Exercises on chapters 4, 5 & 6

1. About evolution PDEs and semigroups (Chapter 4)

Exercice 1.1. *Consider a Banach space X and an unbounded operator Λ on X. We assume that $X = Y'$ for a Banach space Y and that the dual operator Λ^* generates a strongly continuous semigroup T on Y.*
(1) Prove that $S := T^*$ is a (at least) weakly $*\,\sigma(X,Y)$ continuous semigroup on X with generator Λ and that it provides the unique weak solution to the associated evolution equation.
(2) Prove that for any smooth functions $a = a(x)$ and $c = c(x)$, one can define a weakly continuous semigroup $S = S_\Lambda$ on $L^\infty = L^\infty(\mathbb{R}^d)$ associated to the transport operator

$$(\Lambda f)(x) := -a(x) \cdot \nabla f(x) - c(x)\, f(x),$$

as the dual semigroup associated to the dual operator Λ^* defined on $L^1(\mathbb{R}^d)$.
(3) Prove similarly that one can define a weakly continuous semigroup on $M^1(\mathbb{R}^d) := (C_0(\mathbb{R}^d))'$, the space of Radon measures, associated to the transport operator Λ.

Exercice 1.2 (Miyadera-Voigt perturbation theorem)**.** *Given a generator $\mathcal{B}$ on X, we say that $\mathcal{A} \in \mathscr{C}_D(X)$ is $\mathcal{B}$-bounded if*

$$\|\mathcal{A}f\| \le C(\|f\| + \|\mathcal{B}f\|) \quad \forall\, f \in D(\mathcal{B})$$

for some constant $C \in (0,\infty)$. In particular, $D(\mathcal{B}) \subset D(\mathcal{A}) \subset X$.
Consider $S_\mathcal{B}$ a semigroup satisfying the growth estimate $\|S_\mathcal{B}(t)\|_{\mathscr{B}(X)} \le M\, e^{bt}$ and $\mathcal{A}$ a $\mathcal{B}$-bounded operator such that

$$\exists\, T > 0, \quad \int_0^T \|S_\mathcal{B}(t)\mathcal{A}\|_{\mathscr{B}(X)}\, dt \le \frac{1}{2}, \quad \sup_{t\in[0,T]} \|S_\mathcal{B}(t)\mathcal{A}\|_{\mathscr{B}(X,X_{-1})} < \infty, \tag{1.1}$$

where the abstract Sobolev space $X_{-1} = X_{-1}^{\mathcal{B}}$ is defined as the closure of X for the norm

$$\|f\|_{X_{-1}} := \|(\mathcal{B} - b - 1)^{-1} f\|_X.$$

Prove that $\Lambda := \mathcal{A} + \mathcal{B}$ is the generator of a semigroup which satisfies the growth estimate $\|S_\Lambda(t)\|_{\mathscr{B}(X)} \le M'\, e^{b' t}$, with $M' = 2e^{bT}M$ and $b' = (\log 2e^{bT}M)/T$.

Exercice 1.3. *Apply the Hille-Yosida-Lumer-Phillips Theorem on the following equations.*
– Heat equation

$$\partial_t u = \Delta u, \quad u(0) = u_0,$$

on the space $H := L^2(\Omega)$, with $\Lambda u := \Delta u$, $D(\Lambda) = H^1_0(\Omega) \cap H^2(\Omega)$, $\Omega \subset \mathbb{R}^d$.
– Wave equation

$$\partial^2_{tt} u = \Delta u \quad u(0) = u_0,\ \partial_t u(0) = v_0,$$

written as

$$\partial_t U = \Lambda U, \quad U = (u, \partial_t u), \quad \Lambda = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}$$

on the space $H := H^1_0(\Omega) \times L^2(\Omega)$, $D(\Lambda) = (H^1_0(\Omega) \cap H^2(\Omega)) \times H^1_0(\Omega)$, $\Omega \subset \mathbb{R}^d$.
– Scrödinger equation

$$i\partial_t u + \Delta u = 0, \quad u(0) = u_0,$$

on the space $H := L^2(\mathbb{R}^d;\mathbb{C})$, with $\Lambda u := i\Delta u$, $D(\Lambda) = H^2(\mathbb{R}^d)$
– Stokes equation

$$\partial_t u = \Delta u + \nabla p, \quad \operatorname{div} u = 0, \quad u(0) = u_0,$$

on the space $H := \{u \in (L^2(\mathbb{R}^d))^d;\ \operatorname{div} u = 0\}$, with $\Lambda u := \Delta u$ and

$$D(\Lambda) = \{u \in (H^2(\mathbb{R}^d))^d \cap H,\ \Delta u \in H\}.$$

2. About the heat equation and related problems (Chapter 5)

Exercice 2.1. *(1) Prove (1) in Theorem 2.1.*

(2) Establish rigorously the decay estimate about the solutions of the heat equation for weak solutions and using Nash approach.

Exercice 2.2. *Prove that for any* $\lambda < \lambda_P$*, there exists* $\varepsilon > 0$ *so that the following stronger version*

$$\begin{aligned} \int_{\mathbb{R}^d} \left| \nabla \left(\frac{f}{G} \right) \right|^2 G dx \quad &\geq \quad \lambda \int_{\mathbb{R}^d} f^2 \, G^{-1} dx \\ &+\varepsilon \int_{\mathbb{R}^d} \left(f^2 \, |x|^2 + |\nabla f|^2 \right) \, G^{-1} dx \end{aligned}$$

holds for any $f \in \mathcal{D}(\mathbb{R}^d)$ *with* $\langle f \rangle = 0$*. Here* G *denotes de normalized Gaussian function* $e^{-|x|^2/2 + C_0}$.

Exercice 2.3. *Prove that* $0 \leq f_n \to f$ *in* $L^q \cap L^1_k$*,* $q > 1$*,* $k > 0$*, implies that* $H(f_n) \to H(f)$*.*
(Hint. Use the splitting

$$s \, |\log s| \leq \sqrt{s} \, \mathbf{1}_{0 \leq s \leq e^{-|x|^k}} + s \, |x|^k \, \mathbf{1}_{e^{-|x|^k} \leq s \leq 1} + s(\log s)_+ \, \mathbf{1}_{s \geq 1} \quad \forall \, s \geq 0$$

and the dominated convergence theorem).

Exercice 2.4. *Prove the rate of convergence*

$$I(f(t,.)|G) \leq e^{-2\,t} \, I(\varphi|G).$$

for any $\varphi \in \mathbf{P}(\mathbb{R}^d) \cap L^1_2(\mathbb{R}^d)$ *such that* $I(\varphi) < \infty$*.*
(Hint. Compute the equations for the moments of order 1 *and* 2*).*

Exercice 2.5. *Assume that the log-Sobolev inequality*

$$\lambda \, H(f|G) \leq \frac{1}{2} I(f|G) \quad \forall \, f \in D$$

holds for some constant $\lambda > 0$*. Prove that the Poincaré inequality*

$$(\lambda + d) \, \|h\|^2_{L^2(G^{-1/2})} \leq \int |\nabla h|^2 \, G^{-1} \quad \forall \, h \in \mathcal{D}(\mathbb{R}^d), \; \langle h[1, x, |x|^2] \, \rangle = 0,$$

also holds (for the same constant $\lambda > 0$*).*

Exercice 2.6. *1. Give another proof of the Nash inequality by using the Sobolev inequality in dimension* $d \geq 3$*. (Hint. Write the interpolation estimate*

$$\|f\|_{L^2} \leq \|f\|^{\theta}_{L^1} \, \|f\|^{1-\theta}_{L^{2^*}}$$

and then use the Sobolev inequality associated to the Lebesgue exponent $p = 2$*).*
2. Give another proof of the Nash inequality by using the Sobolev inequality in dimension $d = 2$*. (Hint. Prove the interpolation estimate*

$$\|f\|_{L^2} \leq \|f\|^{1/4}_{L^1} \, \|f^{3/2}\|^{1/2}_{L^2},$$

then use the Sobolev inequality associated to the Lebesgue exponent $p = 1$ *and* $p^* := 2$ *and finally the Cauchy-Schwartz inequality in order to bound the second term).*
3. Give another proof of the Nash inequality by using the Sobolev inequality in dimension $d = 1$*. (Hint. Prove the interpolation estimate*

$$\|f\|_{L^2} \leq \|f\|^{1/2}_{L^1} \, \|f^{3/2}\|^{1/3}_{L^\infty},$$

then use the Sobolev inequality associated to the Lebesgue exponent $p = 1$ *and* $p^* := \infty$ *and finally the Cauchy-Schwartz inequality in order to bound the second term).*

Exercice 2.7. *1) Prove the Poincaré-Wirtinger inequality*

$$\|f - f_r\|_{L^2} \leq C \, r \|\nabla f\|_{L^2}, \quad f_r(x) := \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y) \, dy,$$

for any $r > 0$ *and some constant* $C = C(d) > 0$*.*

2) Recover the Nash inequality in any dimension $d \ge 1$. (Hint. Write that $\|f\|_{L^2}^2 = (f, f - f_r) + (f, f_r)$ and decuce that $\|f\|_{L^2}^2 \le C_1\, r\, \|f\|_{L^2}\, \|\nabla f\|_{L^2} + C_2\, r^{-d}\, \|f\|_{L^1}^2$, for any $r > 0$).
3) Generalize (1) by establishing that

$$\|\rho_\varepsilon * f - f\|_{L^2} \le C\, \varepsilon\, \|\nabla f\|_{L^2}, \quad \forall \varepsilon > 0,$$

for a constant C which only depends on the function $\rho \in \mathbf{P}(\mathbb{R}^d) \cap L^1_{comp}(\mathbb{R}^d)$ in the definition of the mollifier (ρ_ε).

Exercice 2.8. *We say that V satisfies a Lyapunov condition if there exists a function W such that $W \ge 1$ and there exist some constants $\theta > 0$, $b, R \ge 0$ such that*

$$(L^*W)(x) := \Delta W(x) - \nabla V \cdot \nabla W(x) \le -\theta\, W(x) + b\, \mathbf{1}_{B_R}(x), \qquad \forall x \in \mathbb{R}^d, \tag{2.1}$$

where $B_R = B(0, R)$ denotes the centered ball of radius R.
Establish (2.1) in the following situations:

- (i) *$V(x) := \langle x \rangle^\alpha$ with $\alpha \ge 1$;*
- (ii) *there exist $\alpha > 0$ and $R \ge 0$ such that*

$$x \cdot \nabla V(x) \ge \alpha \qquad \forall x \notin B_R;$$

- (iii) *there exist $a \in (0,1)$, $c > 0$ and $R \ge 0$ such that*

$$a\, |\nabla V(x)|^2 - \Delta V(x) \ge c \qquad \forall x \notin B_R;$$

- (iv) *V is convex (or it is a compact supported perturbation of a convex function) and satisfies $e^{-V} \in L^1(\mathbb{R}^d)$.*

Exercice 2.9. *Generalize the Poincaré inequality to a general superlinear potential $V(x) = \langle x \rangle^\alpha / \alpha + V_0$, $\alpha \ge 1$, in the following strong (weighted) formulation*

$$\int |\nabla g|^2\, \mathcal{G} \ge \kappa \int |g - \langle g \rangle_{\mathcal{G}}|^2\, (1 + |\nabla V|^2)\, \mathcal{G} \qquad \forall g \in \mathcal{D}(\mathbb{R}^d),$$

where we have defined $\mathcal{G} := e^{-V} \in \mathbf{P}(\mathbb{R}^d)$ (for an appropriate choice of $V_0 \in \mathbb{R}$).

3. More about the longtime asymptotic (Chapter 6)

Exercice 3.1. *We consider a semigroup $S_t = e^{tL}$ of linear and bounded operators on L^1 and we assume that*
(i) $S_t \ge 0$;
(ii) $\exists\, g > 0$ such that $Lg = 0$, or equivalently $S_t g = g$ for any $t \ge 0$;
(iii) $\exists\, \phi$ such that $L^*\phi = 0$, or equivalently $\langle S_t h, \phi \rangle = \langle h, \phi \rangle$ for any $h \in L^1$ and $t \ge 0$.

Our aim is to generalize to that more abstract framework the general relative entropy principle.

(a) Prove that for any real affine function ℓ, there holds $\ell[(S_t f)/g] g = S_t[\ell(f/g) g]$.

(b) Prove that for any convex function H and any $f \ge 0$, there holds $H[(S_t f)/g] g \le S_t[H(f/g) g]$. (Hint. Use the fact that $H = \sup_{\ell \le H} \ell$).

(c) Deduce that

$$\int H[(S_t f)/g] g \phi \le \int H[f/g] g \phi, \quad \forall t \ge 0.$$

Exercice 3.2. *Consider the Fokker-Planck equation*

$$\partial_t f = \mathcal{L} f := \Delta f + div(E f), \tag{3.1}$$

on the density $f = f(t,x)$, $t \ge 0$, $x \in \mathbb{R}^d$, where the force field $E \in \mathbb{R}^d$ is a given fixed (exterior) vectors field.
(1) Give conditions on E so that (3.1) admits a steady state, namely

$$\exists\, G \in L^1(\mathbb{R}^d) \cap \mathbb{P}(\mathbb{R}^d), \quad div(\nabla G + E\, G) = 0.$$

(2) Establish that G is a steady state iff

$$E = \nabla U + E_0, \quad div(E_0\, e^{-U}) = 0, \tag{3.2}$$

for a confinement potential $U : \mathbb{R}^d \to \mathbb{R}$ and a non gradient force field perturbation $E_0 : \mathbb{R}^d \to \mathbb{R}^d$, and then $G(x) := e^{-U(x)+U_0}$ is a stationary state for some $U_0 \in \mathbb{R}$.
(3) Establish that $\mathcal{L}$ is a self-adjoint operator in the Hilbert space $L^2(m)$ if and only if $E = \nabla U$ and $m = e^{U/2}$ for some confinement potential $U : \mathbb{R}^d \to \mathbb{R}$.

Exercice 3.3. *We consider the linear Boltzmann (or scattering) equation , writes*

$$\partial_t f = \mathcal{L}\, f := \int_{\mathcal{V}} b_* f_*\, dv_* - B(v)\, f, \tag{3.3}$$

on the density function $f = f(t,v) \ge 0$, $t \ge 0$, $v \in \mathcal{V} \subset \mathbb{R}^d$, where $f = f(v)$, $f_* = f(v_*)$, $b = b(v,v_*)$ and $b_* = b(v_*,v)$, $b \ge 0$ is a given function (the rate of collisions), and we assume that there exists a function $\phi > 0$ such that

$$\mathcal{L}^*\phi := \int_{\mathcal{V}} b\, \phi_*\, dv_* - B\, \phi = 0, \quad \textit{in other words} \quad B(v) := \int_{\mathcal{V}} \frac{\phi_*}{\phi}\, b\, dv_*,$$

with again $\phi = \phi(v)$ and $\phi_* = \phi(v_*)$.
(1) Establish that there exists a function $0 < F \in L^1(\mathcal{V}) \cap \mathbf{P}(\mathcal{V})$ which is a stationary solution

$$\mathcal{L}F = \int_{\mathcal{V}} b_* F_*\, dv_* - \int_{\mathcal{V}} \frac{\phi_*}{\phi}\, b\, dv_*\, F = 0,$$

when $b > 0$.
(2) Assuming (1), establish that any solution f to the equation (3.3) *satisfies (at least formally)*

$$\frac{d}{dt} \int_{\mathcal{V}} f^2\, \frac{\phi}{F}\, dv = 2 \int_{\mathcal{V}} (\mathcal{L}\, f)\, \frac{f\, \phi}{F}\, dv = -D_2(f) \tag{3.4}$$

with

$$D_2(f) := \int_{\mathcal{V}}\int_{\mathcal{V}} b_*\, F\, \phi \left(\frac{f_*}{F_*} - \frac{f}{F} \right)^2 dv dv_*. \tag{3.5}$$

Exercice 3.4. *Let $S = S_{\mathcal{L}}$ be a strongly continuous semigroup on a Banach space $X \subset L^1$. Show that there is equivalence between*
(a) $S_{\mathcal{L}}$ is a Stochastic semigroup;
(b) $\mathcal{L}^* 1 = 0$ and $\mathcal{L}$ satisfies Kato's inequality

$$(\text{sign}\, f)\mathcal{L}f \le \mathcal{L}|f|, \quad \forall\, f \in D(\mathcal{L}).$$

(Hint. In order to prove (b) $\Rightarrow$ (a), consider $f \in D(\mathcal{L}^2)$ and estimate $|Stf| - |f|$ by introducing a telescopic sum and a Taylor expansion in the time variable).

Exercice 3.5. *Consider $S_{\mathcal{L}^*}$ a (constant preserving) Markov semigroup and $\Phi : \mathbb{R} \to \mathbb{R}$ a concave function. Prove that $\mathcal{L}^*\Phi(m) \le \Phi'(m)\mathcal{L}^* m$. (Hint. Use that $\Phi(a) = \inf\{\ell(a);\, \ell$ affine such that $\ell \ge \Phi\}$ in order to prove $S^*_t(\Phi(m)) \le \Phi(S^*_t m)$ and $\Phi(b) - \Phi(a) \ge \Phi'(a)(b-a)$).*