering Lemma 6.1. Let us then fix $g \in L^1(\mathbb{R}^d)$. For any $\alpha > 0$, we define

$$E_\alpha := \{x; Mg(x) > \alpha\}.$$

By definition of the maximal function, for any $x \in E_\alpha$, there exists a least one ball B_x with center x such that

$$\alpha\,\lambda(B_x) < \int_{B_x} |g(y)|\,dy. \tag{6.5}$$

Because the right hand side is bounded by $\|g\|_{L^1}$ the volume of B_x is necessarily smaller than $\|g\|_{L^1}/\alpha$ independently of $x \in E_\alpha$. Since the union of the balls B_x obviously covers E_α, we may make use of the covering Lemma 6.1 and deduce that there exists a sequence of balls B_k (extracted from the previous family) such that they are mutually disjoint and

$$\lambda(E_\alpha) \le C \sum_k \lambda(B_k) \tag{6.6}$$

Putting together (6.5), (6.6) and the fact that the balls (B_k) are disjoint, we get

$$\alpha\,\lambda(E_\alpha) \le C \sum_k \alpha\lambda(B_k) \le \sum_k \int_{B_k} |g(y)|\,dy = \int_{\cup B_k} |g(y)|\,dy,$$

from which (6.4) immediately follows.

Step 2. We prove (6.3). Let us fix $f \in L^p(\mathbb{R}^d)$. For any given $\alpha > 0$, we define

$$g(x) := f(x)\,\mathbf{1}_{|f(x)| \ge \alpha/2} \quad \text{and} \quad E_\alpha := \{x; Mf(x) > \alpha\}.$$

From $|f(x)| \le |g(x)| + \alpha/2$, we deduce $Mf(x) \le Mg(x) + \alpha/2$, and therefore

$$E_\alpha \subset \{x; Mg(x) > \alpha/2\}.$$

Applying (6.4) to the function g (which belongs to $L^1(\mathbb{R}^d)$), we have

$$\lambda(E_\alpha) \le \lambda(\{x \in \mathbb{R}^d; (Mg)(x) > \alpha\}) \le \frac{2K}{\alpha}\|g\|_{L^1} = \frac{2K}{\alpha}\int_{|f| \ge \alpha/2} |f|\,dx. \tag{6.7}$$

On the other hand, we observe that thanks to the Fubini Theorem, we have

$$\begin{aligned}\int_{\mathbb{R}^d} (Mf)^p\,dx &= \int_{\mathbb{R}^d}\int_0^{Mf(x)} p\alpha^{p-1}\,d\alpha\,dx = \int_0^\infty \int_{\mathbb{R}^d} \mathbf{1}_{Mf(x)>\alpha}\,dx\,p\alpha^{p-1}\,d\alpha \\ &= \int_0^\infty \lambda(\{x \in \mathbb{R}^d; (Mg)(x) > \alpha\})\,p\alpha^{p-1}\,d\alpha.\end{aligned}$$

Together with (6.7) and using again the Fubini Theorem, we deduce

$$\begin{aligned}\int_{\mathbb{R}^d} (Mf)^p\,dx &\le \int_0^\infty \frac{2K}{\alpha}\int_{|f|\ge\alpha/2} |f|\,dx\,p\alpha^{p-1}\,d\alpha.\\ &= 2Kp\int_{\mathbb{R}^d}|f(x)|\Big(\int_0^{2|f(x)|}\alpha^{p-2}\,d\alpha\Big)dx\\ &= K\frac{p}{p-1}2^p\int_{\mathbb{R}^d}|f(x)|^p\,dx,\end{aligned}$$

which ends the proof of (6.3). □

6.2. **Hardy-Littlewood-Sobolev inequality.**

(6.8) $$\Big\|\frac{1}{|z|}*f\Big\|_{L^{2r/(2-r)}(\mathbb{R}^2)} \le C_r\,\|f\|_{L^r(\mathbb{R}^2)},\quad \forall\, r\in(1,2),$$

For $d\ge 2$, we define the functional I by

$$I[f](x) := \int_{\mathbb{R}^d}\frac{f(y)}{|x-y|^{d-1}}\,dy,\quad \forall\, f\in\mathcal{D}(\mathbb{R}^d).$$

For $d\ge 1$, $\alpha\in(0,d)$, we define the functional F by

$$F(x) := \int_{\mathbb{R}^d}\frac{f(y)}{|x-y|^{\alpha}}\,dy,\quad \forall\, f\in\mathcal{D}(\mathbb{R}^d).$$

We assume that $p,q>1$ are such that

(6.9) $$\frac{1}{p}+\frac{\alpha}{d} = 1+\frac{1}{q}.$$

In particular, for $d\ge 2$ and $\alpha=d-1$, we have

$$\frac{1}{p}-\frac{1}{d}=\frac{1}{q},$$

so that $q=p^*$ is the classical Sobolev exponent associated to $W^{1,p}(\mathbb{R}^d)$ when $p\in(1,d)$.

Theorem 6.3. *For any $p\in(1,d)$, there holds*

$$I : L^p(\mathbb{R}^d)\to L^q(\mathbb{R}^d),\quad \frac{1}{q}:=\frac{1}{p}-\frac{1}{d}.$$

In particular, there exists a constant C such that

(6.10) $$\big\|f*\mathcal{K}\big\|_{L^4(\mathbb{R}^2)} \le C\,\|f\|_{L^{4/3}(\mathbb{R}^2)},\quad \forall\, f\in L^{4/3}(\mathbb{R}^2).$$

• Hardy-Littlewood-Sobolev inequality. For any $p\in(1,d)$, there holds

(6.11) $$\|f*G\|_{L^{p^*}} \le C_p\|f\|_{L^p},\quad \forall\, f\in L^p,$$

where G and $p^*\in(1,\infty)$ are defined by

$$G(x) := \frac{1}{|x|^{d-1}},\quad \frac{1}{p^*}=\frac{1}{p}-\frac{1}{d}.$$

It is worth emphasizing that it is a critical case of the Young inequality since G almost belongs to the L^q Lebesgue space with $q:=d/(d-1)$ and that for such a exponent q the exponent r associated to the Young inequality satisfies

$$\frac{1}{p}+\frac{d-1}{d}=1+\frac{1}{r}$$

which is precisely $r = p^*$.

We take $0 \le f \in L^p$ and we define

$$F(x) := \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-1}}\, dy.$$

For a fixed $\delta > 0$, we define $F = F_\delta + F^\delta$ with

$$F_\delta(x) := \int_{|x-y|<\delta} \frac{f(y)}{|x-y|^{d-1}} dy, \quad F^\delta(x) := \int_{|x-y|\ge\delta} \frac{f(y)}{|x-y|^{d-1}} dy.$$

On the one hand, for F_δ, we write

$$\begin{aligned} F_\delta(x) &:= \int_{|x-y|<\delta} \frac{f(y)}{|x-y|^\alpha} dy \\ &= \int_{|x-y|<\delta} f(y)\Big(\frac{1}{1+\alpha}\int_{|x-y|}^\infty r^{-\alpha-1} dr\Big) dy \\ &= \frac{1}{1+\alpha}\int_0^\delta r^{d-\alpha-1}\Big(\frac{1}{r^d}\int_{|x-y|<r} f(y)dy\Big) dr \\ &\le \frac{\lambda_d(B_1)}{1+\alpha}\int_0^\delta r^{d-\alpha-1} dr Mf(x) \\ &= C_1\, \delta^{d-\alpha}\, Mf(x), \qquad C_1 := \frac{\lambda_d(B_1)}{(1+\alpha)(d-\alpha)}, \end{aligned}$$

where we have used the Fubini theorem at the third line and the definition of the maximal function Mf at the fourth line.

On the other hand, for F^δ, we write

$$\begin{aligned} F^\delta(x) &:= \int_{|x-y|>\delta} \frac{f(y)}{|x-y|^\alpha} dy \\ &\le \|f\|_{L^p}\Big(\int_{|x-y|>\delta} \frac{1}{|x-y|^{\alpha p'}} dy\Big)^{1/p'} = C_2\, \delta^{d/p'-\alpha}\|f\|_{L^p}, \end{aligned}$$

where we have used

$$\begin{aligned} \Big(\int_{|z|>\delta} \frac{dz}{|z|^{\alpha p'}} dy\Big)^{1/p'} &= \lambda_{d-1}(S_1)^{1/p'}\Big(\int_\delta^\infty r^{d-1-\alpha p'} dr\Big)^{1/p'} \\ &= \frac{\lambda_{d-1}(S_1)^{1/p'}}{(\alpha p'-d)^{1/p'}}\, \delta^{(d-\alpha p')/p'}. \end{aligned}$$

It is worth emphasizing that the last integral converges because from (6.9), there holds

$$d - \alpha p' = dp'(1/p' - \alpha/d) = -dp'/q < 0.$$

Both estimates together imply

$$\begin{aligned} F(x) &\lesssim \delta^{d-\alpha} Mf(x) + \delta^{d/p'-\alpha}\|f\|_{L^p} \\ &\lesssim (Mf(x))^{1-(p/d)(d-\alpha)}\|f\|_{L^p}^{(p/d)(d-\alpha)} \\ &= (Mf(x))^{p(1/p-1+\alpha/d)}\|f\|_{L^p}^{p(1-\alpha/d)} \end{aligned}$$

with the choice $\delta := (\|f\|_{L^p}/Mf(x))^{p/d}$. As a consequence, we have

$$\begin{aligned}\int F^r \, dx &= \int F^{(\frac{1}{p}+\frac{\alpha}{p}-1)^{-1}} \, dx \\ &\lesssim \int (Mf(x))^p \, dx \|f\|_{L^p}^{r-p}.\end{aligned}$$

We immediately conclude thanks to Theorem 6.2. $\square$