Exercises on chapters 1, 2 & 3

Modifications are written in blue color.

1. About the Gronwall lemma (Chapter 1)

Exercice 1.1. *We assume that* $u \in C([0,T);\mathbb{R})$, $T \in (0,\infty)$, *satisfies the integral inequality*

$$u(t) \le B(t) + \int_0^t a(s)u(s)\,ds \quad on \quad [0,T), \tag{1.1}$$

for some $B \in C([0,T))$ *and* $0 \le a \in L^1(0,T)$. *Prove that* u *satisfies the pointwise estimate*

$$u(t) \le B(t) + \int_0^t a(s)B(s)e^{A(t)-A(s)}\,ds, \quad \forall\, t \in (0,T).$$

1) By considering the function

$$v(t) := \int_0^t a(s)u(s)\,ds.$$

2) By considering the function

$$v(t) := \int_0^t a(s)u(s)\,ds\, e^{-A(t)} - \int_0^t a(s)B(s)e^{-A(s)}\,ds.$$

Recover the fact that $u \in C([0,T);\mathbb{R})$, $T \in (0,\infty)$, *satisfies the integral inequality*

$$u(t) \le u_0 + \int_0^t a(s)u(s)\,ds + \int_0^t b(s)\,ds \quad on \quad [0,T),$$

for some $0 \le a \in L^1(0,T)$ *and* $b \in L^1(0,T)$, *implies that* u *satisfies the pointwise estimate*

$$u(t) \le u_0\, e^{A(t)} + \int_0^t b(s)e^{A(t)-A(s)}\,ds, \quad \forall\, t \in (0,T),$$

3) first in the case when $b = 0$, *next in the general case.*

Exercice 1.2. *We consider the ODE*

$$\dot{x}(t) = a(t, x(t)), \quad x(s) = x \in \mathbb{R}^d, \quad s \ge 0, \tag{1.2}$$

associated to a vector field $a : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ *which is* C^1 *and satisfies the globally Lipschitz estimate*

$$|a(t,x) - a(t,y)| \le L\,|x-y|, \quad \forall\, t \ge 0,\ x,y \in \mathbb{R}^d, \tag{1.3}$$

for some constant $L \in (0,\infty)$. *From the Cauchy-Lipschitz theorem we know that this one admits a unique solution* $t \mapsto x(t) = \Phi_{t,s}(x) \in C^1(\mathbb{R}_+;\mathbb{R}^d)$. *Moreover, for any* $s,t \ge 0$, *the vectors valued function* $\Phi_{t,s} : \mathbb{R}^d \to \mathbb{R}^d$ *is a* C^1*-diffeomorphism which satisfies the semigroup properties* $\Phi_{0,0} = \mathrm{Id}$, $\Phi_{t_3,t_2} \circ \Phi_{t_2,t_1} = \Phi_{t_3,t_1}$ *for any* $t_3, t_2, t_1 \ge 0$. *We denote* $\Phi_t = \Phi_{t,0}$.

1) Establish that $|\Phi_t(y) - \Phi_t(x)| \le e^{Lt}|x-y|$ *for any* $t \ge 0$, $x,y \in \mathbb{R}^d$. *(Hint. Use the Gronwall lemma).*

2) Establish that $|\Phi_t(x)| \le (|x| + B(t))e^{tL}$, *with* $B(t) := \int_0^t |a(s,0)|\,ds$, *for any* $t \ge 0$, $x \in \mathbb{R}^d$.

3) Deduce that $|\Phi_t(x) - \Phi_0(x)| \le tC(T)(1+|x|)$ *for any* $t \in [0,T]$, $x \in \mathbb{R}^d$. *(Hint. A possible choice is* $C(T) := Le^{LT}(1 + B(T)) + \|a(\cdot,0)\|_{L^\infty(0,T)}$ and a possible way to proceed is to use 2)).

4) Prove that for any $R > 0$, *there exists* R_t *such that* $\Phi_t^{-1}(B_R) \subset B_{R_t}$ *and deduce that if supp* $f_0 \subset B_R$ *then the function* $f(t,x) := f_0(\Phi_t^{-1}(x))$ *is such that supp* $f(t,\cdot) \subset B_{R_t}$. *(Hint. Observe that* $B_R \cap (\Phi_t^{-1})^{-1}(B_{R_t}^c) = \emptyset$*).*

2. About variational solutions (Chapter 2)

Exercice 2.1. *Consider* $f \in L^1(\mathbb{R}^d)$ *such that* $\mathrm{div} f \in L^1(\mathbb{R}^d)$. *Show that*

$$\int_{\mathbb{R}^d} \mathrm{div} f \, dx = 0.$$

Exercice 2.2. *Consider the parabolic type equation*

$$\partial_t f = \partial_i (a_{ij} \, \partial_j f) + b_i \, \partial_i f + \partial_i (\beta_i u) + cf + \int k(t,x,y) \, f(t,y) \, dy + G, \tag{2.1}$$

with time dependent coefficients and source term

$$a, \, b, \, \beta, \, c \in L^\infty((0,T) \times \mathbb{R}^d), \quad k \in L^\infty(0,T; L^2(\mathbb{R}^d \times \mathbb{R}^d)), \quad G \in L^2(0,T; H^{-1}(\mathbb{R}^d)),$$

and under uniformly elliptic condition

$$\forall \, t \in (0,T), \; \forall \, x \in \mathbb{R}^d, \; \forall \, \xi \in \mathbb{R}^d \quad a_{ij}(t,x) \, \xi_i \xi_j \ge \alpha \, |\xi|^2, \quad \alpha > 0. \tag{2.2}$$

For any $g_0 \in L^2(\mathbb{R}^d)$, *generalize J.-L. Lions theorem about the existence and uniqueness of variational solutions* $f \in X_T$. *(Hint. Define*

$$a_i := \frac{n}{T} \int_{t_{i-1}}^{t_i} a(t, \cdot) \, dt, \quad i = 1, \dots, n, \quad t_i := iT/n,$$

and a similar way b_i, β_i, c_i, k_i, *and prove that there exists a unique variational solution* $g_i \in X_{T/n}$ *associated to the* a_i, b_i, c_i, k_i *and the initial condition* g_0 *when* $i = 1$, $g_{i-1}(T/n)$ *when* $i \ge 2$. *Build next a solution* $g^n \in X_T$ *to the equation (2.1) associated to the piecewise constant functions* $a^n(t) = a_i$ *if* $t \in [t_i, t_{i+1})$, $i = 0, \dots, n-1$, *and* b^n, β^n, c^n, k^n *defined similarly. Conclude by passing to the limit* $n \to \infty$*).*

Exercice 2.3. *For the above problem, show that* $f \ge 0$ *if* $f_0, G, k \ge 0$. *(Hint. Show that the sequence* (g_k) *defined in step 2 of the proof of the existence part is such that* $g_k \ge 0$ *for any* $k \in \mathbb{N}$*).*

Exercice 2.4. *We consider the nonlinear McKean-Vlasov (McKV) equation*

$$\partial_t f = \Lambda_f f := \Delta f + div(a_f f), \quad f(0) = f_0 \ge 0, \tag{2.3}$$

with

$$a_f := a * f, \quad a \in L^\infty(\mathbb{R}^d)^d.$$

1) Prove that a nonnegative solution f *is (at least formally) mass conserving*

$$\|f(t)\|_{L^1} = \|f_0\|_{L^1}, \qquad \forall \, t \ge 0.$$

2) We define the weighted Lebesgue space L^2_k *by its norm* $\|f\|_{L^2_k} := \|f \langle x \rangle^k\|_{L^2}$, $\langle x \rangle := (1 + |x|^2)^{1/2}$, *for* $k \ge 0$. *For any nice functions* f *and* g, *observe that*

$$\begin{aligned} \int f \Lambda_g f \langle x \rangle^{2k} &= -\int |\nabla f|^2 \langle x \rangle^{2k} + \int \nabla f \cdot a_g \, f \langle x \rangle^{2k} - \int f^2 \big[\frac{1}{2} \Delta \langle x \rangle^{2k} + a_g \cdot \nabla \langle x \rangle^{2k} \big] \\ &\le -\frac{1}{2} \int |\nabla f|^2 \langle x \rangle^{2k} + C \int f^2 \nabla \langle x \rangle^{2k}, \end{aligned}$$

for a constant $C := C(k, \|a\|_{L^\infty}, \|g\|_{L^1})$. *Deduce that a nonnegative solution* f *satisfies*

$$\|f(t)\|^2_{L^2_k} + \int_0^t \|\nabla f(s)\|^2_{L^2_k} \, ds \le e^{C_0 t} \, \|f_0\|_{L^2_k}, \quad \forall \, t \ge 0, \tag{2.4}$$

for a constant $C_0 := C_0(k, \|a\|_{L^\infty}, \|f_0\|_{L^1})$.

3) We set $H := L^2_k$, $k > d/2$, *and* $V := H^1_k$, *where we define the weighted Sobolev space* H^1_k *by its norm* $\|f\|^2_{H^1_k} := \|f\|^2_{L^2_k} + \|\nabla f\|^2_{L^2_k}$. *Observe that for any* $g \in H$, *the distribution* $\Lambda_g f$ *is well defined in* V' *thanks to the identity*

$$\langle \Lambda_g f, \varphi \rangle := -\int_{\mathbb{R}^d} (\nabla f + a_g f) \cdot \nabla (\varphi \langle x \rangle^{2k}) \, dx, \quad \forall \, \varphi \in V.$$

(Hint. prove that $L^2_k \subset L^1$*) and that* $-\Lambda_g$ *is bounded and satisfies a Gårding's inequality. Write the variational formulation associated to the nonlinear McKV equation in this framework. Establish that if* $f \in X_T$ *(with usual definition) is a nonnegative variational solution to the nonlinear McKV equation then it is mass conserving and it satisfies* (2.4)*. (Hint. One may observe that* $\varphi(x) := \langle x\rangle^{-2k} \in V \subset X_T$*).*

4) Prove that for any $0 \le f_0 \in H$ *and* $g \in C([0,T];H)$ *there exists a unique mass preserving variational solution* $0 \le f \in X_T$ *to the linear McKV equation*

$$\partial_t f = \Delta f + div(a_g f), \quad f(0) = f_0.$$

5) For two solutions f_1 *and* f_2 *associated respectively to* g_1 *and* g_2*, observe that* $f = f_2 - f_1$ *satisfies*

$$\partial_t f = \Delta f + \mathrm{div}(a_g f_2) + \mathrm{div}(a_{g_1} f),$$

with $g := g_2 - g_1$*, and prove that*

$$\frac{d}{dt}\|f_t\|^2_H \le C_1(1 + \|g_{1t}\|^2_{L^1})\|f_t\|^2_H + C_2\|g_t\|^2_{L^1}\|f_{2t}\|^2_H.$$

(Hint. Write

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|f\|^2_H &= -\|\nabla f\|^2_H - \int \nabla f \cdot a_g f_2 \langle x\rangle^{2k} - \int \nabla f \cdot a_{g_1} f \langle x\rangle^{2k} \\ &\quad + \frac{1}{2}\int f^2 \Delta\langle x\rangle^{2k} - \int f f_2 a_g \cdot \nabla\langle x\rangle^{2k} - \int f^2 a_{g_1} \cdot \nabla\langle x\rangle^{2k},\end{aligned}$$

and in the three first terms kill the ∇f *contribution). We define*

$$\mathcal{E}_T := \{f \in C([0,T];H);\ f \ge 0,\ \|f_t\|_{L^1} = \|f_0\|_{L^1}\},$$

that we endow with the $C([0,T];H)$ *norm. Deduce that the mapping* $g \mapsto f$ *is a contraction in* $\mathcal{E}_T$ *for* $T > 0$ *small enough. Conclude to the existence and uniqueness of a global (in time) variational solution to the nonlinear McKean-Vlasov equation.*

Exercice 2.5. *For* $a, c \in L^\infty(\mathbb{R}^d)$*,* $f_0 \in L^p(\mathbb{R}^d)$*,* $1 \le p \le \infty$*, we consider the linear parabolic equation*

$$\partial_t f = \Lambda f := \Delta f + a \cdot \nabla f + c f, \quad f(0) = f_0. \tag{2.5}$$

We introduce the usual notations $H := L^2$*,* $V := H^1$ *and* X_T *the associated space for some given* $T > 0$*.*

1) Prove that for $\gamma \in C^1(\mathbb{R})$*,* $\gamma(0) = 0$*,* $\gamma' \in L^\infty$*, there holds* $\gamma(f) \in H$ *for any* $f \in H$ *and* $\gamma(f) \in V$ *for any* $f \in V$*.*

2) Prove that $f \in X_T$ *is a variational solution to* (2.5) *if and only if*

$$\frac{d}{dt} f = \Lambda f \ \text{ in } V' \ \text{ a.e. on } (0,T).$$

3) On the other hand, prove that for any $f \in X_T$ *and any function* $\beta \in C^2(\mathbb{R})$*,* $\beta(0) = \beta'(0) = 0$*,* $\beta'' \in L^\infty$*, there holds*

$$\frac{d}{dt}\int_{\mathbb{R}^d} \beta(f) = \langle \frac{d}{dt} f, \beta'(f)\rangle_{V',V} \ \text{ a.e. on } (0,T).$$

(Hint. Consider $f_\varepsilon = f *_t \rho_\varepsilon \in C^1([0,T];H^1)$ *and pass to the limit* $\varepsilon \to 0$*).*

4) Consider a convex function $\beta \in C^2(\mathbb{R})$ *such that* $\beta(0) = \beta'(0) = 0$ *and* $\beta'' \in L^\infty$*. Prove that any variational solution* $f \in X_T$ *to the above linear parabolic equation satisfies*

$$\int_{\mathbb{R}^d} \beta(f_t)\,dx \le \int_{\mathbb{R}^d} \beta(f_0)\,dx + \int_0^t \int_{\mathbb{R}^d} \{c\,f\,\beta'(f) - (div\,a)\,\beta(f)\}\,dxds,$$

for any $t \ge 0$*.*

5) Assuming moreover that $\beta \ge 0$ *and there exists a constant* $K \in (0,\infty)$ *such that* $0 \le s\,\beta'(s) \le K\beta(s)$ *for any* $s \in \mathbb{R}$*, deduce that for some constant* $C := C(a,c,K)$*, there holds*

$$\int_{\mathbb{R}^d} \beta(f_t)\,dx \le e^{Ct}\int_{\mathbb{R}^d} \beta(f_0)\,dx, \quad \forall\, t \ge 0.$$

6) Prove that for any $p \in [1,2]$, for some constant $C := C(a,c)$ and for any $f_0 \in L^2 \cap L^p$, there holds

$$\|f(t)\|_{L^p} \le e^{Ct}\|f_0\|_{L^p}, \quad \forall\, t \ge 0.$$

(Hint. Define β on $\mathbb{R}_+$ and extend it to $\mathbb{R}$ by symmetry. More precisely, define $\beta''_\alpha(s) = 2\theta \mathbf{1}_{s\le\alpha} + p(p-1)s^{p-2}\mathbf{1}_{s>\alpha}$, with $2\theta = p(p-1)\alpha^{p-2}$ and then the primitives which vanish at the origin, which are thus defined by $\beta'_\alpha(s) = 2\theta s \mathbf{1}_{s\le\alpha} + (ps^{p-1} + p(p-2)\alpha^{p-1})\mathbf{1}_{s>\alpha}$, $\beta_\alpha(s) = \theta s^2 \mathbf{1}_{s\le\alpha} + (s^p + p(p-2)\alpha^{p-1}s + A\alpha^p)\mathbf{1}_{s>\alpha}$, $A := p(p-1)/2 - 1 - p(p-2)$. Observe that $s\beta'_\alpha(s) \le 2\beta_\alpha(s)$ because $s\beta''_\alpha(s) \le \beta'_\alpha(s)$ and $\beta_\alpha(s) \le \beta(s)$ because $\beta''_\alpha(s) \le \beta''(s)$).

7) Prove that for any $p \in [2,\infty]$ and for some constant $C := C(a,c,p)$ there holds

$$\|f(t)\|_{L^p} \le e^{Ct}\|f_0\|_{L^p}, \quad \forall\, t \ge 0.$$

(Hint. Define $\beta''_R(s) = p(p-1)s^{p-2}\mathbf{1}_{s\le R} + 2\theta\mathbf{1}_{s>R}$, with $2\theta = p(p-1)R^{p-2}$, and then the primitives which vanish in the origin and which are thus defined by $\beta'_R(s) = ps^{p-1}\mathbf{1}_{s\le R} + (pR^{p-1} + 2\theta(s-R))\mathbf{1}_{s>R}$, $\beta_R(s) = s^p\mathbf{1}_{s\le R} + (R^p + pR^{p-1}(s-R) + \theta(s-R)^2)\mathbf{1}_{s>R}$. Observe that $s\beta'_R(s) \le p\beta_R(s)$ because $s\beta''_R(s) \le (p-1)\beta'_R(s)$ and $\beta_R(s) \le \beta(s)$ because $\beta''_R(s) \le \beta''(s)$. Pass to the limit $p \to \infty$ in order to deal with the case $p = \infty$).

8) Prove that for any $f_0 \in L^p(\mathbb{R}^d)$, $1 \le p \le \infty$, there exists at least one weak (in the sense of distributions) solution to the linear parabolic equation (2.5). (Hint: Consider $f_{0,n} \in L^1 \cap L^\infty$ such that $f_{0,n} \to f_0$ in L^p, $1 \le p < \infty$, and prove that the associate variational solution $f_n \in X_T$ is a Cauchy sequence in $C([0,T];L^p)$. Conclude the proof by passing to the limit $p \to \infty$).

9) Prove that if $0 \le f_0 \in L^p(\mathbb{R}^d)$, $p \in (1,\infty)$, there exists a weak solution $f \in C([0,T];L^p(\mathbb{R}^d))$ such that $f(t,\cdot) \ge 0$ for any $t \in (0,T)$. (Hint. Choose $\beta(s) := s_-^p$). Generalize to the case $p \in [1,\infty]$. Discuss the "a posteriori result": $f_0 \ge 0$ implies $f(t) \ge 0$ for any $t \in (0,T)$.

10) Prove the existence of a weak solution to the McKean-Vlasov equation (2.3) for any initial datum $f_0 \in L^1(\mathbb{R}^d)$.

11) Establish the L^p estimates with "optimal" constant C (that is the one given by the formal computations).

12) Extend the above result to an equation with an integral term and/or a source term.

3. About transport equations (Chapter 3)

Exercice 3.1. *Make explicit the construction and formulas in the three following cases:*
(1) $a(x) = a \in \mathbb{R}^d$ is a constant vector. (Hint. One must find $f(t,x) = f_0(x - at)$).
(2) $a(x) = x$. (Hint. One must find $f(t,x) = f_0(e^{-t}x)$).
(3) $a(x,v) = v$, $f_0 = f_0(x,v) \in C^1(\mathbb{R}^d \times \mathbb{R}^d)$ and look for a solution $f = f(t,x,v) \in C^1((0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d)$. (Hint. One must find $f(t,x,v) = f_0(x - vt, v)$).

Exercice 3.2. *(1) Show that for any characteristics solution f to the transport equation associated to an initial datum $f_0 \in C^1_c(\mathbb{R}^d)$, for any times $T > 0$ and radius R, there exists some constants $C_T, R_T \in (0,\infty)$ such that*

$$\sup_{t\in[0,T]} \int_{B_R} |f(t,x)|\,dx \le C_T \int_{B_{R_T}} |f_0(x)|\,dx.$$

(Hint. Use the property of finite speed propagation of the transport equation).

(2) Adapt the proof of existence to the case $f_0 \in L^\infty$.

(3) Prove that for any $f_0 \in C_0(\mathbb{R}^d)$ there exists a global weak solution f to the transport equation which furthermore satisfies $f \in C([0,T];C_0(\mathbb{R}^d))$.

Exercice 3.3. *Consider the relaxation equation*

$$\partial_t f + v \cdot \nabla_x f = M(v)\rho_f - f$$

on the function $f = f(t,x,v)$, $t \ge 0$, $x, v \in \mathbb{R}^d$, where we denote

$$\rho_f := \int_{\mathbb{R}^d} f\,dv, \quad M(v) := (2\pi)^{-d/2}\exp(-|v|^2/2).$$

Prove the existence and uniqueness of a solution $f \in C(\mathbb{R}_+; L^1(\mathbb{R}^d \times \mathbb{R}^d))$ to the relaxation equation for any initial datum $f_0 \in L^1(\mathbb{R}^d \times \mathbb{R}^d)$.

Exercice 3.4. *1) Consider the transport equation with boundary condition*

$$\begin{cases} \partial_t f + \partial_x f + af = 0 \\ f(t,0) = b(t), \quad f(0,x) = f_0(x), \end{cases} \tag{3.1}$$

where $f = f(t,x)$, $t \geq 0$, $x \geq 0$. Assume $a \in L^\infty(\mathbb{R}_+)$. For any $f_0 \in L^1(\mathbb{R}_+)$ and $b \in C([0,T])$, establish that there exists a unique weak solution $f \in C([0,T]; L^1(\mathbb{R}_+))$ associated to equation (3.1). (Hint. One may observe that both

$$\frac{d}{dt}(e^{A(t+x)} f(t, t+x)) = 0, \quad \frac{d}{dx}(e^{A(x)} f(t+x, x)) = 0, \quad A(x) := \int_0^x a(u)\, du,$$

and a possible solution is

$$\bar f(t,x) := e^{A(x-t)-A(x)} f_0(x-t) \mathbf{1}_{x>t} + e^{-A(x)} b(t-x) \mathbf{1}_{t>x}).$$

2) Consider the renewal equation

$$\begin{cases} \partial_t f + \partial_x f + af = 0 \\ f(t,0) = \rho_{f(t)}, \quad f(0,x) = f_0(x), \end{cases} \tag{3.2}$$

where $f = f(t,x)$, $t \geq 0$, $x \geq 0$, and

$$\rho_g := \int_0^\infty g(y)\, a(y)\, dy.$$

Assume $a \in L^\infty(\mathbb{R}_+)$. For any $f_0 \in L^1(\mathbb{R}_+)$, establish that there exists a unique weak solution $f \in C([0,T]; L^1(\mathbb{R}_+))$ associated to equation (3.2).