Exercises on chapters 1 & 2

1. About the Gronwall lemma

Exercice 1.1. *Prove in full generality the following classical differential version of Gronwall lemma.*
Lemma. *We assume that $u \in C([0,T);\mathbb{R})$, $T \in (0,\infty)$, satisfies the differential inequality*

$$u' \le a(t)u + b(t) \quad \text{on} \quad (0,T), \tag{1.1}$$

in the distributional sense, for some $a, b \in L^1(0,T)$. Then, u satisfies the pointwise estimate

$$u(t) \le e^{A(t)}u(0) + \int_0^t b(s)e^{A(t)-A(s)}\,ds, \quad \forall\, t \in [0,T), \tag{1.2}$$

where we have defined the primitive function

$$A(t) := \int_0^t a(s)\,ds.$$

Exercice 1.2. *We assume that $u \in C([0,T);\mathbb{R})$, $T \in (0,\infty)$, satisfies the integral inequality*

$$u(t) \le B(t) + \int_0^t a(s)u(s)\,ds \quad \text{on} \quad [0,T), \tag{1.3}$$

for some $B \in C([0,T))$ and $0 \le a \in L^1(0,T)$. Prove that u satisfies the pointwise estimate

$$u(t) \le B(t) + \int_0^t a(s)B(s)e^{A(t)-A(s)}\,ds, \quad \forall\, t \in (0,T).$$

1) By considering the function

$$v(t) := \int_0^t a(s)u(s)\,ds.$$

2) By considering the function

$$v(t) := \int_0^t a(s)u(s)\,ds\, e^{-A(t)} - \int_0^t a(s)B(s)e^{-A(s)}\,ds.$$

Recover the fact that $u \in C([0,T);\mathbb{R})$, $T \in (0,\infty)$, satisfies the integral inequality

$$u(t) \le u_0 + \int_0^t a(s)u(s)\,ds + \int_0^t b(s)\,ds \quad \text{on} \quad [0,T),$$

for some $0 \le a \in L^1(0,T)$ and $b \in L^1(0,T)$, implies that u satisfies the pointwise estimate

$$u(t) \le u_0\, e^{A(t)} + \int_0^t b(s)e^{A(t)-A(s)}\,ds, \quad \forall\, t \in (0,T)$$

3) first in the case when $b = 0$;
4) next, in the general case.

Exercice 1.3. *We consider the ODE*

$$\dot{x}(t) = a(t, x(t)), \quad x(s) = x \in \mathbb{R}^d, \quad s \ge 0, \tag{1.4}$$

associated to a vector field $a : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ which is C^1 and satisfies the globally Lipschitz estimate

$$|a(t,x) - a(t,y)| \le L\,|x-y|, \quad \forall\, t \ge 0,\ x, y \in \mathbb{R}^d, \tag{1.5}$$

for some constant $L \in (0,\infty)$. From the Cauchy-Lipschitz theorem we know that this one admits a unique solution $t \mapsto x(t) = \Phi_{t,s}(x) \in C^1(\mathbb{R}_+;\mathbb{R}^d)$. Moreover, for any $s, t \ge 0$, the vectors valued function $\Phi_{t,s} : \mathbb{R}^d \to \mathbb{R}^d$ is a C^1-diffeomorphism which satisfies the semigroup properties $\Phi_{0,0} = \mathrm{Id}$, $\Phi_{t_3,t_2} \circ \Phi_{t_2,t_1} = \Phi_{t_3,t_1}$ for any $t_3, t_2, t_1 \ge 0$. We denote $\Phi_t = \Phi_{t,0}$.

1) Establish that $|\Phi_t(y) - \Phi_t(x)| \le e^{Lt}|x-y|$ *for any* $t \ge 0$, $x, y \in \mathbb{R}^d$. *(Hint. Use the Gronwall lemma).*

2) Establish that $|\Phi_t(x)| \le (|x| + B(t))e^{tL}$, *with* $B(t) := \int_0^t |a(s,0)|\,ds$, *for any* $t \ge 0$, $x \in \mathbb{R}^d$.

3) Deduce that $|\Phi_t(x) - \Phi_0(x)| \le tC(T)(1+|x|)$ *for any* $t \in [0,T]$, $x \in \mathbb{R}^d$. *(Hint. A possible choice is* $C(T) := Le^{LT}(1 + B(T)) + \|a(\cdot,0)\|_{L^\infty(0,T)}$ *and a possible way to proceed is to use 2)).*

4) Prove that for any $R > 0$, *there exists* R_t *such that* $\Phi_t^{-1}(B_R) \subset B_{R_t}$ *and deduce that if* $\operatorname{supp} f_0 \subset B_R$ *then the function* $f(t,x) := f_0(\Phi_t^{-1}(x))$ *is such that* $\operatorname{supp} f(t,\cdot) \subset B_{R_t}$. *(Hint. Observe that* $B_R \cap (\Phi_t^{-1})^{-1}(B^c_{R_t}) = \emptyset$*).*

2. About variational solutions

Exercice 2.1. *Consider* $f \in L^1(\mathbb{R}^d)$ *such that* $\operatorname{div} f \in L^1(\mathbb{R}^d)$. *Show that*

$$\int_{\mathbb{R}^d} \operatorname{div} f\,dx = 0.$$

Exercice 2.2. *We consider the nonlinear McKean-Vlasov equation*

$$\partial_t f = \Lambda_f f := \Delta f + div(a_f f), \quad f(0) = f_0 \ge 0, \tag{2.1}$$

with

$$a_f := a * f, \quad a \in L^\infty(\mathbb{R}^d)^d.$$

1) Prove that a nonnegative solution f *is (at least formally) mass conserving*

$$\|f(t)\|_{L^1} = \|f_0\|_{L^1}, \qquad \forall\, t \ge 0.$$

2) We define the weighted Lebesgue space L^2_k *by its norm* $\|f\|_{L^2_k} := \|f\langle x\rangle^k\|_{L^2}$, $\langle x\rangle := (1+|x|^2)^{1/2}$, *for* $k \ge 0$. *Observe that*

$$\begin{aligned} \int f\Lambda_f f\langle x\rangle^{2k} &= -\int |\nabla f|^2\langle x\rangle^{2k} + \int \nabla f \cdot a_f\, f\langle x\rangle^{2k} - \int f^2\big[\frac{1}{2}\Delta\langle x\rangle^{2k} + a_f\cdot\nabla\langle x\rangle^{2k}\big] \\ &\le -\frac{1}{2}\int |\nabla f|^2\langle x\rangle^{2k} + C\int f^2\nabla\langle x\rangle^{2k}, \end{aligned}$$

for a constant $C := C(k, \|a\|_{L^\infty}, \|f\|_{L^1})$. *Deduce that a nonnegative solution* f *satisfies the a priori estimate*

$$\|f(t)\|^2_{L^2_k} + \frac{1}{2}\int_0^t \|\nabla f(s)\|^2_{L^2_k}\,ds \le e^{C_0 t}\,\|f_0\|_{L^2_k} \quad \forall\, t \ge 0, \tag{2.2}$$

for a constant $C_0 := C_0(k, \|a\|_{L^\infty}, \|f_0\|_{L^1})$.

3) We set $H := L^2_k$, $k > d/2$, *and* $V := H^1_k$, *where we define the weighted Sobolev space* H^1_k *by its norm* $\|f\|^2_{H^1_k} := \|f\|^2_{L^2_k} + \|\nabla f\|^2_{L^2_k}$. *Observe that for any* $g \in H$, *the distribution* $\Lambda_g f$ *is well defined in* V' *thanks to the identity*

$$\langle \Lambda_g f, h\rangle := -\int_{\mathbb{R}^d} \langle x\rangle^k(\nabla f + a_g f)\cdot(\nabla h\langle x\rangle^k + h2kx\langle x\rangle^{k-2})\,dx, \quad \forall\, h \in V.$$

(Hint. Prove that $L^2_k \subset L^1$*) and that* $-\Lambda_g$ *is bounded and satisfies a Gårding's inequality. Write the variational formulation associated to the nonlinear McKean-Vlasov equation in this framework. Establish that if* $f \in X_T$ *(with the usual definition) is a nonnegative variational solution to the nonlinear McKean-Vlasov equation then it is mass conserving and it satsifies* (2.2). *(Hint. For proving the mass consevation, take* $\chi_M\,\langle x\rangle^{-2k}$ *as a test function in the variational formulation, with* $\chi_M(x) := \chi(x/M)$, $\chi \in \mathcal{D}(\mathbb{R}^d)$, $\mathbf{1}_{B(0,1)} \le \chi \le \mathbf{1}_{B(0,2)}$*).*

4) Prove that for any $0 \le f_0 \in H$ *and* $g \in C([0,T]; H)$ *there exists a unique mass preserving variational solution* $0 \le f \in X_T$ *to the linear McKean-Vlasov equation*

$$\partial_t f = \Delta f + div(a_g f), \quad f(0) = f_0.$$

For two solutions f_1 and f_2 associated respectively to g_1 and g_2, observe that $f = f_2 - f_1$ satisfies

$$\partial_f f = \Delta f + \mathrm{div}(a_g f_2) + \mathrm{div}(a_{g_1} f),$$

with $g := g_2 - g_1$, and prove that

$$\frac{d}{dt}\|f_t\|_H^2 \le C_1 \|f_t\|_H^2 + C_2 \|g_t\|_H^2 e^{bt} \|f_0\|_H^2.$$

Deduce that the mapping $g \mapsto f$ is a contraction in $C([0,T];H)$ for $T > 0$ small enough. Conclude to the existence and uniqueness of a global (in time) variational solution to the nonlinear McKean-Vlasov equation.

Exercice 2.3. *For $a, c \in L^\infty(\mathbb{R}^d)$, $f_0 \in L^p(\mathbb{R}^d)$, $1 \le p \le \infty$, we consider the linear parabolic equation*

$$\partial_t f = \Lambda f := \Delta f + a \cdot \nabla f + cf, \quad f(0) = f_0. \tag{2.3}$$

We introduce the usual notations $H := L^2$, $V := H^1$ and X_T the associated space for some given $T > 0$.
1) Prove that for $\gamma \in C^1(\mathbb{R})$, $\gamma(0) = 0$, $\gamma' \in L^\infty$, there holds $\gamma(f) \in H$ for any $f \in H$ and $\gamma(f) \in V$ for any $f \in V$.
2) Prove that $f \in X_T$ is a variational solution to (2.3) if and only if

$$\frac{d}{dt} f = \Lambda f \ \text{ in } V' \text{ a.e. on } (0,T).$$

3) On the other hand, prove that for any $f \in X_T$ and any function $\beta \in C^2(\mathbb{R})$, $\beta(0) = \beta'(0) = 0$, $\beta'' \in L^\infty$, there holds

$$\frac{d}{dt}\int_{\mathbb{R}^d} \beta(f) = \langle \frac{d}{dt} f, \beta'(f)\rangle_{V',V} \ \text{ a.e. on } (0,T).$$

(Hint. Consider $f_\varepsilon = f *_t \rho_\varepsilon \in C^1([0,T];H^1)$ and pass to the limit $\varepsilon \to 0$).
4) Consider a convex function $\beta \in C^2(\mathbb{R})$ such that $\beta(0) = \beta'(0) = 0$ and $\beta'' \in L^\infty$. Prove that any variational solution $f \in X_T$ to the above linear parabolic equation satisfies

$$\int_{\mathbb{R}^d} \beta(f_t)\,dx \le \int_{\mathbb{R}^d} \beta(f_0)\,dx + \int_0^t \int_{\mathbb{R}^d} \{c\, f\, \beta'(f) - (div\, a)\, \beta(f)\}\,dxds,$$

for any $t \ge 0$.
5) Assuming moreover that there exists a constant $K \in (0,\infty)$ such that $0 \le s\,\beta'(s) \le K\beta(s)$ for any $s \in \mathbb{R}$, deduce that for some constant $C := C(a,c,K)$, there holds

$$\int_{\mathbb{R}^d} \beta(f_t)\,dx \le e^{Ct} \int_{\mathbb{R}^d} \beta(f_0)\,dx, \quad \forall\, t \ge 0.$$

6) Prove that for any $p \in [1,2]$, for some constant $C := C(a,c)$ and for any $f_0 \in L^2 \cap L^p$, there holds

$$\|f(t)\|_{L^p} \le e^{Ct} \|f_0\|_{L^p}, \quad \forall\, t \ge 0.$$

(Hint. Define β on $\mathbb{R}_+$ and extend it to $\mathbb{R}$ by symmetry. More precisely, define $\beta''_\alpha(s) = 2\theta \mathbf{1}_{s\le\alpha} + p(p-1)s^{p-2}\mathbf{1}_{s>\alpha}$, with $2\theta = p(p-1)\alpha^{p-2}$ and then the primitives which vanish at the origin, which are thus defined by $\beta'_\alpha(s) = 2\theta s \mathbf{1}_{s\le\alpha} + (ps^{p-1} + p(p-2)\alpha^{p-1})\mathbf{1}_{s>\alpha}$, $\beta_\alpha(s) = \theta s^2 \mathbf{1}_{s\le\alpha} + (s^p + p(p-2)\alpha^{p-1}s + A\alpha^p)\mathbf{1}_{s>\alpha}$, $A := p(p-1)/2 - 1 - p(p-2)$. Observe that $s\beta'_\alpha(s) \le 2\beta_\alpha(s)$ because $s\beta''_\alpha(s) \le \beta'_\alpha(s)$ and $\beta_\alpha(s) \le \beta(s)$ because $\beta''_\alpha(s) \le \beta''(s)$).
7) Prove that for any $p \in [2,\infty]$ and for some constant $C := C(a,c,p)$ there holds

$$\|f(t)\|_{L^p} \le e^{Ct} \|f_0\|_{L^p}, \quad \forall\, t \ge 0.$$

(Hint. Define $\beta''_R(s) = p(p-1)s^{p-2}\mathbf{1}_{s\le R} + 2\theta \mathbf{1}_{s>R}$, with $2\theta = p(p-1)R^{p-2}$, and then the primitives which vanish in the origin and which are thus defined by $\beta'_R(s) = ps^{p-1}\mathbf{1}_{s\le R} + (pR^{p-1} + 2\theta(s-R))\mathbf{1}_{s>R}$, $\beta_R(s) = s^p \mathbf{1}_{s\le R} + (R^p + pR^{p-1}(s-R) + \theta(s-R)^2)\mathbf{1}_{s>R}$. Observe that $s\beta'_R(s) \le p\beta_R(s)$ because $s\beta''_R(s) \le (p-1)\beta'_R(s)$ and $\beta_R(s) \le \beta(s)$ because $\beta''_R(s) \le \beta''(s)$. Pass to the limit $p \to \infty$ in order to deal with the case $p = \infty$).
8) Prove that for any $f_0 \in L^p(\mathbb{R}^d)$, $1 \le p \le \infty$, there exists at least one weak (in the sense of distributions) solution to the linear parabolic equation (2.3). (Hint: Consider $f_{0,n} \in L^1 \cap L^\infty$ such that $f_{0,n} \to f_0$ in

L^p, $1 \le p < \infty$, and prove that the associate variational solution $f_n \in X_T$ is a Cauchy sequence in $C([0,T];L^p)$. Conclude the proof by passing to the limit $p \to \infty$).

9) Establsih the L^p estimates with "optimal" constant C (that is the one given by the formal computations).

10) Extend the above result to an equation with an integral term and/or a source term.

11) Prove the existence of a weak solution to the McKean-Vlasov equation (2.1) for any initial datum $f_0 \in L^1(\mathbb{R}^d)$.

12) Prove that $f_0 \ge 0$ implies $f(t) \ge 0$ for any $t \in (0,T)$. (Hint. Choose $\beta(s) := s_-$).

3. About transport equations

Exercice 3.1. *Make explicit the construction and formulas in the three following cases:*
(1) $a(x) = a \in \mathbb{R}^d$ is a constant vector. (Hint. One must find $f(t,x) = f_0(x - at)$).
(2) $a(x) = x$. (Hint. One must find $f(t,x) = f_0(e^{-t}x)$).
(3) $a(x,v) = v$, $f_0 = f_0(x,v) \in C^1(\mathbb{R}^d \times \mathbb{R}^d)$ and look for a solution $f = f(t,x,v) \in C^1((0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d)$. (Hint. One must find $f(t,x,v) = f_0(x - vt, v)$).

Exercice 3.2. *(1) Show that for any characterictics solution f to the transport equation associated to an initial datum $f_0 \in C^1_c(\mathbb{R}^d)$, for any times $T > 0$ and radius R, there exists some constants $C_T, R_T \in (0,\infty)$ such that*

$$\sup_{t\in[0,T]} \int_{B_R} |f(t,x)|\,dx \le C_T \int_{B_{R_T}} |f_0(x)|\,dx.$$

(Hint. Use the property of finite speed propagation of the transport equation).

(2) Adapt the proof of existence to the case $f_0 \in L^\infty$.

(3) Prove that for any $f_0 \in C_0(\mathbb{R}^d)$ there exists a global weak solution f to the transport equation which furthermore satisfies $f \in C([0,T];C_0(\mathbb{R}^d))$.

Exercice 3.3. *Consider the relaxation equation*

$$\partial_t f + v\cdot\nabla_x f = M(v)\rho_f - f$$

on the function $f = f(t,x,v)$, $t \ge 0$, $x, v \in \mathbb{R}^d$, where we denote

$$\rho_f := \int_{\mathbb{R}^d} f\,dv, \quad M(v) := (2\pi)^{-d/2}\exp(-|v|^2/2).$$

Prove the existence and uniqueness of a solution $f \in C(\mathbb{R}_+; L^1(\mathbb{R}^d\times\mathbb{R}^d))$ to the relaxation equation for any initial datum $f_0 \in L^1(\mathbb{R}^d\times\mathbb{R}^d)$.

Exercice 3.4. *1) Consider the transport equation with boundary condition*

$$\begin{cases} \partial_t f + \partial_x f + af = 0 \\ f(t,0) = b(t), \quad f(0,x) = f_0(x), \end{cases} \tag{3.1}$$

where $f = f(t,x)$, $t \ge 0$, $x \ge 0$. Assume $a \in L^\infty(\mathbb{R}_+)$. For any $f_0 \in L^1(\mathbb{R}_+)$ and $b \in L^1(0,T)$, establish that there exists a unique weak solution $f \in C([0,T];L^1(\mathbb{R}_+))$ associated to equation (3.1).

2) Consider the renewal equation

$$\begin{cases} \partial_t f + \partial_x f + af = 0 \\ f(t,0) = \rho_{f(t)}, \quad f(0,x) = f_0(x), \end{cases} \tag{3.2}$$

where $f = f(t,x)$, $t \ge 0$, $x \ge 0$, and

$$\rho_g := \int_0^\infty g(y)\,a(y)\,dy.$$

Assume $a \in L^\infty(\mathbb{R}_+)$. For any $f_0 \in L^1(\mathbb{R}_+)$, establish that there exists a unique weak solution $f \in C([0,T];L^1(\mathbb{R}_+))$ associated to equation (3.2).