Some exercises

Exercise 0.1 (Chapter 5) *Prove that for any $f \in H^1(\mathbb{R}^d)$, there holds*

$$\|\rho_\varepsilon * f - f\|_{L^2} \le C\,\varepsilon\,\|\,\nabla f\|_{L^2},$$

for a constant $C > 0$ which only depends on the function $\rho \in \mathbf{P}(\mathbb{R}^d) \cap \mathcal{D}(\mathbb{R}^d)$ used in the definition of the mollifier (ρ_ε). Deduce the Nash inequality. (Hint. Write $f = f - \rho_\varepsilon * f + \rho_\varepsilon * f$).

Exercise 0.2 (Chapter 6) *Let $S = S_{\mathcal{L}}$ be a strongly continuous semigroup on a Banach space $X \subset L^1$. Show that there is equivalence between*
(a) $S_{\mathcal{L}}$ is a stochastic semigroup;
(b) $\mathcal{L}^*1 = 0$ and $\mathcal{L}$ satisfies Kato's inequality

$$(\text{sign}\, f)\mathcal{L}f \le \mathcal{L}|f|, \quad \forall\, f \in D(\mathcal{L}).$$

(Hint. In order to prove (b) $\Rightarrow$ (a), consider $f \in D(\mathcal{L}^2)$ and estimate $|S_t f| - |f|$ by introducing a telescopic sum and a Taylor expansion in the time variable).

Exercise 0.3 (Chapter 6) *Consider $S_{\mathcal{L}^*}$ a (constant preserving) Markov semigroup and $\Phi : \mathbb{R} \to \mathbb{R}$ a concave function. Prove that $\mathcal{L}^*\Phi(m) \le \Phi'(m)\mathcal{L}^* m$. (Hint. Use that $\Phi(a) = \inf\{\ell(a);\, \ell \text{ affine such that } \ell \ge \Phi\}$ in order to prove $S_t^*(\Phi(m)) \le \Phi(S_t^* m)$ and $\Phi(b) - \Phi(a) \ge \Phi'(a)(b-a)$).*

Exercise 0.4 *We say that $(S_t)_{t\ge 0}$ is a dynamical system on a metric space $(\mathcal{Z}, d)$ if*
(S1) $\forall t \ge 0$, $S_t \in C(\mathcal{Z}, \mathcal{Z})$ (continuously defined on $\mathcal{Z}$);
(S2) $\forall x \in \mathcal{Z}$, $t \mapsto S_t\, x \in C([0,\infty), \mathcal{Z})$ (trajectories are continuous);
(S3) $S_0 = I$; $\forall\, s, t \ge 0$, $S_{t+s} = S_t\, S_s$ (semigroup property).
We say that $\bar z \in \mathcal{Z}$ is invariant (a steady state, a stationary point) if $S_t \bar z = \bar z$ for any $t \ge 0$.

Consider a bounded and convex subset $\mathcal{Z}$ of a Banach space X which is sequentially compact when it is endowed with the metric associated to the norm $\|\cdot\|_X$ (strong topology), to the weak topology $\sigma(X, X')$ or to the weak-$\star$ topology $\sigma(X, Y)$, $Y' = X$. Prove that any dynamical system $(S_t)_{t\ge 0}$ on $\mathcal{Z}$ admits at least one steady state
(Hint. Observe taht for any dyadic number $t > 0$, there exists $z_t \in \mathcal{Z}$ such that $S_t z_t = z_t$ thanks to the Schauder point fixed theorem).

Exercise 0.5 *(i) Prove that* $L^p \cap L^q \subset L^r$ *for any* $p \leq r \leq q$ *and*

$$\|u\|_{L^r} \leq \|u\|_{L^p}^{\theta} \|u\|_{L^q}^{1-\theta}, \quad \frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}, \quad \forall\, u \in L^p \cap L^q.$$

(ii) Prove that $L^{p_1}(L^{p_2}) \cap L^{q_1}(L^{q_2}) \subset L^{r_1}(L^{r_2})$*, and more precisely*

$$\|u\|_{L^{r_1}(L^{r_2})} \leq \|u\|_{L^{p_1}(L^{p_2})}^{\theta} \|u\|_{L^{q_1}(L^{q_2})}^{1-\theta}, \quad \frac{1}{r_i} = \frac{\theta}{p_i} + \frac{1-\theta}{q_i}$$

for any $u \in L^{p_1}(L^{p_2}) \cap L^{q_1}(L^{q_2})$ *for* <u>*a same*</u> $\theta \in (0,1)$.

(iii) Prove that $\dot{H}^{s_1} \cap \dot{H}^{s_2} \subset \dot{H}^s$ *for any* $s_1 \leq s \leq s_2$ *and*

$$\|u\|_{\dot{H}^s} \leq \|u\|_{\dot{H}^{s_1}}^{\theta} \|u\|_{\dot{H}^{s_2}}^{1-\theta}, \quad s = \theta s_1 + (1-\theta) s_2, \quad \forall\, u \in \dot{H}^{s_1} \cap \dot{H}^{s_2}.$$

(iv) Prove that $L^p(\dot{H}^a) \cap L^q(\dot{H}^b) \subset L^r(\dot{H}^c)$*, and more precisely*

$$\|u\|_{L^r(\dot{H}^c)} \leq \|u\|_{L^p(\dot{H}^a)}^{\theta} \|u\|_{L^q(\dot{H}^b)}^{1-\theta}, \quad \frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}, \ c = \theta a + (1-\theta) b$$

for any $u \in L^p(\dot{H}^a) \cap L^q(\dot{H}^b)$ *for* <u>*a same*</u> $\theta \in (0,1)$.

(v) Prove that $\dot{H}^{1/2}(\mathbb{R}^2) \subset L^4(\mathbb{R}^2)$, $\dot{H}^{1/2}(\mathbb{R}^3) \subset L^3(\mathbb{R}^2)$. *(Hint. We will use the classical Sobolev embedding and the following interpolation theorem in order to prove (at least when* $s \in (0, d-1]$*)*

$$\dot{H}^s(\mathbb{R}^d) \subset L^p(\mathbb{R}^d), \quad \text{with} \quad \frac{1}{p} = \frac{1}{2} - \frac{s}{d},$$

when $s \in [0, d)$*).*

Theorem 0.6 *(Interpolation) Assume that* T *is a linear mapping such that* $T : W^{s_0,p_0} \to W^{\sigma_0,q_0}$ *and* $T : W^{s_1,p_1} \to W^{\sigma_1,q_1}$ *are bounded, for some* $p_0, p_1, q_0, p_1 \in [1, \infty]$ *and some* $s_0, s_1, \sigma_0, \sigma_1 \in \mathbb{R}$*. Then* $T : W^{s_\theta,p_\theta} \to W^{\sigma_\theta,q_\theta}$ *for any* $\theta \in [0,1]$*, with*

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1},$$

and

$$s_\theta = (1-\theta) s_0 + \theta s_1, \quad \sigma_\theta = (1-\theta)\sigma_0 + \theta \sigma_1,$$

when ($p_0 = p_1$ *or* $s_0 = s_1$) *and* ($q_0 = q_1$ *or* $\sigma_0 = \sigma_1$).

Exam 2015 - Three problems

Problem I - the fragmentation equation

We consider the fragmentation equation

$$\partial_t f(t,x) = (\mathcal{F} f(t,.))(x)$$

on the density function $f = f(t,x) \geq 0$, $t, x > 0$, where the fragmentation operator is defined by

$$(\mathcal{F} f)(x) := \int_x^\infty b(y,x)\, f(y)\, dy - B(x) f(x).$$

We assume that the total fragmentation rate B and the fragmentation rate b satisfy

$$B(x) = x^\gamma,\ \gamma > 0, \quad b(x,y) = x^{\gamma-1}\wp(y/x),$$

with

$$0 < \wp \in C((0,1]),\ \int_0^1 z\,\wp(z)\,dz = 1, \quad \int_0^1 z^k\,\wp(z)\,dz < \infty,\ \forall\, k < 1.$$

1) Prove that for any $f, \varphi : \mathbb{R}_+ \to \mathbb{R}$, the following identity

$$\int_0^\infty (\mathcal{F} f)(x)\varphi(x)\,dx = \int_0^\infty f(x) \int_0^x b(x,y) \Big(\varphi(y) - \frac{y}{x}\varphi(x)\Big)\, dy dx$$

holds, whenever the two integrales at the RHS are absolutely convergent.

1) We define the moment function

$$M_k(f) = \int_0^\infty x^k\, f(x)\,dx, \quad k \in \mathbb{R}.$$

Prove that any solution f $(\in C^1([0,T); L^1_k) \cap L^1(0,T; L^1_{k+\gamma}),\ \forall T > 0)$ to the fragmentation equation formally (rigorously) satisfies

$$M_1(f(t)) = \text{cst}, \quad M_k(f(t,.)) \nearrow \text{ if } k < 1, \quad M_k(f(t,.)) \searrow \text{ if } k > 1.$$

Deduce that any solution $f \in C([0,\infty); L^1_{k+\gamma})$ to the fragmentation equation asymptotically satisfies

$$f(t,x)\,x \rightharpoonup M_1(f(0,.))\,\delta_{x=0} \text{ weakly in } (C_c([0,\infty))' \text{ as } t \to \infty.$$

2) For which values of $\alpha, \beta \in \mathbb{R}$, the function

$$f(t,x) = t^\alpha\, G(t^\beta\, x)$$

is a (self-similar) solution of the fragmentation equation such that $M_1(f(t,.)) = \text{cst}$. Prove then that the profile G satisfies the stationary equation

$$\gamma\,\mathcal{F} G = x\,\partial_x G + 2\,G.$$

3) Prove that for any solution f to the fragmentation equation, the rescalled density

$$g(t,x) := e^{-2t}\, f\left(e^{\gamma\, t} - 1, x\, e^{-t}\right)$$

solves the fragmentation equation in self-similar variable

$$\partial_t g + x\,\partial_x g + 2\, g = \gamma \mathcal{F}\, g.$$

Problem II - the discret Fokker-Planck equation

In all the problem, we consider the discrete Fokker-Planck equation

$$\partial_t f = \mathcal{L}_\varepsilon f := \Delta_\varepsilon f + \mathrm{div}_x(xf) \quad \text{ in } (0,\infty)\times\mathbb{R}, \tag{0.1}$$

where

$$\Delta_\varepsilon f = \frac{1}{\varepsilon^2}(k_\varepsilon * f - f) = \int_{\mathbb{R}} \frac{1}{\varepsilon^2} k_\varepsilon(x-y)(f(y)-f(x))\, dx,$$

and where $k_\varepsilon(x) = 1/\varepsilon k(x/\varepsilon)$, $0 \le k \in W^{1,1}(\mathbb{R}) \cap L^1_3(\mathbb{R})$,

$$\int_{\mathbb{R}} k(x) \begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix} dx = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

that we complement with an initial condition

$$f(0,x) = \varphi(x) \quad \text{ in } \mathbb{R}. \tag{0.2}$$

Question 1

Establish the formula

$$\int (\Delta_\varepsilon f)\, \beta'(f)\, m\, dx = \int \beta(f)\, \Delta_\varepsilon m\, dx - \iint \frac{1}{\varepsilon^2} k_\varepsilon(y-x) J(x,y)\, m(x)\, dy dx$$

with

$$J(x,y) := \beta(f(y)) - \beta(f(x)) - (f(y)-f(x))\beta'(f(x)).$$

Formally prove that the discrete Fokker-Planck equation is mass conservative and satisfies the (weak) maximum principle. Explain (quickly) why for $\varphi \in L^2_k(\mathbb{R}^d)$, $k \ge 0$, the equation (0.1)-(0.2) has a (unique) solution $f(t)$ in some functional space to be specified. We recall that

$$\|g\|_{L^p_k} := \|g\, \langle\cdot\rangle^k\|_{L^p}, \quad \langle x\rangle := (1+|x|^2)^{1/2}.$$

Establish that if $L^2_k(\mathbb{R}^d) \subset L^1(\mathbb{R}^d)$, then the solution satisfies

$$\sup_{t\ge 0} \|f(t)\|_{L^1} \le \|\varphi\|_{L^1}.$$

Question 2

We define

$$M_k(t) = M_k(f(t)), \quad M_k(f) := \int_{\mathbb{R}} f(x) \langle x \rangle^k \, dx.$$

Prove that if $0 \le \varphi \in L^2_k(\mathbb{R}^d) \subset L^1(\mathbb{R}^d)$, then the solution satisfies

$$M_0(t) \equiv M_0(\varphi) \quad \forall t \ge 0.$$

Prove that when $L^2_k(\mathbb{R}^d) \subset L^1_2(\mathbb{R}^d)$, then the solution satisfies

$$\frac{d}{dt} M_2(t) = M_0(0) - M_2(t).$$

Question 3 (discrete Nash inequality)

Prove that there exist $\theta, \eta \in (0,1)$, $\rho > 0$ such that

$$|\hat{k}(r)| < \theta \;\forall\, |r| > \rho, \quad 1 - \hat{k}(r) > \eta r^2 \;\forall\, |r| < \rho.$$

Deduce that for any $f \in L^1 \cap L^2$ and any $R > 0$, there holds

$$\|k_\varepsilon * f\|^2_{L^2} \le \theta \|f\|^2_{L^2} + R \|f\|^2_{L^1} + \frac{1}{\eta R^2} \, I_\varepsilon[f]$$

with

$$I_\varepsilon[f] := \int_{\mathbb{R}} \frac{1 - \hat{k}(\varepsilon\xi)}{\varepsilon^2} \, |\hat{f}|^2 \, d\xi.$$

Question 4

Prove that for any $\alpha > 0$, the solution satisfies

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} f^2 \;\; &= \;\; -(\frac{1}{\varepsilon^2} - \alpha) \iint k_\varepsilon(y-x) \, (f(y) - f(x))^2 \, dx dy \\ &\quad + 2\alpha \int (k_\varepsilon * f) f - 2\alpha \int f^2 + \int f^2, \end{aligned}$$

and then

$$\frac{d}{dt} \int_{\mathbb{R}} f^2 \;\; \le \;\; -(1 - \alpha\varepsilon^2) \, I_\varepsilon[f] + \alpha \|k_\varepsilon * f\|^2_{L^2} - (\alpha - 1) \|f\|^2_{L^2}.$$

Deduce that for $\varepsilon \in (0, \varepsilon_0)$, $\varepsilon_0 > 0$ small enough, the solution satisfies

$$\frac{d}{dt} \|f\|^2_{L^2} \le -\frac{1}{2} \, I_\varepsilon[f] - C_1 \|f\|^2_{L^2} + C_2 \|f\|^2_{L^1}.$$

Also prove that

$$\frac{d}{dt} \int_{\mathbb{R}} f^2 |x|^2 \le \int_{\mathbb{R}} f^2 - \int_{\mathbb{R}} f^2 |x|^2.$$

Question 5

Prove that there exists a constant C such that the set

$$\mathcal{Z} := \{f \in L^2_1(\mathbb{R});\ f \ge 0,\ \|f\|_{L^1} = 1,\ \|f\|_{L^2_1} \le C\}$$

is invariant under the action of the semigroup associated to discrete Fokker-Planck equation. Deduce that there exists a unique solution G_ε to

$$0 < G_\varepsilon \in L^2_1, \quad \Delta_\varepsilon G_\varepsilon + \mathrm{div}(x G_\varepsilon) = 0, \quad M_0(G_\varepsilon) = 1.$$

Prove that for any $\varphi \in L^2_1$, $M_0(\varphi) = 1$, the associated solution satisfies

$$f(t) \rightharpoonup G \quad \text{as} \quad t \to \infty.$$

Problem III - the Fokker-Planck equation with weak confinement

In all the problem, we consider the Fokker-Planck equation

$$\partial_t f = \Lambda f := \Delta_x f + \mathrm{div}_x(f\, \nabla a(x)) \quad \text{in } (0,\infty) \times \mathbb{R}^d \tag{0.3}$$

for the confinement potential

$$a(x) = \frac{\langle x \rangle^\gamma}{\gamma}, \quad \gamma \in (0,1), \quad \langle x \rangle^2 := 1 + |x|^2,$$

that we complement with an initial condition

$$f(0,x) = \varphi(x) \quad \text{in } \mathbb{R}^d. \tag{0.4}$$

Question 1

Exhibit a stationary solution $G \in \mathbf{P}(\mathbb{R}^d)$. Formally prove that this equation is mass conservative and satisfies the (weak) maximum principle. Explain (quickly) why for $\varphi \in L^p_k(\mathbb{R}^d)$, $p \in [1,\infty]$, $k \ge 0$, the equation (0.3)-(0.4) has a (unique) solution $f(t)$ in some functional space to be specified. Establish that if $L^p_k(\mathbb{R}^d) \subset L^1(\mathbb{R}^d)$ then the solution satisfies

$$\sup_{t \ge 0} \|f(t)\|_{L^1} \le \|\varphi\|_{L^1}.$$

Can we affirm that $f(t) \to G$ as $t \to \infty$? and that convergence is exponentially fast?

Question 2

We define

$$\mathcal{B} f := \Lambda f - M \chi_R f$$

with $\chi_R(x) := \chi(x/R)$, $\chi \in \mathcal{D}(\mathbb{R}^d)$, $0 \le \chi \le 1$, $\chi(x) = 1$ for any $|x| \le 1$, and with $M, R > 0$ to be fixed.

We denote by $f_{\mathcal{B}}(t) = S_{\mathcal{B}}(t)\varphi$ the solution associated to the evolution PDE corresponding to the operator $\mathcal{B}$ and the initial condition (0.4).
(1) Why such a solution is well defined (no more than one sentence of explanation)?
(2) Prove that there exists $M, R > 0$ such that for any $k \geq 0$ there holds

$$\frac{d}{dt}\int_{\mathbb{R}^d} f_{\mathcal{B}}(t)\langle x\rangle^k\,dx \leq -c_k\int_{\mathbb{R}^d} f_{\mathcal{B}}(t)\langle x\rangle^{k+\gamma-2} \leq 0,$$

for some constant $c_k \geq 0$, $c_k > 0$ if $k > 0$, and

$$\|S_{\mathcal{B}}(t)\|_{L^1_k \to L^1_k} \leq 1.$$

(3) Establish that if $u \in C^1(\mathbb{R}_+)$ satisfies

$$u' \leq -c\,u^{1+1/\alpha}, \quad c, \alpha > 0,$$

there exists $C = C(c, \alpha, u(0))$ such that

$$u(t) \leq C/t^\alpha \quad \forall\, t > 0.$$

(4) Prove that for any $k_1 < k < k_2$ there exists $\theta \in (0,1)$ such that

$$\forall\, f \geq 0 \quad M_k \leq M_{k_1}^\theta\, M_{k_2}^{1-\theta}, \quad M_\ell := \int_{\mathbb{R}^d} f(x)\,\langle x\rangle^\ell\,dx$$

and write θ as a function of k_1, k and k_2.
(5) Prove that if $\ell > k > 0$ there exists $\alpha > 0$ such that

$$\|S_{\mathcal{B}}(t)\|_{L^1_\ell \to L^1} \leq \|S_{\mathcal{B}}(t)\|_{L^1_\ell \to L^1_k} \leq C/\langle t\rangle^\alpha,$$

and that $\alpha > 1$ if ℓ is large enough (to be specified).
(6) Prove that

$$S_{\mathcal{L}} = S_{\mathcal{B}} + S_{\mathcal{B}} * (\mathcal{A}S_{\mathcal{L}}),$$

and deduce that for k large enough (to be specified)

$$\|S_{\mathcal{L}}\|_{L^1_k \to L^1_k} \leq C.$$

Remark. You have recovered (with a simpler and more general proof) a result established by Toscani and Villani in 2001.

Question 3 (difficult)

Establish that there exists $\kappa > 0$ such that for any $h \in \mathcal{D}(\mathbb{R}^d)$ satisfying

$$\langle h\rangle_\mu := \int_{\mathbb{R}^d} h\,d\mu = 0, \quad \mu(dx) := G(x)\,dx,$$

there holds

$$\int_{\mathbb{R}^d} |\nabla h|^2\,d\mu \geq \kappa\int_{\mathbb{R}^d} h^2\,\langle x\rangle^{2(\gamma-1)}\,d\mu.$$

Question 4

Establish that for any $\varphi \in \mathcal{D}(\mathbb{R}^d)$ and any convex function $j \in C^2(\mathbb{R})$ the associated solution $f(t) = S_\Lambda(t)\varphi$ satisfies

$$\frac{d}{dt}\int_{\mathbb{R}^d} j(f(t)/G)\, G\, dx = -\mathcal{D}_j \le 0$$

and give the expression of the functional $\mathcal{D}_j$. Deduce thgat

$$\|f(t)/G\|\|_{L^\infty} \le \|f_0/G\|\|_{L^\infty} \quad \forall\, t \ge 0.$$

Question 5 (difficult)

Prove that for any $\varphi \in \mathcal{D}(\mathbb{R}^d)$ and for any $\alpha > 0$ there exists C such that

$$\|f(t) - \langle\varphi\rangle\, G\|_{L^2} \le C/t^\alpha.$$

Exam 2016 - Two problems

Problem I - Nash estimates

We consider the evolution PDE

$$\partial_t f = \mathrm{div}(A\nabla f), \tag{0.1}$$

on the unknown $f = f(t,x)$, $t \ge 0$, $x \in \mathbb{R}^d$, with $A = A(x)$ a symmetric, uniformly bounded and coercive matrix, in the sense that

$$\nu\, |\xi|^2 \le \xi \cdot A(x)\xi \le C\, |\xi|^2, \quad \forall\, x, \xi \in \mathbb{R}^d.$$

It is worth emphasizing that we do not make any regularity assumption on A. We complement the equation with an initial condition

$$f(0,x) = f_0(x).$$

1) Existence. What strategy can be used in order to exhibit a semigroup $S(t)$ in $L^p(\mathbb{R}^d)$, $p = 2$, $p = 1$, which provides solutions to (0.1) for initial date in $L^p(\mathbb{R}^d)$? Is the semigroup positive? mass conservative?

In the sequel we will not try to justify rigorously the a priori estimates we will establish, but we will carry on the proofs just as if there do exist nice (smooth and fast decaying) solutions. We denote by C or C_i some constants which may differ from line to line.

2) Uniform estimate. a) Prove that any solution f to (0.1) satisfies

$$\|f(t)\|_{L^2} \le C\, t^{-d/4}\, \|f_0\|_{L^1}, \quad \forall\, t > 0.$$

b) We define the dual semigroup $S^*(t)$ by

$$\langle S^*(t)g_0, f_0\rangle = \langle g_0, S(t)f_0\rangle, \quad \forall\, t \geq 0,\ f_0 \in L^p,\ g_0 \in L^{p'}.$$

Identify $S^*(t)$ and deduce that any solution f to (0.1) satisfies

$$\|f(t)\|_{L^\infty} \leq C\, t^{-d/4}\, \|f_0\|_{L^2}, \quad \forall\, t > 0.$$

c) Conclude that any solution f to (0.1) satisfies

$$\|f(t)\|_{L^\infty} \leq [C^2 2^{d/2}]\, t^{-d/2}\, \|f_0\|_{L^1}, \quad \forall\, t > 0.$$

3) Entropy and first moment. For a given and (nice) probability measure f, we define the (mathematical) entropy and the first moment functional by

$$H := \int_{\mathbb{R}^d} f \log f \, dx, \quad M := \int_{\mathbb{R}^d} f\, |x|\, dx.$$

a) Prove that for any $\lambda \in \mathbb{R}$, there holds

$$\min_{s \geq 0} \{s \log s + \lambda\, s\} = -e^{-\lambda - 1}.$$

Deduce that there exists a constant $D = D(d)$ such that for any (nice) probability measure f and any $a \in \mathbb{R}_+$, $b \in \mathbb{R}$, there holds

$$H + aM + b \geq -e^{-b-1}\, a^{-d}\, D.$$

b) Making the choice $a := d/M$ and $e^{-b} := (e/D)\, a^d$, deduce that

$$M \geq \kappa\, e^{-H/d}, \tag{0.2}$$

for some $\kappa = \kappa(d) > 0$.

From now on, **we restrict ourself** to consider an initial datum which is a (nice) probability measure:

$$f_0 \geq 0, \quad \int_{\mathbb{R}^d} f_0 \, dx = 1,$$

and we denote by $f(t)$ the nonnegative and normalized solution to the evolution PDE (0.1) corresponding to f_0. We also denote by $H = H(t)$, $M = M(t)$ the associated entropy and first moment.

3) Dynamic estimate on the entropy. Deduce from 2) that f satisfies

$$H(t) \leq K - \frac{d}{2} \log t, \quad \forall\, t > 0, \tag{0.3}$$

for some constant $K \in \mathbb{R}$ (independent of f_0).

4) Dynamic estimate on the entropy and the first moment. a) Prove that

$$\Big|\frac{d}{dt}M(t)\Big| \le C \int |\nabla f(t)|,$$

for some positive constant $C = C(A)$.
b) Deduce that there exists a constant $\theta = \theta(C, \nu) > 0$ such that

$$\Big|\frac{d}{dt}M(t)\Big| \le \theta \left(-\frac{d}{dt}H(t)\right)^{1/2}, \quad \forall t > 0. \tag{0.4}$$

c) Prove that for the heat equation (when $A = I$), we have

$$\frac{d}{dt}\int f|x|^2\,dx = 2,$$

and next

$$M(t) \le C\,\langle t\rangle^{1/2}, \quad t \ge 0. \tag{0.5}$$

From now on, **we will always restrict ourself** to consider the Dirac mass initial datum

$$f_0 = \delta_0(dx),$$

and our goal is to establish a similar estimate as (0.5) (and in fact, a bit sharper estimate than (0.5)) for the corresponding solution.

5) Dynamic estimate on the first moment. a) Deduce from 1) that there exists (at least) one function $f \in C((0,\infty); L^1) \cap L^\infty_{loc}((0,\infty); L^\infty)$ which is a solution to the evolution PDE (0.1) associated to the initial datum δ_0. Why does that solution satisfy the same above estimates for positive times?
b) We define

$$R = R(t) := K/d - H(t)/d - \frac{1}{2}\log t \ge 0,$$

where K is defined in (0.3). Observing that $M(0) = 0$, deduce from the previous estimates that

$$C_1\,t^{1/2}\,e^R \le M \le C_2 \int_0^t \left(\frac{1}{2s} + \frac{dR}{ds}\right)^{1/2} ds, \quad \forall t > 0.$$

c) Observe that for $a > 0$ and $a + b > 0$, we have $(a+b)^{1/2} \le a^{1/2} + b/(2a^{1/2})$, and deduce that

$$C_1\,e^R \le Mt^{-1/2} \le C_2(1+R), \quad \forall t > 0.$$

d) Deduce from the above estimate that R must be bounded above, and then

$$C_1\,t^{1/2} \le M \le C_2\,t^{1/2}, \quad \forall t > 0.$$

You have recovered one of the most crucial step of Nash's article "Continuity of solutions of parabolic and elliptic equations", Amer. J. Math. (1958).

Problem II - The fractional Fokker-Planck equation

We consider the fractional Fokker-Planck equation

$$\partial_t f = \mathcal{L} f := I[f] + \mathrm{div}_x(E f) \quad \text{in } (0,\infty)\times\mathbb{R}, \tag{0.1}$$

where

$$I[f](x) = \int_{\mathbb{R}} k(y-x)\,(f(y)-f(x))\,dy, \quad k(z) = \frac{1}{|z|^{1+\alpha}}, \quad \alpha\in(0,1),$$

and where E is a smooth vectors field such that

$$\forall\,|x|\geq 1, \quad |E(x)|\leq C\,\langle x\rangle, \quad \mathrm{div} E(x)\leq C, \quad x\cdot E\geq |x|^2.$$

We complement the equation with an initial condition

$$f(0,x) = \varphi(x) \quad \text{in} \quad \mathbb{R}. \tag{0.2}$$

We denote by $\mathcal{F}$ the Fourier transform operator, and next $\hat{f} = \mathcal{F} f$ for a given function f on the real line.

Question 1. Preliminary issues (if not proved, theses identities can be accepted). Here all the functions (f, φ, β) are assumed to be suitably nice so that all the calculations are licit.

a) - Establish the formula

$$\int (I[f])\,\beta'(f)\,\varphi\,dx = \int \beta(f)\,I[\varphi]\,dx - \iint k(y-x) J(x,y)\,\varphi(x)\,dydx$$

with

$$J(x,y) := \beta(f(y)) - \beta(f(x)) - (f(y)-f(x))\beta'(f(x)).$$

b) - Prove that there exists a positive constant C_1 such that

$$\mathcal{F}(I[f])(\xi) = \int_{\mathbb{R}} f(x)\,e^{-ix\xi}\Big\{\int_{\mathbb{R}} \frac{\cos(z\xi)-1}{|z|^{1+\alpha}}\,dz\Big\}dx = C_1\,|\xi|^\alpha\,\hat{f}(\xi), \quad \forall\,\xi\in\mathbb{R}.$$

c) - Denote $s := \alpha/2$. Prove that

$$|||f|||^2_{\dot{H}^s} := \int_{\mathbb{R}^2} \frac{|f(x)-f(y)|^2}{|x-y|^{1+2s}}\,dxdy = \int_{\mathbb{R}} \Big\|\frac{f(z+\cdot)-f(\cdot)}{|z|^{s+1/2}}\Big\|^2_{L^2(\mathbb{R})}\,dz,$$

and then that there exists a positive constant C_2 such that

$$|||f|||^2_{\dot{H}^s} = C_2 \int_{\mathbb{R}} |\xi|^{2s}\,|\hat{f}(\xi)|^2\,d\xi =: C_2 \|f\|^2_{\dot{H}^s}.$$

d) - Prove that

$$\int_{\mathbb{R}} I[f]\,f\,dx = -\frac{1}{2}|||f|||^2_{\dot{H}^s}.$$

From questions 2 to 4, we consider f (and g) a solution to the fractional Fokker-Planck equation (0.1) and we establish formal a priori estimates.

Question 2. Moment estimates. For any $k \geq 0$, we define

$$M_k = M_k(t) = M_k(f(t)), \quad \text{with} \quad M_k = M_k(f) := \int_{\mathbb{R}} f(x) \langle x \rangle^k \, dx.$$

Prove that the solution f satisfies

$$M_0(t) \equiv M_0(\varphi), \quad \forall \, t \geq 0.$$

Prove that for any $k \in (0, \alpha)$, there exists $C > 0$ such that

$$C^{-1} |\langle y \rangle^k - \langle x \rangle^k| \leq \left| |y|^2 - |x|^2 \right|^{k/2} \leq C \, \left(|y - x|^{k/2} \, |x|^{k/2} + |y - x|^k \right),$$

and deduce that there exist $C_1, C_2 > 0$ such that the solution f satisfies

$$\frac{d}{dt} M_k \leq C_1 \, M_{k/2} - C_2 \, M_k.$$

Conclude that there exists $A_k = A_k(M_0(\varphi))$ such that the solution f satisfies

$$\sup_{t \geq 0} M_k(t) \leq \max(M_k(0), A_k). \tag{0.3}$$

Question 3. Fractional Nash inequality and L^2 estimate. Prove that there exists a constant $C > 0$ such that

$$\forall \, h \in \mathcal{D}(\mathbb{R}), \quad \|h\|_{L^2} \leq C \, \|h\|_{L^1}^{\frac{\alpha}{1+\alpha}} \, \|h\|_{\dot{H}^s}^{\frac{1}{1+\alpha}}.$$

Deduce that the square of the L^2-norm $u := \|f(t)\|_{L^2}^2$ of the solution f satisfies

$$\frac{d}{dt} u \leq -C_1 u^{1+\alpha} + C_2,$$

for some constants $C_i = C_i(M_0(\varphi)) > 0$. Conclude that there exists $A_2 = A_2(M_0(\varphi))$ such that

$$\sup_{t \geq 0} u(t) \leq \max(u(0), A_2). \tag{0.4}$$

Question 4. Around generalized entropies and the L^1-norm. Consider a convex function β and define the entropy $\mathcal{H}$ and the associated dissipation of entropy $\mathcal{D}$ by

$$\mathcal{H}(f|g) := \int_{\mathbb{R}} \beta(X) \, g \, dx,$$

$$\mathcal{D}(f|g) := \int_{\mathbb{R}} \int_{\mathbb{R}} k \, g_* \left\{ \beta(X_*) - \beta(X) - \beta'(X)(X_* - X) \right\} dx dx_*,$$

where $k = k(x - x_*)$, $g_* = g(x_*)$, $X = f(x)/g(x)$ and $X_* = f(x_*)/g(x_*)$. Why do two solutions f and g satisfy

$$\frac{d}{dt}\mathcal{H}(f(t)|g(t)) \le -\mathcal{D}(f(t)|g(t))? \tag{0.5}$$

Deduce that for $\beta(s) = s_+$ and $\beta(s) = |s|$, any solution f satisfies

$$\int \beta(f(t,\cdot))\,dx \le \int \beta(\varphi)\,dx, \quad \forall\, t \ge 0. \tag{0.6}$$

Question 5. Well-posedness. Explain briefly how one can establish the existence of a weakly continuous semigroup $S_{\mathcal{L}}$ defined in the space $X = L^1_s \cap L^2$ such that it is a contraction for the L^1 norm and such that for any $\varphi \in X$ the function $f(t) := S_{\mathcal{L}}(t)\varphi$ is a (weak) solution to the fractional Fokker-Planck equation (0.1). Why is $S_{\mathcal{L}}$ mass and positivity preserving and why does any associated trajectory satisfy (0.3), (0.4) and (0.6)?
(*Ind.* One may consider the sequence of kernels $k_n(z) := k(z)\,_{n^{-1}<|z|<n}$).

Question 6. Prove that there exists a constant C such that the set

$$\mathcal{Z} := \{f \in L^1(\mathbb{R});\ f \ge 0,\ \|f\|_{L^1} = 1,\ \|f\|_X \le C\}$$

is invariant under the action of $S_{\mathcal{L}}$. Deduce that there exists at least one function $G \in X$ such that

$$IG + \mathrm{div}_x(E\,G) = 0, \quad G \ge 0, \quad M_0(G) = 1. \tag{0.7}$$

Question 7. We accept that for any convex function β, any nonnegative solution $f(t)$ and any nonnegative stationary solution G the following inequality holds

$$\mathcal{H}(f(t)|G) + \int_0^t \mathcal{D}(f(s)|G)\,ds \le \mathcal{H}(\varphi|G). \tag{0.8}$$

Deduce that $\mathcal{D}(g|G) = 0$ for any (other) stationary solution g. *(Ind. First consider the case when $\beta \in W^{1,\infty}(\mathbb{R})$ and next use an approximation argument).* Deduce that

$$G(y)\left(\frac{g(y)}{G(y)} - \frac{g(x)}{G(x)}\right)^2 = 0 \quad \text{for a.e. } x, y \in \mathbb{R},$$

and then that the solution to (0.7) is unique. Prove that for any $\varphi \in X$, there holds

$$S_{\mathcal{L}}(t)\varphi \rightharpoonup M_0(\varphi)\,G \ \text{ weakly in } X, \quad \text{as} \quad t \to \infty.$$

Question 8. (a) Introducing the splitting

$$\mathcal{A} := \lambda\, I,\ \lambda \in \mathbb{R}, \quad \mathcal{B} := \mathcal{L} - \mathcal{A},$$

explain why

$$S_{\mathcal{L}} = S_{\mathcal{B}} + S_{\mathcal{B}}\mathcal{A} * S_{\mathcal{L}},$$

and next for any $n \geq 1$

$$S_{\mathcal{L}} = S_{\mathcal{B}} + ... + (S_{\mathcal{B}}\mathcal{A})^{*(n-1)} * S_{\mathcal{L}} + (S_{\mathcal{B}}\mathcal{A})^{*n} * S_{\mathcal{L}},$$

where the convolution on $\mathbb{R}_+$ is defined by

$$(u * v)(t) := \int_0^t u(t-s)\, v(s)\, ds,$$

and the itareted convolution by $u^{*1} = u$, $u^{*k} = u^{*(k-1)} * u$ if $k \geq 2$.
(b) Prove that for $\lambda > 0$, large enough, there holds

$$\|S_{\mathcal{B}}\|_{Y \to Y} \leq e^{-t}, \quad Y = L^1_k,\ k \in (0, \alpha),\ Y = L^2,$$

as well as

$$\|S_{\mathcal{B}}(t)\|_{L^1 \to L^2} \leq \frac{C}{t^{1/(2\alpha)}}\, e^{-t}, \quad \forall\, t \geq 0, \qquad \int_0^\infty \|S_{\mathcal{B}}(s)\|^2_{L^2 \to \dot{H}^s}\, ds \leq C,$$

and deduce that for n large enough

$$\int_0^\infty \|(\mathcal{A}S_{\mathcal{B}})^{(*n)}(s)\|_{L^1 \to \dot{H}^s}\, ds \leq C.$$

(c) Establish that for any $\varphi \in L^1_s$, $k > 0$, the associated solution $f(t) = S_{\mathcal{L}}(t)\varphi$ splits as

$$f(t) = g(t) + h(t), \quad \|g(t)\|_{L^1} \leq e^{-t}, \quad \|h(t)\|_{L^1_k \cap \dot{H}^s} \leq C(M_0(\varphi)).$$

(d) Conclude that

$$\forall\, \varphi \in L^1(\mathbb{R}), \quad \|S_{\mathcal{L}}(t)\varphi - M_0(\varphi) G\|_{L^1} \to 0 \quad \text{as} \quad t \to \infty.$$

Question 9. Justification of (0.8). We define the regularized operator

$$\mathcal{L}_{\varepsilon,n} f := \varepsilon \Delta_x f + I_n[f] + \mathrm{div}_x(Ef)$$

with $\varepsilon > 0$, $n \in \mathbb{N}^*$ and I_n associated to the kernel k_n (introduced in question 5). Why is there a (unique) solution $G_{\varepsilon,n}$ to the stationary problem

$$G_{\varepsilon,n} \in X, \quad \mathcal{L}_{\varepsilon,n} G_{\varepsilon,n} = 0, \quad G_{\varepsilon,n} \geq 0, \quad M_0(G_{\varepsilon,n}) = 1,$$

and why does a similar inequality as (0.8) hold? Prove that there exist $G_\varepsilon, G \in X$, $\varepsilon > 0$, such that (up to the extraction of a subsequence) $G_{n,\varepsilon} \to G_\varepsilon$ and $G_\varepsilon \to G$ strongly in L^1. Prove that a similar strong convergence result holds for the familly $S_{\mathcal{L}_{\varepsilon,n}}(t)\varphi$, $\varphi \in X$. Conclude that (0.8) holds.

Exam 2017 - Two problems

Problem I - A general Fokker-Planck equation with strong confinement

We consider the evolution PDE

$$\partial_t f = \Delta f + \mathrm{div}(Ef), \tag{0.1}$$

on the unknown $f = f(t,x)$, $t \geq 0$, $x \in \mathbb{R}^d$, with $E = E(x)$ a given smooth force field which satisfies for some $\gamma \geq 1$

$$\forall\, |x| \geq 1, \quad |E(x)| \leq C\,|x|^{\gamma-1}, \quad \mathrm{div}E(x) \leq C|x|^{\gamma-2}, \quad x \cdot E \geq |x|^\gamma.$$

We complement the equation with an initial condition

$$f(0,x) = f_0(x).$$

Question 1. Which strategy can be used in order to exhibit a semigroup $S(t)$ in $L^p(\mathbb{R}^d)$, which provides solutions to (0.1) for initial data in $L^p(\mathbb{R}^d)$? Is the semigroup positive? mass conservative? Explain briefly why there exists a function $G = G(x)$ such that

$$0 \leq G \in L^2(m), \quad \langle G \rangle := \int G = 1, \quad \mathcal{L}G = 0.$$

We accept that $G > 0$. For any nice function $f : \mathbb{R}^d \to \mathbb{R}$ we denote $h := f/G$ and, reciprocally, for any nice function $h : \mathbb{R}^d \to \mathbb{R}$ we denote $f := Gh$.

In the sequel we will not try to justify rigorously the a priori estimates we will establish, but we will carry on the proofs just as if there do exist nice (smooth and fast decaying) solutions. We denote by C or C_i some constants which may differ from line to line.

Question 2. Prove that for any weight function $m : \mathbb{R}^d \to [1,\infty)$ and any nice function $f : \mathbb{R}^d \to \mathbb{R}$, there holds

$$\int (\mathcal{L}f) f m = -\int |\nabla f|^2 m + \frac{1}{2}\int f^2 \mathcal{L}^* m,$$

where we will make explicit the expression of $\mathcal{L}^*$.

Question 3. Prove that there exist $w : \mathbb{R}^d \to [1,\infty)$, $\alpha > 0$ and $b, R_0 \geq 0$ such that

$$\mathcal{L}^* w \leq -\alpha\, w + b\mathbf{1}_{B_{R_0}}.$$

Question 4. For some constant $\lambda \geq 0$ to be specified later, we define $W := w + \lambda$. Deduce from the previous question that

$$\int h^2 w\, G \leq \frac{1}{\alpha}\int h^2\,(b\,\mathbf{1}_{B_{R_0}} - \mathcal{L}^* W)\, G,$$

for any nice function $h : \mathbb{R}^d \to \mathbb{R}$.

Question 5. Take a nice function $h : \mathbb{R}^d \to \mathbb{R}$ such that $\langle hG \rangle = 0$ and denote $G(\Omega) := \langle G\, \mathbf{1}_\Omega \rangle$. Prove that for any $R \geq R_0$ there exists $\kappa_R \in (0, \infty)$ such that

$$\int h^2\, \mathbf{1}_{B_R}\, G \leq \kappa_R \int_{B_R} |\nabla h|^2\, G + \frac{1}{G(B_R)} \Big(\int_{B_R^c} h\, G \Big)^2$$

and deduce that

$$\int h^2\, \mathbf{1}_{B_R}\, G \leq \frac{\kappa_R}{1+\lambda} \int |\nabla h|^2\, W\, G + \frac{G(B_R^c)}{G(B_R)} \int h^2\, w\, G.$$

Question 6. Establish finally that there exist some constants $\lambda, K_1 \in (0, \infty)$ such that

$$\frac{2}{K_1} \int h^2 w G \leq \int \Big(W |\nabla h|^2 - \frac{1}{2} h^2 \mathcal{L}^* W \Big) G = \int (-\mathcal{L} f) f G^{-1} W,$$

for all nice function h such that $\langle hG \rangle = 0$.

Question 7. Consider a nice solution f to (0.1) associated to an initial datum f_0 such that $\langle f_0 \rangle = 0$. Establish that f satisfies

$$\frac{1}{2} \frac{d}{dt} \int h_t^2\, W G = - \int |\nabla h|^2\, W G + \frac{1}{2} \int h^2 G \mathcal{L}^* W.$$

Deduce that there exists $K_2 \in (0, \infty)$ such that f satisfies the decay estimate

$$\int f_t^2\, W G^{-1}\, dx \leq e^{-K_2 t} \int f_0^2\, W G^{-1}\, dx, \quad \forall\, t \geq 0.$$

Problem II - Estimates for the relaxation equation

We consider the relaxation equation

$$\partial_t f = \mathcal{L} f := -v \cdot \nabla f + \rho_f M - f \quad \text{in} \quad (0, \infty) \times \mathbb{R}^{2d}, \tag{0.1}$$

on the unknown $f = f(t, x, v)$, $t \geq 0$, $x, v \in \mathbb{R}^d$, with

$$\rho_f(t, x) = \int_{\mathbb{R}^d} f(t, x, v)\, dv, \quad M(v) := \frac{1}{(2\pi)^{d/2}} \exp(-|v|^2/2).$$

We complement the equation with an initial condition

$$f(0, x, v) = f_0(x, v) \quad \text{in} \quad \mathbb{R}^{2d}. \tag{0.2}$$

Question 1. A priori estimates and associated semigroup. We denote by f a nice solution to the relaxation equation (0.1)–(0.2).

(a) Prove that f is mass conserving.

(b) Prove that

$$|\rho_g| \le \|g\|_{L^2_v(M^{-1/2})}, \quad \forall\, g = g(v) \in L^2_v(M^{-1/2})$$

and deduce that

$$\|f(t,\cdot)\|_{L^2_{xv}(M^{-1/2})} \le \|f_0\|_{L^2_{xv}(M^{-1/2})}.$$

(c) Consider $m = \langle v \rangle^k$, $k > d/2$. Prove that there exists a constant $C \in (0,\infty)$ such that

$$|\rho_g| \le C\|g\|_{L^p_v(m)}, \quad \forall\, g = g(v) \in L^p_v(m), \quad p = 1, 2,$$

and deduce that

$$\|f(t,\cdot)\|_{L^p_{xv}(m)} \le e^{\lambda t}\|f_0\|_{L^p_{xv}(m)},$$

for a constant $\lambda \in [0,\infty)$ that we will express in function of C.

(d) What strategy can be used in order to exhibit a semigroup $S(t)$ in $L^p_{xv}(m)$, $p = 2$, $p = 1$, which provides solutions to (0.1) for initial date in $L^p_{xv}(m)$? Is the semigroup positive? mass conservative? a contraction in some spaces?

The aim of the problem is to prove that the associated semigroup $S_\mathcal{L}$ to (0.1) is bounded in $L^p(m)$, $p = 1, 2$, without using the estimate proved in question (1b).

In the sequel we will not try to justify rigorously the a priori estimates we will establish, but we will carry on the proofs just as if there do exist nice (smooth and fast decaying) solutions.

We define

$$\mathcal{A}f := \rho_f M, \quad \mathcal{B}f = \mathcal{L}f - \mathcal{A}f.$$

Question 2. Prove that $S_\mathcal{B}$ satisfies a growth estimate $\mathcal{O}(e^{-t})$ in any $L^p_{xv}(m)$ space. Using the Duhamel formula

$$S_\mathcal{L} = S_\mathcal{B} + S_\mathcal{B}\mathcal{A} * S_\mathcal{L}$$

prove that $S_\mathcal{L}$ is bounded in $L^1_{xv}(m)$.

Question 3. Establish that $\mathcal{A} : L^1_{xv}(m) \to L^1_x L^\infty_v(m)$ where

$$\|g\|_{L^1_x L^p_v(m)} := \int_{\mathbb{R}^d} \|g(x,)\|_{L^p(m)}\,dx.$$

Prove that

$$\frac{d}{dt}\int \Big(\int f^p\,dx\Big)^{1/p} dv = \int \Big(\int (\partial_t f) f^{p-1}\,dx\Big)\Big(\int f^p\,dx\Big)^{1/p-1} dv.$$

Deduce that $S_\mathcal{B}$ satisfies a growth estimate $\mathcal{O}(e^{-t})$ in any $L^1_x L^p_v(m)$ space for $p \in (1,\infty)$, and then in $L^1_x L^\infty_v(m)$. Finally prove that $S_\mathcal{B}(t)\mathcal{A}$ is appropriately bounded in $\mathscr{B}(L^1, L^1_x L^\infty_v(m))$ and that $S_\mathcal{L}$ is bounded in $L^1_x L^\infty_v(m)$.

Question 4. We define $u(t) := \mathcal{A}S_\mathcal{B}(t)$. Establish that

$$(u(t)f_0)(x,v) = M(v)e^{-t}\int_{\mathbb{R}^d} f_0(x - v_* t, v_*)\,dv_*.$$

Deduce that

$$\|u(t)f_0\|_{L^\infty_{xv}(m)} \le C\,\frac{e^{-t}}{t^d}\,\|f_0\|_{L^1_xL^\infty_v(m)}.$$

Question 5. Establish that there exists some constants $n \ge 1$ and $C \in [1,\infty)$ such that

$$\|u^{(*n)}(t)\|_{L^1_{xv}(m)\to L^\infty_{xv}(m)} \le C\,e^{-t/2}.$$

Deduce that $S_{\mathcal{L}}$ is bounded in $L^\infty_{xv}(m)$.

Question 6. How to prove that $S_{\mathcal{L}}$ is bounded in $L^2_{xv}(m)$ in a similar way? How to shorten the proof of that last result by using question (1b)? Same question for the space $L^\infty_{xv}(m)$.

Exam 2018 - Local in time estimate (from Nash)

Consider a smooth and fast decaying initial datum f_0, the associated solution $f = f(t,x)$, $t \ge 0$, $x \in \mathbb{R}^d$, to heat equation

$$\partial_t f = \frac{1}{2}\,\Delta f, \quad f(0,.) = f_0,$$

and for a given $\alpha \in \mathbb{R}^d$, define

$$g := f\,e^{\psi}, \quad \psi(x) := \alpha\cdot x.$$

(1) Establish that

$$\partial_t g = \frac{1}{2}\Delta g - \alpha\cdot\nabla g + \frac{1}{2}|\alpha|^2 g.$$

(2) Establish that $\|\, g(t,.)\|_{L^1} \le e^{\alpha^2 t/2}\,\|\, g_0\|_{L^1}$ for any $t \ge 0$.

(3) Establish that

$$\|\, g(t)\|^2_{L^2}\,e^{-\alpha^2 t} \le \frac{\|\, g_0\|^2_{L^1}}{(2/d\,C_N\,t)^{d/2}}, \quad \forall\, t > 0.$$

(4) Denoting by $T(t)$ the semigroup associated to the parabolic equation satisfies by g, prove successively that

$$T(t) : L^1 \to L^2, \quad L^2 \to L^\infty, \quad L^1 \to L^\infty,$$

for some constants $Ct^{-d/4}e^{\alpha^2 t/2}$, $Ct^{-d/4}e^{\alpha^2 t/2}$ and $Ct^{-d/2}e^{\alpha^2 t/2}$.

(5) Denoting by S the heat semigroup and by $F(t,x,y) := (S(t)\delta_x)(y)$ the fundamental solution associated to the heat equation when starting from the Dirac function in $x \in \mathbb{R}^d$, deduce

$$F(t,x,y) \le \frac{C}{t^{d/2}}\,e^{\alpha\cdot(x-y)+\alpha^2 t/2}, \quad \forall\, t > 0, \forall\, x, y, \alpha \in \mathbb{R}^d,$$

and then

$$F(t,x,y) \le \frac{C}{t^{d/2}}\,e^{-\frac{|x-y|^2}{2t}}, \quad \forall\, t > 0, \forall\, x, y \in \mathbb{R}^d.$$

(6) May we prove a similar result for the parabolic equation

$$\partial_t f = \mathrm{div}_x(A(x)\nabla_x f), \quad 0 < \nu \le A \in L^\infty\,?$$

Exam 2019 - two problems about subgeometric convergence

Problem I - Subgeometric Harris estimate

In this part, we consider a Markov semigroup $S = S_{\mathcal{L}}$ on $L^1(\mathbb{R}^d)$ which fulfills
(H1) there exist some weight functions $m_i : \mathbb{R}^d \to [1,\infty)$ satisfying $m_1 \ge m_0$, $m_0(x) \to \infty$ as $x \to \infty$ and there exists constant $b > 0$ such that

$$\mathcal{L}^* m_1 \le -m_0 + b;$$

(H2) there exists a constant $T > 0$ and for any $R \ge R_0 \ge 0$ there exists a positive and not zero measure ν such that

$$S_T f \ge \nu \int_{B_R} f, \quad \forall\, f \in L^1,\ f \ge 0;$$

(H3) there exists $m_2 \ge m_1$ such that and for any $\lambda > 0$ there exists ξ_λ such that

$$m_1 \le \lambda m_0 + \xi_\lambda m_2, \quad \xi_\lambda \to 0 \ \text{ as } \lambda \to \infty.$$

(H4) We also assume that

$$\sup_{t \ge 0} \|S_t f\|_{L^1(m_i)} \le M_i \|f\|_{L^1(m_i)}, \quad M_i \ge 1,\ i = 1, 2.$$

(1) Prove

$$\|S_T f_0\|_{L^1} \le \|f_0\|_{L^1}, \quad \forall\, T > 0,\ \forall\, f_0 \in L^1.$$

In the sequel, we fix $f_0 \in L^1(m_2)$ such that $\langle f_0 \rangle = 0$ and we denote $f_t := S_t f_0$.
(2) Prove that

$$\frac{d}{dt}\|f_t\|_{L^1(m_1)} \le -\|f_t\|_{L^1(m_0)} + b\|f_t\|_{L^1},$$

and deduce that

$$\|S_T f_0\|_{L^1(m_1)} + \frac{T}{M_0}\|S_T f_0\|_{L^1(m_0)} \le \|f_0\|_{L^1(m_1)} + K\|f_0\|_{L^1}.$$

We define

$$\|f\|_\beta := \|f\|_{L^1} + \beta \|f\|_{L^1(m_1)}, \quad \beta > 0.$$

We fix $R \ge R_0$ large enough such that $A := m(R)/4 \ge 3M_0/T$, and we observe that the following alternative holds

$$\|f_0\|_{L^1(m_0)} \le A\|f_0\|_{L^1} \tag{0.1}$$

or

$$\|f_0\|_{L^1(m_0)} > A\|f_0\|_{L^1}. \tag{0.2}$$

(3) We assume that condition (0.1) holds. Prove that

$$\|S_T f_0\|_{L^1} \le \gamma_1 \|f_0\|_{L^1},$$

with $\gamma_1 \in (0,1)$. Deduce that

$$\|S_T f_0\|_\beta \le \gamma_1 \|f_0\|_{L^1} - \frac{\beta T}{M_0}\|S_T f_0\|_{L^1(m_0)} + \beta \|f_0\|_{L^1(m_1)} + \beta K \|f_0\|_{L^1}$$

and next

$$\|S_T f_0\|_\beta + \frac{\beta T}{M_0}\|S_T f_0\|_{L^1(m_0)} \le \|f_0\|_\beta,$$

for $\beta > 0$ small enough.

(4) We assume that condition (0.2) holds. Prove that

$$\|S_T f_0\|_{L^1(m_1)} + \frac{T}{M_0}\|S_T f_0\|_{L^1(m_0)} \le \|f_0\|_{L^1(m_1)} + \frac{T}{3M_0}\|f_0\|_{L^1(m_0)},$$

and deduce

$$\|S_T f_0\|_\beta + \frac{\beta T}{M_0}\|S_T f_0\|_{L^1(m_0)} \le \|f_0\|_\beta + \frac{\beta T}{3M_0}\|f_0\|_{L^1(m_0)}.$$

(5) Observe that in both cases (0.1) and (0.2), there holds

$$\|S_T f_0\|_\beta + 3\alpha \|S_T f_0\|_{L^1(m_0)} \le \|f_0\|_\beta + \alpha \|f_0\|_{L^1(m_0)},$$

where from now β and α are fixed constants. Deduce that

$$Z(u_1 + \alpha v_1) \le u_0 + \alpha v_0 + \frac{\xi_\lambda}{\lambda}\alpha w_1,$$

with

$$u_n := \|S_{nT} f_0\|_\beta, \quad v_n := \|S_{nT} f_0\|_{L^1(m_0)}, \quad w_n := \|S_{nT} f_0\|_{L^1(m_2)}$$

and for $\lambda \ge \lambda_0 \ge 1$ large enough

$$Z := 1 + \frac{\delta}{\lambda} \le 2, \quad \delta := \frac{\alpha}{1+\beta}.$$

Deduce that for any $n \ge 1$, there holds

$$u_n \le Z^{-n}(u_0 + \alpha v_0) + \frac{Z}{Z-1}\frac{\xi_\lambda \alpha}{\lambda}\sup_{i\ge 1} w_i,$$

and next

$$\|S_{nT} f_0\|_\beta \le \Big(e^{-\frac{nT}{\lambda}\frac{\delta}{2T}} + \xi_\lambda\Big) C \|f_0\|_{L^1(m_2)}, \quad \forall\, \lambda \ge \lambda_0.$$

(6) Prove that

$$\|S_t f_0\|_{L^1} \le \Theta(t) \|f_0\|_{L^1(m_2)}, \quad \forall\, t \ge 0,\ \forall\, f_0 \in L^1(m_2),\ \langle f \rangle = 0,$$

for the function Θ given by

$$\Theta(t) := C \inf_{\lambda > 0} \{ e^{-\kappa t/\lambda} + \xi_\lambda \}.$$

What is the value of Θ when $m_0 = 1$, $m_1 = \langle x \rangle$, $m_2 = \langle x \rangle^2$?

Problem II - An application to the Fokker-Planck equation with weak confinement

In all the problem, we consider the Fokker-Planck equation

$$\partial_t f = \mathcal{L} f := \Delta_x f + \mathrm{div}_x(f\, E) \quad \text{in } (0,\infty) \times \mathbb{R}^d \tag{0.3}$$

for the confinement potential $E := \nabla \phi$, $\phi := \langle x \rangle^\gamma / \gamma$, $\langle x \rangle^2 := 1 + |x|^2$, that we complement with an initial condition

$$f(0,x) = f_0(x) \quad \text{in } \mathbb{R}^d. \tag{0.4}$$

Question 1

Give a strategy in order to build solutions to (0.3) when $f_0 \in L^p_k(\mathbb{R}^d)$, $p \in [1,\infty]$, $k \ge 0$.

We assume from now on that $f_0 \in L^1_k(\mathbb{R}^d)$, $k > 0$, and that we are able to build a unique weak (and renormalized) solution $f \in C([0,\infty); L^1_k)$ to equation (0.3)-(0.4).
We also assume that $\gamma \ge 2$.

Question 2

Prove

$$\langle f(t) \rangle = \langle f_0 \rangle \quad \text{and} \quad f(t,.) \ge 0 \text{ if } f_0 \ge 0.$$

Question 3

Prove that there exist $\alpha > 0$ and $K \ge 0$ such that

$$\mathcal{L}^* \langle x \rangle^k \le -\alpha \langle x \rangle^k + K,$$

and deduce that

$$\sup_{t \ge 0} \|f(t,\cdot)\|_{L^1_k} \le C_1 \|f_0\|_{L^1_k}.$$

(Hint. A possible constant is $C_1 := \max(1, K/\alpha)$).

Question 4

Prove that

$$\sup_{t\geq 0} \|f(t,\cdot)\|_{L^2_k} \leq C_2 \|f_0\|_{L^2_k},$$

at least for $k > 0$ large enough.

Question 5

Prove that

$$\sup_{t\geq 0} \|f(t,\cdot)\|_{H^1_k} \leq C_3 \|f_0\|_{H^1_k},$$

at least for $k > 0$ large enough.

Question 6

Prove that

$$\|f(t,\cdot)\|_{H^1} \leq \frac{C_4}{t^\alpha} \|f_0\|_{L^1_k},$$

at least for $k > 0$ large enough and for some constant $\alpha > 0$ to be specified.
(Hint. Consider the functional $\mathcal{F}(t) := \|f(t)\|_{L^1_k} + t^\alpha \|\nabla_x f(t)\|^2_{L^2}$).

We assume from now on that $d = 1$, so that $C^{0,1/2} \subset H^1$.

Question 7

We fix $f_0 \in L^1$ such that $f_0 \geq 0$ and $\operatorname{supp} f_0 \subset B_R$, $R > 0$. Using question 3, prove that

$$\int_{B_\rho} f(t) \geq \frac{1}{2} \int_{B_R} f_0,$$

for any $t \geq 0$ by choosing $\rho > 0$ large enough. Using question 6, prove that there exist $r, \kappa > 0$ and for any $t > 0$ there exists $x_0 \in B_R$ such that

$$f(t) \geq \kappa, \quad \forall\, x \in B(x_0, r).$$

We accept the spreading of the positivity property, namely that for ant $r_0, r_1 > 0$, $x_0 \in \mathbb{R}^d$, there exist $t_1, \kappa_1 > 0$ such that

$$f_0 \geq \mathbf{1}_{B(x_0,r_0)} \quad \Rightarrow \quad f(t_1,\cdot) \geq \kappa_1 \mathbf{1}_{B(x_0,r_1)}.$$

Deduce that there exist $\theta > 0$ and $T > 0$ such that

$$f(T,\cdot) \geq \theta \mathbf{1}_{B(0,R)} \int_{B_R} f_0 \, dx.$$

Question 8

Prove that for any $k > 0$, there exists $C, \lambda > 0$ such that $f_0 \in L^1_k$ satisfying $\langle f_0 \rangle = 0$, there holds

$$\forall\, t > 0, \qquad \|f(t,.)\|_{L^1_k} \le C e^{-\lambda t} \|f_0\|_{L^1_k}.$$

Question 9

We assume now $\gamma \in (0,2)$. We define

$$\mathcal{B} f := \mathcal{L} f - M \chi_R f$$

with $\chi_R(x) := \chi(x/R)$, $\chi \in \mathcal{D}(\mathbb{R}^d)$, $0 \le \chi \le 1$, $\chi(x) = 1$ for any $|x| \le 1$, and with $M, R > 0$ to be fixed.
We denote by $f_\mathcal{B}(t) = S_\mathcal{B}(t) f_0$ the solution associated to the evolution PDE corresponding to the operator $\mathcal{B}$ and the initial condition f_0.
(1) Why such a solution is well defined (no more than one sentence of explanation)?
(2) Prove that there exists $M, R > 0$ such that for any $k \ge 0$ there holds

$$\frac{d}{dt} \int_{\mathbb{R}^d} f_\mathcal{B}(t) \langle x \rangle^k \, dx \le -c_k \int_{\mathbb{R}^d} f_\mathcal{B}(t) \langle x \rangle^{k+\gamma-2} \le 0,$$

for some constant $c_k \ge 0$, $c_k > 0$ if $k > 0$, and

$$\|S_\mathcal{B}(t)\|_{L^1_k \to L^1_k} \le 1.$$

(3) Prove that for any $k_1 < k < k_2$ there exists $\theta \in (0,1)$ such that

$$\forall\, f \ge 0 \quad M_k \le M_{k_1}^\theta \, M_{k_2}^{1-\theta}, \quad M_\ell := \int_{\mathbb{R}^d} f(x) \, \langle x \rangle^\ell \, dx$$

and write θ as a function of k_1, k and k_2.
(4) Prove that if $\ell > k > 0$ there exists $\alpha > 0$ such that

$$\|S_\mathcal{B}(t)\|_{L^1_\ell \to L^1} \le \|S_\mathcal{B}(t)\|_{L^1_\ell \to L^1_k} \le C / \langle t \rangle^\alpha,$$

and that $\alpha > 1$ if ℓ is large enough (to be specified).
(6) Prove that

$$S_\mathcal{L} = S_\mathcal{B} + S_\mathcal{B} * (\mathcal{A} S_\mathcal{L}),$$

and deduce that for k large enough (to be specified)

$$\|S_\mathcal{L}\|_{L^1_k \to L^1_k} \le C.$$

Question 10

Still in the case $\gamma \in (0,2)$, what can we say about the decay of

$$\|f(t,.)\|_{L^1}$$

when $f_0 \in L^1_k$, $k > 0$, satisfies $\langle f_0 \rangle = 0$?

Exam 2020 - On the Landau equation

We aim to establish the existence of solutions to the Landau equation

$$\partial_t f(t,v) = Q(f,f)(t,v), \quad f(0,v) = f_0(v), \tag{0.1}$$

on the density function $f = f(t,v) \geq 0$, $t \geq 0$, $v \in \mathbb{R}^d$, $d \geq 2$, where the Landau kernel is defined by the formula

$$Q(f,f)(v) := \frac{\partial}{\partial v_i}\Big\{\int_{\mathbb{R}^d} a_{ij}(v-v_*)\Big(f(v_*)\frac{\partial f}{\partial v_j}(v) - f(v)\frac{\partial f}{\partial v_j}(v_*)\Big)\,dv_*\Big\}.$$

Here and the sequel we use Einstein's convention of sommation of repeated indices. The matrix $a = (a_{ij})$ is defined by

$$a(z) = |z|^2\Pi(z), \quad \Pi_{ij}(z) := \delta_{ij} - \hat z_i \hat z_j, \quad \hat z_k := \frac{z_k}{|z|},$$

so that Π is the is the orthogonal projection on the hyperplan $z^\perp := \{y \in \mathbb{R}^d;\, y\cdot z = 0\}$.

Part I - Physical properties and a priori estimates.

(1) Observe that $a(z)z = 0$ for any $z \in \mathbb{R}^d$ and $a(z)\xi\xi \geq 0$ for any $z, \xi \in \mathbb{R}^d$. Here and below, we use the bilinear form notation $auv = {}^t vau = v\cdot au$. In particular, the symmetric matrix a is positive but not strictly positive.

(2) For any nice functions $f, \varphi : \mathbb{R}^d \to \mathbb{R}$, $f \geq 0$, prove that

$$\int Q(f,f)\varphi\,dv = \frac{1}{2}\iint a(v-v_*)(f\nabla_* f_* - f_*\nabla f)(\nabla\varphi - \nabla_*\varphi_*)\,dvdv_*,$$

where $f_* = f(v_*)$, $\nabla_*\psi_* = (\nabla\psi)(v_*)$. Deduce that

$$\int Q(f,f)\varphi\,dv = 0, \quad \text{for } \varphi = 1, v_i, |v|^2,$$

and

$$-D(f) := \int Q(f,f)\log f\,dv \leq 0.$$

Establish then

$$\Big|\int Q(f,f)\varphi\,dv\Big| \leq D(f)^{1/2}\Big(\frac{1}{2}\iint ff_* a(v-v_*)(\nabla\varphi - \nabla_*\varphi_*)(\nabla\varphi - \nabla_*\varphi_*)\,dvdv_*\Big)^{1/2}.$$

(3) For $H_0 \in \mathbb{R}$, we define $\mathcal{E}_{H_0}$ the set of functions

$$\mathcal{E}_{H_0} := \Big\{ f \in L^1_2(\mathbb{R}^d);\ f \ge 0, \int f\,dv = 1, \int f\,v\,dv = 0, \\ \int f\,|v|^2\,dv \le d,\ H(f) := \int f\,\log f\,dv \le H_0 \Big\}.$$

Prove that there exists a constant C_0 such that

$$H_-(f) := \int f(\log f)_- dv \le C_0, \quad \forall\, f \in \mathcal{E}_{H_0},$$

and define $D_0 := H_0 + C_0$. Deduce that for any nice positive solution f to the Landau equation such that $f_0 \in \mathcal{E}_{H_0}$, there holds

$$f \in \mathcal{F}_T := \Big\{ g \in C([0,T]; L^1_2);\ g(t) \in \mathcal{E}_{H_0}, \forall\, t \in (0,T), \int_0^T D(g(t))\,dt \le D_0 \Big\}.$$

We say that $f \in C([0,T); L^1)$ is a weak solution to the Landau equation if $f \in \mathcal{F}_T$ and (0.1) holds in the distributional sense. Why the definition is meaningful?

(4) Prove that

$$Q(f,f) = \partial_i(\bar a_{ij}\partial_j f - \bar b_i f) = \partial^2_{ij}(\bar a_{ij} f) - 2\partial_i(\bar b_i f) = \bar a_{ij}\partial^2_{ij} f - \bar c f,$$

with

$$\bar a_{ij} = \bar a^f_{ij} := a_{ij} * f, \quad \bar b_i = \bar b^f_i := b_i * f, \quad \bar c = \bar c^f := c * f, \tag{0.2}$$

and

$$b_i := \sum_{j=1}^d \partial_j a_{ij} = -(d-1) z_i, \quad c := \sum_{i=1}^d \partial_i b_i = -(d-1)d.$$

Prove that there existe $C \in (0,\infty)$ such that

$$|\bar a_{ij}| \le C(1 + |v|^2), \quad |\bar b_i| \le C(1+|v|),$$

Part II - On the ellipticity of $\bar a$.

We fix $H_0 \in \mathbb{R}$ and $f \in \mathcal{E}_{H_0}$.

(5a) Show that there exists a function $\eta \ge 0$ (only depending of D_0) such that

$$\forall\, A \subset \mathbb{R}^d, \quad \int_A f\,dv \le \eta(|A|)$$

and $\eta(r) \to 0$ when $r \to 0$. Here $|A|$ denotes the Lebesgue measure of A. Deduce that

$$\forall\, R, \varepsilon > 0, \quad \int f\,\mathbf{1}_{|v|\le R}\,\mathbf{1}_{|v_i|\le\varepsilon}\,dv \le \eta_R(\varepsilon)$$

and $\eta_R(r) \to 0$ when $r \to 0$.

(5b) Show that

$$\int f\mathbf{1}_{|v|\le R} \ge 1 - \frac{d}{R^2}.$$

(5c) Deduce from the two previous questions that

$$\forall\, i = 1,\dots,d, \quad T_i := \int f v_i^2 dv \ge \lambda,$$

for some constant $\lambda > 0$ which only depends on D_0. Generalize the last estimate into

$$\forall\, \xi \in \mathbb{R}^d, \quad T(\xi) := \int f|v\cdot\xi|^2 dv \ge \lambda|\xi|^2.$$

(6) Deduce that

$$\forall\, v,\xi \in \mathbb{R}^d, \quad \bar{a}(v)\xi\xi := \sum_{i,j=1}^{d} \bar{a}_{ij}(v)\xi_i\xi_j \ge (d-1)\lambda\,|\xi|^2.$$

Prove that any weak solution formally satisfies

$$\frac{d}{dt}H(f) = -\int \bar{a}_{ij}\frac{\partial_i f\partial_j f}{f} - \int \bar{c}f,$$

and thus the following bound on the Fisher information

$$I(f) := \int \frac{|\nabla f|^2}{f} \in L^1(0,T)$$

Part III - Weak stability.

We consider here a sequence of weak solutions (f_n) to the Landau equation such that $f_n \in \mathcal{F}_T$ for any $n \ge 1$.

(7) Prove that

$$\int_0^T \int |\nabla_v f_n|\, dvdt \le C_T$$

and that

$$\frac{d}{dt}\int f_n\varphi\, dv \text{ is bounded in } L^\infty(0,T), \quad \forall\, \varphi \in C_c^2(\mathbb{R}^d).$$

Deduce that (f_n) belongs to a compact set of $L^1((0,T)\times\mathbb{R}^d)$. Up to the extraction of a subsequence, we then have

$$f_n \to f \text{ strongly in } L^1((0,T)\times\mathbb{R}^d).$$

Deduce that

$$Q(f_n,f_n) \rightharpoonup Q(f,f) \text{ weakly in } \mathcal{D}((0,T)\times\mathbb{R}^d)$$

and that f is a weak solution to the Landau equation.
(8) (Difficult, here $d = 3$) Take $f \in \mathcal{E}_{H_0}$ with energy equals to d. Establish that $D(f) = 0$ if, and only if,

$$\frac{\nabla f}{f} - \frac{\nabla f_*}{f_*} = \lambda(v, v_*)(v - v_*), \quad \forall\, v, v_* \in \mathbb{R}^d,$$

for some scalar function $(v, v_*) \mapsto \lambda(v, v_*)$. Establish then that the last equation is equivalent to

$$\log f = \lambda_1 |v|^2/2 + \lambda_2 v + \lambda_3, \quad \forall\, v \in \mathbb{R}^d,$$

for some constants $\lambda_1 \in \mathbb{R}$, $\lambda_2 \in \mathbb{R}^d$, $\lambda_3 \in \mathbb{R}$. Conclude that

$$D(f) = 0 \text{ if, and only if, } f = M(v) := (2\pi)^{-3/2} \exp(-|v|^2/2).$$

(9) (very difficult, because needs many steps) Prove that for any global weak solution f associated to $f_0 \in L^1_3 \cap \mathcal{E}_{H_0}$ with energy equals d, there holds $f(t) \rightharpoonup M$ when $t \to \infty$. (Hint. Accept that the energy $M_2(f(t)) = d$ and prove that the third moment $M_3(f(t))$ is uniformly bounded).

Part IV - Existence.

(10) We fix $k = d + 4$. Show that $\mathcal{H} := L^2_k \subset L^1_3$ and that $H_0 := H(f_0) \in \mathbb{R}$ if $0 \le f_0 \in L^2_k$. In the sequel, we first assume that $f_0 \in \mathcal{E}_{H_0} \cap \mathcal{H}$.
(11) For $f \in C([0, T]; \mathcal{E}_{H_0})$, we define $\bar a$, $\bar b$ and $\bar c$ thanks to (0.2) and then

$$\tilde a_{ij} := \bar a_{ij} + \varepsilon |v|^2 \delta_{ij}, \quad \tilde b_i := \bar b_i - \varepsilon \frac{d+2}{2} v_i, \quad \varepsilon \in (0, \lambda).$$

We define $\mathcal{V} := H^1_{k+2}$ and then

$$\forall\, g \in \mathcal{V}, \quad Lg := \partial_i (\tilde a_{ij} \partial_j g - \tilde b_i g) \in \mathcal{V}'.$$

Show that for some constant $C_i \in (0, \infty)$, there hold

$$(Lg, g)_{\mathcal{H}} \le -\varepsilon \|g\|^2_{\mathcal{V}} + C_1 \|g\|^2_{\mathcal{H}}, \quad |(Lg, h)_{\mathcal{H}}| \le C_2 \|g\|_{\mathcal{V}} \|h\|_{\mathcal{V}}, \quad \forall\, g, h \in \mathcal{V}.$$

Deduce that there exists a unique variational solution

$$g \in \mathcal{X}_T := C([0, T]; \mathcal{H}) \cap L^2(0, T; \mathcal{V}) \cap H^1(0, T; \mathcal{V}')$$

to the parabolic equation

$$\partial_t g = Lg, \quad g(0) = f_0.$$

Prove furthermore that $g \in \mathcal{F}_T$.
(12) Prove that there exists a unique fonction

$$f_\varepsilon \in C([0, T]; L^2_k) \cap L^2(0, T; H^1_k) \cap \mathcal{F}_T$$

solution to the nonlinear parabolic equation

$$\partial_t f_\varepsilon = \partial_i(\tilde{a}_{ij}^{f_\varepsilon} \partial_j f_\varepsilon + \tilde{b}_i^{f_\varepsilon} f_\varepsilon), \quad f_\varepsilon(0) = f_0,$$

where $\tilde{a}_{ij}^{f_\varepsilon}$ denotes the
(13) For $f_0 \in \mathcal{E}_{H_0}$ and $T > 0$, prove that there exists at least one weak solution $f \in \mathcal{F}_T$ to the Landau equation.