CHAPTER 2
VARIATIONAL SOLUTION FOR PARABOLIC EQUATION

I write in blue color what has been taught during the classes.

We present the theory of variational solutions for abstract evolution equations associated to a coercive operator and we apply the theory to the case of uniformly elliptic parabolic equations.

Contents

1. Introduction

In this chapter we will focus on the question of existence (and uniqueness) of a solution $f = f(t, x)$ to the (linear) evolution PDE of "parabolic type"

$$\partial_t f = \Lambda f \quad \text{on} \quad (0, \infty) \times \mathbb{R}^d, \tag{1.1}$$

where Λ is the following integro-differential operator

$$(\Lambda f)(x) = \Delta f(x) + a(x) \cdot \nabla f(x) + c(x)\, f(x) + \int_{\mathbb{R}^d} b(y, x)\, f(y)\, dy, \tag{1.2}$$

that we complement with an initial condition

$$f(0, x) = f_0(x) \quad \text{in} \quad \mathbb{R}^d. \tag{1.3}$$

Here $t \ge 0$ stands for the *"time"* variable, $x \in \mathbb{R}^d$ stands for the *"position"* variable (for instance), $d \in \mathbb{N}^*$.

In order to develop the variational approach for the equation (1.1)-(1.2), we assume that

$$f_0 \in L^2(\mathbb{R}^d) =: H, \quad \text{which is an Hilbert space,}$$

and that the coefficients satisfy

$$a, c \in L^\infty(\mathbb{R}^d), \quad b \in L^2(\mathbb{R}^d \times \mathbb{R}^d).$$

The **main result** we will present in this chapter is the existence of a weak (variational) solution (which sense will be specified below)

$$f \in X_T := C([0,T); L^2) \cap L^2(0,T; H^1) \cap H^1(0,T; H^{-1}),$$

to the evolution equation (1.1)–(1.3). We mean variational solution because the space of "test functions" is the same as the space in which the solution lives. It also refers to the associated stationary problem which is of "variational type" (see [1, chapters VIII & IX]).

The existence of solutions issue is tackled by following a scheme of proof that we will repeat for all the other evolution equations that we will consider in the next chapters.

(1) We look for **a priori estimates** by performing (formal) differential and integral calculus.

(2) We deduce a possible natural **functional space** in which lives a solution and we propose a **definition of a solution**, that is a (weak) sense in which we may understand the evolution equation.

(3) We state and prove the associated **existence theorem**. For the existence proof we typically argue as follows: we introduce a *"regularized problem"* for which we are able to construct a solution and we are allowed to rigorously perform the calculus leading to the *"a priori estimates"*, and then we pass to the limit in the sequence of regularized solutions.

2. A priori estimates

We explain how we may obtain *"a priori estimates"* for solutions to the parabolic equation (1.1)-(1.2) and more general, but related, abstract *"coercive+dissipative"* type equations. The term *"a priori estimates"* means that we do not seek in this first step to establish the estimates with full mathematical rigor but we rather try to perform formally some reasonable and usual computations (typically: derivation, integration, summation and inversion of theses operations). This step is fundamental in order to bring out what kind of information is reasonable to hope for. Of course, in some next steps, these bounds will have to be justified.

Define $V = H^1(\mathbb{R}^d)$ endowed with its usual norm and recall the definition (1.2) of Λ. We first observe that for any nice function f and for any $\alpha \in (0,1)$

$$\begin{aligned}\langle \Lambda f, f\rangle &= -\int |\nabla f|^2 + \int a f \cdot \nabla_x f + \int c\, f^2 + \int\!\!\int b(y,x)\, f(x) f(y)\, dx dy.\\ &\le -\alpha \|f\|_V^2 + \Big(\alpha + \frac{1}{4\alpha}\|a\|^2_{L^\infty} + \|c_+\|_{L^\infty} + \|b\|_{L^2}\Big)\, \|f\|^2_H,\end{aligned}$$

thanks to the Cauchy-Schwarz inequality in $L^2(\mathbb{R}^d)$ and $L^2(\mathbb{R}^d \times \mathbb{R}^d)$ and to the Young inequality $st \le \alpha s^2/2 + t^2/(2\alpha)$, $\forall\, s,t > 0$.

Exercise 2.1. *Prove that in the case* $\operatorname{div} a \in L^\infty$, *the following estimate holds*

$$\langle \Lambda f, f\rangle \le -\|f\|^2_V + \Big(1 + \|(c - \frac{1}{2}\operatorname{div} a)_+\|_{L^\infty} + \|b\|_{L^2}\Big)\, \|f\|^2_H.$$

We also observe that for any nice functions f, g

$$\begin{aligned}|\langle \Lambda f, g\rangle| &\le \|\nabla f\|_{L^2}\, \|\nabla g\|_{L^2} + \|a\|_{L^\infty}\, \|\nabla f\|_{L^2}\, \|g\|_{L^2} + (\|c\|_\infty + \|b\|_{L^2})\, \|f\|_{L^2}\, \|g\|_{L^2}\\ &\le (1 + \|a\|_\infty + \|c\|_{L^\infty} + \|b\|_{L^2})\, \|f\|_V \|g\|_V.\end{aligned}$$

We easily deduce from the two preceding estimates that our parabolic operator falls into the following abstract variational framework.

Abstract variational framework. We consider a Hilbert space H endowed with the scalar product $(\cdot,\cdot) = (\cdot,\cdot)_H$ and the norm $|\cdot| = |\cdot|_H$. We identify H with its dual space $H' = H$. We consider another Hilbert space V endowed with a norm $\|\cdot\| = \|\cdot\|_V$. We assume $V \subset H$ with dense and bounded embedding. Observing that for any $u \in H$, the mapping

$$v \in V \mapsto (u, v)$$

defines a linear and continuous form on V, we may identify u as an element of V'. In other words, we have

$$V \subset H \subset V' \quad \text{and} \quad \langle u, v\rangle = (u, v), \quad \forall\, u \in H,\, v \in V,$$

where we denote $\langle .,. \rangle = \langle .,. \rangle_{V',V}$ the duality product on V.

We consider a linear operator $\Lambda : V \to V'$ which is bounded (or continuous) in the sense

(i) $\exists\, M > 0$ such that

$$|\langle \Lambda g, h\rangle| \le M\, \|g\|\, \|h\|, \qquad \forall\, g, h \in V;$$

and which is *"coercive+dissipative"*[1] (or $-\Lambda$ satisfies a *"Gårding's inequality"*) in the sense

(ii) $\exists\, \alpha > 0$, $b \in \mathbb{R}$ such that

$$\langle \Lambda g, g\rangle \le -\alpha\, \|g\|^2 + b\, |g|^2, \qquad \forall\, g \in V.$$

We consider then the associated abstract evolution equation

$$\frac{dg}{dt} = \Lambda g \quad \text{on} \quad (0, T), \tag{2.1}$$

on a function $g : [0, T) \to H$ with prescribed initial value

$$g(0) = g_0 \in H. \tag{2.2}$$

A priori bound in the abstract variational framework. With the above assumptions and notations, any solution g to the abstract evolution equation (2.1) (formally) satisfies the following estimate

$$|g(T)|_H^2 + 2\alpha \int_0^T \|g(s)\|_V^2\, ds \le e^{2bT}\, |g_0|_H^2 \quad \forall\, T. \tag{2.3}$$

Indeed, using just the *coercivity+dissipativity* assumption (ii), we have (at least formally)

$$\frac{d}{dt} \frac{|g(t)|_H^2}{2} = \langle \Lambda g, g\rangle \le -\alpha \|g(t)\|_V^2 + b\, |g(t)|_H^2,$$

and we conclude to (2.3) thanks to the Gronwall lemma.

[1] We commonly say that (the bilinear form associated to) $-\Lambda$ is coercive if (ii) holds with $\alpha > 0$ and $b = 0$, and that $\Lambda - b$ is dissipative if (ii) holds with $\alpha = 0$ and $b \in \mathbb{R}$. Our assumption (ii) is then more general than a coercivity condition (on $-\Lambda$) but less general than a dissipativity condition (on Λ).

From estimate (2.3) together with equation (2.1) and the continuity estimate (i) on Λ, we deduce

$$\Big\| \frac{dg}{dt} \Big\|_{V'} = \|\Lambda g\|_{V'} \le M \, \|g\|_V \in L^2(0,T),$$

and we conclude with

$$g \in L^\infty(0,T;H) \cap L^2(0,T;V) \cap H^1(0,T;V'). \tag{2.4}$$

The two first Lebesgue spaces are defined through the Bochner integral which main features are explained in Appendix B while the last Sobolev space is defined as explained below. It is worth emphasizing that (almost) everything in the Bochner integral for functions with values in Banach spaces holds similarly as for the usual Lebesgue integral for functions with values in $\mathbb{R}$. The only results about the Bochner integral we will explicitly used in the sequel are the claims B.6, B.7 and B.8.

Definition 2.2. *For a Banach space $\mathcal{X}$ and an exponent $1 \le p \le \infty$, we say that $g \in W^{1,p}(0,T;\mathcal{X})$ if $g \in L^p(0,T;\mathcal{X})$ and there exists $w \in L^p(0,T;\mathcal{X})$ such that*

$$-\int_0^T g \, \varphi' \, dt = \int_0^T w \, \varphi \, dt \ \text{ in } \mathcal{X}, \quad \forall \, \varphi \in \mathcal{D}(0,T).$$

We note $w = g'$ or $w = \frac{d}{dt} g$, and we define the Sobolev norm

$$\|g\|_{H^1(0,T;\mathcal{X})} := (\|g\|^2_{L^2(0,T;\mathcal{X})} + \|g'\|^2_{L^2(0,T;\mathcal{X})})^{1/2},$$

as well as the usual modification for the $W^{1,p}(0,T;\mathcal{X})$ norm when $p \ne 2$.

3. Variational solutions

We develop the theory of variational solution for evolution equation associated to coercive+dissipative operator and we state our main existence and uniqueness result.

Definition 3.1. *For any given $g_0 \in H$, $T > 0$, we say that*

$$g = g(t) \in X_T := C([0,T];H) \cap L^2(0,T;V) \cap H^1(0,T;V')$$

is a **variational solution** *to the Cauchy problem (2.1)–(2.2) on the time interval $[0,T]$ if it is a solution in the following weak sense*

$$(g(t), \varphi(t))_H = (g_0, \varphi(0))_H + \int_0^t \big\{ \langle \Lambda g(s), \varphi(s) \rangle_{V',V} + \langle \varphi'(s), g(s) \rangle_{V',V} \big\} \, ds, \tag{3.1}$$

for any $\varphi \in X_T$ and any $0 \le t \le T$. We say that g is a global solution if it is a solution on $[0,T]$ for any $T > 0$. It is worth emphasizing that (3.1) is meaningful because of the assumptions made on g, φ and Λ.

Theorem 3.2 (J.-L. Lions)**.** *With the above definition and assumptions for any $g_0 \in H$, there exists a unique global variational solution to the Cauchy problem (2.1)-(2.2). As a consequence, any solution satisfies (2.3).*

We start with some remarks and we postpone the proof of the existence part of Theorem 3.2 to the next section.

3.1. **Parabolic equation.** As a consequence of Theorem 3.2, for any $f_0 \in L^2(\mathbb{R}^d)$ there exists a unique function

$$f = f(t) \in C([0,T];L^2) \cap L^2(0,T;H^1) \cap H^1(0,T;H^{-1}), \quad \forall T > 0,$$

which is a solution to the parabolic equation (1.1)-(1.2) in the variational sense.

3.2. **About the functional space.** The space obtained thanks to the a priori estimates established on g is nothing but X_T as a consequence of the following result.

Lemma 3.3. *The following inclusion*

$$L^2(0,T;V) \cap H^1(0,T;V') \subset C([0,T];H) \tag{3.2}$$

holds true. Moreover, for any $g \in L^2(0,T;V) \cap H^1(0,T;V')$ there holds

$$t \mapsto |g(t)|_H^2 \in W^{1,1}(0,T)$$

and

$$\frac{d}{dt}|g(t)|_H^2 = 2\,\langle g'(t), g(t)\rangle_{V',V} \quad a.e. \ on \quad (0,T). \tag{3.3}$$

Proof of Lemma 3.3. We first establish (3.2) thanks to a regularization trick and using in a fundamental way that $C([0,T];H)$ is a Banach space. The same regularization trick and a weak formulation allow us to end the proof.

Step 1. We define the function $\bar g = g$ on $[0,T]$, $\bar g = 0$ on $\mathbb{R}\backslash[0,T]$, next for a mollifier $\rho : \mathbb{R} \to \mathbb{R}$ with compact support included in $(-1,-1/2)$, we define the approximation to the identity sequence (ρ_ε) by setting $\rho_\varepsilon(t) := \varepsilon^{-1}\rho(\varepsilon^{-1}t)$ and finally the sequence $g_\varepsilon(t) := \bar g *_t \rho_\varepsilon$ where $*$ stands for the usual convolution operator on $\mathbb{R}$. We observe that $g_\varepsilon \in C^1(\mathbb{R};H)$, $g_\varepsilon \to g$ a.e. on $[0,T]$ and in $L^2(0,T;V)$ from the claim B.7. For a fixed $\tau \in (0,T)$ and for any $t \in (0,\tau)$ and any $0 < \varepsilon < T - \tau$, we have $s \mapsto \rho_\varepsilon(t-s) \in \mathcal{D}(0,T)$, since

$$\operatorname{supp} \rho_\varepsilon(t-\cdot) \subset [t+\varepsilon/2, t+\varepsilon] \subset [\varepsilon/2, \tau+\varepsilon] \subset [0,T],$$

and we have then

$$g_\varepsilon = \int_{\mathbb{R}} \rho_\varepsilon(t-s)\,\bar g(s)\,ds = \int_0^T \rho_\varepsilon(t-s)\,g(s)\,ds.$$

Similarly, we compute

$$\begin{aligned} g_\varepsilon' &= \int_{\mathbb{R}} \partial_t \rho_\varepsilon(t-s)\,\bar g(s)\,ds \\ &= -\int_0^T (\partial_s \rho_\varepsilon(t-s))\,g(s)\,ds \\ &= \int_0^T \rho_\varepsilon(t-s)\,g'(s)\,ds = \rho_\varepsilon * (\overline{g'}). \end{aligned}$$

As a consequence $g_\varepsilon' \to g'$ a.e. and in $L^2(0,\tau;V')$, from the claim B.7.

Step 2. We observe that for $t \mapsto u(t) \in C^1((0,T);H)$ and because $h \mapsto |h|_H^2$ is $C^1(H;\mathbb{R})$, we have $t \mapsto |u(t)|_H^2$ is $C^1((0,T);\mathbb{R})$ and

$$\frac{d}{dt}|u(t)|_H^2 = 2(u'(t),u(t))_H = 2\langle u'(t),u(t)\rangle_{V',V}.$$

We fix $\tau \in (0,T)$ and $\varepsilon, \varepsilon' \in (0, T-\tau)$, and we the above computation gives

$$\frac{d}{dt}|g_\varepsilon(t) - g_{\varepsilon'}(t)|_H^2 = 2\,\langle g'_\varepsilon - g'_{\varepsilon'}, g_\varepsilon - g_{\varepsilon'}\rangle_{V',V},$$

so that for any $t_1, t_2 \in [0,\tau]$

$$|g_\varepsilon(t_2) - g_{\varepsilon'}(t_2)|^2 = |g_\varepsilon(t_1) - g_{\varepsilon'}(t_1)|^2 + 2\int_{t_1}^{t_2} \langle g'_\varepsilon - g'_{\varepsilon'}, g_\varepsilon - g_{\varepsilon'}\rangle ds. \tag{3.4}$$

Since $g_\varepsilon \to g$ a.e. on $[0,\tau]$ in $V \subset H$, we may fix $t_1 \in [0,\tau]$ such that

$$g_\varepsilon(t_1) \to g(t_1) \quad \text{in } H. \tag{3.5}$$

As a consequence of (3.4), (3.5) as well as $g_\varepsilon \to g$ in $L^2(0,\tau;V)$ and $g'_\varepsilon \to g'$ in $L^2(0,\tau;V')$, we have

$$\limsup_{\varepsilon,\varepsilon'\to 0} \sup_{[0,\tau]} |g_\varepsilon(t) - g_{\varepsilon'}(t)|_H^2 \le \lim_{\varepsilon,\varepsilon'\to 0} \int_0^\tau \|g'_\varepsilon - g'_{\varepsilon'}\|_{V'}\, \|g_\varepsilon - g_{\varepsilon'}\|_V\, ds = 0,$$

where we have used Fatou's lemma in the last equality. We thus deduce that (g_ε) is a Cauchy sequence in $C([0,\tau];H)$, and then g_ε converges in $C([0,\tau];H)$ to a limit $\tilde g \in C([0,\tau];H)$. That proves $g = \tilde g$ a.e. and thus $g \in C([0,\tau];H)$ (up to modifying g on a set of zero Lebesgue measure). We prove similarly that $g \in C([\tau,T];H)$ for any $\tau \in (0,T)$ and thus $g \in C([0,T];H)$.

Step 3. Similarly as for (3.4), we have

$$|g_\varepsilon(t_2)|_H^2 = |g_\varepsilon(t_1)|_H^2 + 2\int_{t_1}^{t_2} \langle g'_\varepsilon, g_\varepsilon\rangle ds,$$

and passing to the limit $\varepsilon \to 0$, we get

$$|g(t_2)|_H^2 = |g(t_1)|_H^2 + 2\int_{t_1}^{t_2} \langle g', g\rangle ds.$$

Using again that $\langle g', g\rangle \in L^1(0,T)$, we easily deduce from the above identity the two remaining claims of the Lemma. □

Exercise 3.4 (A weak solution is a variational solution)**.** *Take $g_0 \in H$ and assume that $g \in L^2(0,T;V)$ satisfies*

$$(g_0, \varphi(0)) = \int_0^T (g\varphi' + \Lambda g\varphi) dt \quad in \quad V',$$

for any $\varphi \in C^1_c([0,T);\mathbb{R})$. Prove that $g \in X_T$ and g is a variational solution. (Hint. Use Lemma 3.3 and Steps 4 & 5 of the existence part of the proof of Theorem 3.2).

3.3. **A posteriori estimate and uniqueness.** Taking $\varphi = g \in X_T$ as a test function in (3.1), we deduce from Lemma 3.3,

$$\begin{aligned}
\frac12 |g(t)|_H^2 - \frac12 |g_0|_H^2 &= |g(t)|_H^2 - |g_0|_H^2 - \int_0^t \langle g'(s), g(s)\rangle\, ds \\
&= \int_0^t \langle \Lambda g, g\rangle\, ds \\
&\le \int_0^t (-\alpha\,\|g\|_V^2 + b\,|g|_H^2)\, ds,
\end{aligned}$$

where we have used (3.3) at the first line, the variational formulation (3.1) at the second line and the *"coercive+dissipative"* assumption on Λ at the last line. We then obtain (2.3) as an **a posteriori estimate** thanks to the Gronwall.

Let us prove now the **uniqueness of the variational solution** g associated to a given initial datum $g_0 \in H$. In order to do so, we consider two variational solutions g and f associated to the same initial datum. Since the equation (2.1), (2.2) is linear, or more precisely, the variational formulation (3.1) is linear in the solution, the function $g - f$ satisfies the same variational formulation (3.1) but associated to the initial datum $g_0 - f_0 = 0$. The a posteriori estimate (2.3) then holds for $g - f$ and implies that $g - f = 0$.

4. Proof of the existence part of Theorem 3.2.

We first prove thanks to a compactness argument in step 1 to step 3 that there exists a function $g \in L^2(0,T;V)$ such that

$$\langle g_0, \varphi(0)\rangle + \int_0^T \big\{ \langle \Lambda g(s), \varphi(s)\rangle_{V',V} + \langle \varphi'(s), g(s)\rangle_{V',V} \big\}\, ds = 0, \tag{4.1}$$

for any $\varphi \in C^1_c([0,T);V)$. The proof relies on an approximation scheme and a compactness argument by taking advantage that the Bochner integral space $L^2(0,T;V)$ is a Hilbert space. We next deduce by some "regularization tricks" in step 4 and step 5 that the above weak solution is a variational solution.

Step 1. For a given $g_0 \in H$ and $\varepsilon > 0$, we seek $g_1 \in V$ such that

$$g_1 - \varepsilon \Lambda g_1 = g_0. \tag{4.2}$$

We introduce the bilinear form $a : V \times V \to \mathbb{R}$ defined by

$$a(u,v) := (u,v) - \varepsilon\, \langle \Lambda u, v\rangle.$$

Thanks to the assumptions made on Λ, we have

$$|a(u,v)| \le |u|\,|v| + \varepsilon\, M\, \|u\|\, \|v\|,$$

and

$$a(u,u) \ge |u|^2 + \varepsilon\, \alpha\, \|u\|^2 - \varepsilon\, b\, |u|^2 \ge \varepsilon\, \alpha\, \|u\|^2, \tag{4.3}$$

whenever $\varepsilon\, b < 1$, what we assume from now. On the other hand, the mapping $v \in V \mapsto (g_0, v)$ is a linear and continuous form. We may thus apply the Lax-Milgram theorem which implies

$$\exists !\, g_1 \in V \qquad (g_1, v) - \varepsilon \langle \Lambda g_1, v\rangle = (g_0, v) \quad \forall\, v \in V.$$

Step 2. Fix $\varepsilon > 0$ as in the preceding step and build by induction the sequence (g_k) in $V \subset H$ defined by the family of equations

$$\forall\, k \qquad \frac{g_{k+1} - g_k}{\varepsilon} = \Lambda\, g_{k+1}. \tag{4.4}$$

Observe that from the identity

$$(g_{k+1}, g_{k+1}) - \varepsilon\, \langle \Lambda g_{k+1}, g_{k+1}\rangle = (g_k, g_{k+1}),$$

we deduce (that is (4.3) again)

$$|g_{k+1}|^2 + \varepsilon\, \alpha\, \|g_{k+1}\|^2 - \varepsilon\, b\, |g_{k+1}|^2 \le |g_k|\, |g_{k+1}| \le \frac{1}{2}|g_k|^2 + \frac{1}{2}|g_{k+1}|^2,$$

and then

$$|g_{k+1}|^2 + 2\varepsilon\alpha\,\|g_{k+1}\|^2 \le (1-2\varepsilon\, b)^{-1}\,|g_k|^2, \quad \forall\, k \ge 0.$$

Thanks to the discrete version of the Gronwall lemma, we get

$$|g_n| + 2\alpha \sum_{k=1}^{n} \varepsilon \|g_k\|^2 \le (1-2\varepsilon\, b)^{-n} |g_0| \le e^{2b\varepsilon n}\,|g_0|, \quad \forall\, n \ge 1.$$

We now fix $T > 0$, $n \in \mathbb{N}^*$, and we define

$$\varepsilon := T/n, \quad t_k = k\,\varepsilon, \quad g^\varepsilon(t) := g_k \text{ on } [t_k, t_{k+1}).$$

The last estimate writes then

$$2\alpha \int_0^T \|g^\varepsilon\|_V^2\, dt \le (e^{2bT} + 2\alpha\varepsilon)\,|g_0|^2. \tag{4.5}$$

Step 3. Consider a test function $\varphi \in C^1_c([0,T);V)$ and define $\varphi_k := \varphi(t_k)$, so that $\varphi_n = \varphi(T) = 0$. Multiplying the equation (4.4) by φ_k and summing up from $k = 0$ to $k = n-1$, we get

$$-(\varphi_0, g_0) - \sum_{k=1}^{n} \langle \varphi_k - \varphi_{k-1}, g_k\rangle = \sum_{k=0}^{n} \varepsilon\,\langle \Lambda g_{k+1}, \varphi_k\rangle = \sum_{k=1}^{n} \varepsilon\,\langle \Lambda g_k, \varphi_{k-1}\rangle,$$

where in the LHS we use the duality production $\langle , \rangle$ in $V' \times V$ instead of the scalar product $(,)$ in H thanks to the inclusions $V \subset H = H' \subset V'$. Introducing the two functions $\varphi^\varepsilon, \varphi_\varepsilon : [0,T) \to V$ defined by

$$\varphi^\varepsilon(t) := \varphi_{k-1} \quad \text{and} \quad \varphi_\varepsilon(t) := \frac{t_{k+1}-t}{\varepsilon}\,\varphi_{k-1} + \frac{t-t_k}{\varepsilon}\,\varphi_k \qquad \text{for} \quad t \in [t_k, t_{k+1}),$$

in such a way that

$$\varphi_\varepsilon'(t) = \frac{\varphi_k - \varphi_{k-1}}{\varepsilon} \qquad \text{for} \quad t \in (t_k, t_{k+1}),$$

the above equation also writes

$$-\langle \varphi(0), g_0\rangle - \int_\varepsilon^T \langle \varphi_\varepsilon', g^\varepsilon\rangle\, dt = \int_0^T \langle \Lambda g^\varepsilon, \varphi^\varepsilon\rangle\, dt. \tag{4.6}$$

On the one hand, from (4.5) and the claim B.8 about Bochner integral, we know that up to the extraction of a subsequence, there exists $g \in L^2(0,T;V)$ such that $g^\varepsilon \rightharpoonup g$ weakly in $L^2(0,T;V)$. On the other hand, from the above construction, we have $\varphi_\varepsilon' \to \varphi'$ and $\varphi_\varepsilon \to \varphi$ both uniformly in $L^\infty(0,T;V)$ (using that φ and φ' belong to $C([0,T];V)$ and thus are uniformly continuous). We may then pass to the limit as $\varepsilon \to 0$ in (4.6) and we get (4.1).

Step 4. We prove that $g \in X_T$. Taking $\varphi := \chi(t)\,\psi$ with $\chi \in C^1_c((0,T))$ and $\psi \in V$ in equation (4.1) and using claim B.6, we get

$$\Big\langle \int_0^T g\chi' dt, \psi \Big\rangle = \int_0^T \langle g, \psi\rangle \chi'\, dt = -\int_0^T \langle \Lambda g, \psi\rangle \chi\, dt = \Big\langle -\int_0^T \Lambda g \chi\, dt, \psi \Big\rangle.$$

This equation holding true for any $\psi \in V$, it is equivalent to

$$\int_0^T g\chi' dt = -\int_0^T \Lambda g \chi\, dt \quad \text{in} \quad V' \quad \text{for any} \quad \chi \in \mathcal{D}(0,T),$$

or in other words

$$g' = \Lambda g \;\; \text{in the sense of distributions in } V'.$$

Since $g \in L^2(0,T;V)$, we get that $\Lambda g \in L^2(0,T;V')$ and the above relation precisely means that $g \in H^1(0,T;V')$ as defined in Definition 2.2. We conclude thanks to Lemma 3.3 that $g \in X_T$.

Step 5. Assume first $\varphi \in C_c([0,T);H) \cap L^2(0,T;V) \cap H^1(0,T;V')$. We define $\varphi_\varepsilon(t) := \varphi *_t \rho_\varepsilon$ for a mollifier (ρ_ε) with compact support included in $(0,\infty)$ so that from claim B.7, $\varphi_\varepsilon \in C^1_c([0,T);V)$ for any $\varepsilon > 0$ small enough and

$$\varphi_\varepsilon \to \varphi \quad \text{in} \quad C([0,T];H) \cap L^2(0,T;V) \cap H^1(0,T;V').$$

Writing the equation (4.1) for φ_ε and passing to the limit $\varepsilon \to 0$ we get that (4.1) also holds true for φ.

Assume next that $\varphi \in X_T$. We fix $\chi \in C^1(\mathbb{R})$ such that $\operatorname{supp} \chi \subset (-\infty, 0)$, $\chi' \le 0$, $\chi' \in C_c(]-1,0[)$ and $\int_{-1}^0 \chi' = -1$, and we define $\chi_\varepsilon(t) := \chi((t-T)/\varepsilon)$ so that $\varphi_\varepsilon := \varphi \chi_\varepsilon \in C_c([0,T);H)$ and $\chi_\varepsilon \to \mathbf{1}_{[0,T]}$ a.e., $\chi'_\varepsilon \to -\delta_T$ in $\mathcal{D}'(\mathbb{R})$ as $\varepsilon \to 0$. Equation (4.1) for the test function φ_ε writes

$$-\langle g_0, \varphi(0)\rangle - \int_0^T \chi'_\varepsilon \langle \varphi, g\rangle \, ds = \int_0^T \chi_\varepsilon \big\{ \langle \Lambda g, \varphi\rangle + \langle \varphi', g\rangle \big\} \, ds,$$

and we obtain the variational formulation (3.1) for $t_1 = 0$ and $t_2 = T$ by passing to the limit $\varepsilon \to 0$ in the above equation. □

Appendix A. Exercises

Exercise A.1. *Following Definition 2.2, we say that $g \in L^2(0,T;V)$ is a solution to the abstract evolution equation* (2.1)-(2.2) *if*

$$\text{(A.1)} \qquad - g_0 \, \varphi(0) - \int_0^T g \, \varphi' \, dt = \int_0^T \Lambda g \, \varphi \, dt \ \text{ in } V', \quad \forall \varphi \in C^1_c([0,T)).$$

Prove that under the hypothesis of Theorem 3.2, a function $g \in L^2(0,T;V)$ is a solution to the abstract evolution equation (2.1)-(2.2) *in the above sense if, and only if, it is a variational solution (so that in particular $g \in X_T$). (Hint. Use some arguments presented in Lemma 3.3 and in steps 3, 4 and 5 of the proof of Theorem 3.2).*

Exercise A.2. *Prove that $f \ge 0$ if $f_0 \ge 0$ and $b \ge 0$ for the solution of the parabolic equation* (1.1). *(Hint. Show that the sequence (g_k) defined in step 2 of the proof of the existence part is such that $g_k \ge 0$ for any $k \in \mathbb{N}$).*

Exercise A.3. *Prove the existence of a solution $g \in X_T$ to the equation*

$$\text{(A.2)} \qquad \frac{dg}{dt} = \Lambda g + G \quad \text{in} \quad (0,T), \quad g(0) = g_0,$$

for any initial datum $g_0 \in H$ and any source term $G \in L^2(0,T;V')$.
(Hint. Repeat the same proof as for the Theorem 3.2 where for the a priori bound one can use

$$\int_0^T \langle g, G\rangle \, dt \le \frac{\alpha}{2} \int_0^T \|g(t)\|_V^2 \, dt + \frac{1}{2\alpha} \int_0^T \|G(t)\|_{V'}^2 \, dt,$$

and for the approximation scheme one can define

$$\varepsilon^{-1} (g_{k+1} - g_k) = \Lambda g_{k+1} + G_k, \quad G_k := \int_{t_k}^{t_{k+1}} G(s) \, ds).$$

Exercise A.4. *Generalize the existence and uniqueness result to the PDE equation*

$$\text{(A.3)} \qquad \partial_t f = \partial_i (a_{ij} \, \partial_j f) + b_i \, \partial_i f + \partial_i (\beta_i f) + c f + \int k(t,x,y) \, f(t,y) \, dy + G$$

where a_{ij}, b_i, β_i, c and k are times dependent coefficients and where a_{ij} is uniformly elliptic in the sense that

$$\text{(A.4)} \qquad \forall \, t \in (0,T), \ \forall \, x \in \mathbb{R}^d, \ \forall \, \xi \in \mathbb{R}^d \quad a_{ij}(t,x) \, \xi_i \xi_j \ge \alpha \, |\xi|^2, \quad \alpha > 0.$$

More precisely, establish the following result:

Theorem A.5 (J.-L. Lions - the time dependent case)**.** *Assume that*

$$a,\, b,\, \beta,\, c \in L^\infty((0,T)\times\mathbb{R}^d), \quad k \in L^\infty(0,T;L^2(\mathbb{R}^d\times\mathbb{R}^d)),$$

and that a satisfies the uniformly elliptic condition (A.4)*. For any* $g_0 \in L^2(\mathbb{R}^d)$ *and* $G \in L^2(0,T; H^{-1}(\mathbb{R}^d))$*, there exists a unique variational solution to the Cauchy problem associated to* (A.3) *in the sense that*

$$f \in X_T := C([0,T);L^2)\cap L^2(0,T;H^1)\cap H^1(0,T;H^{-1}),$$

such that for any $\varphi \in X_T$ *and any* $t \in (0,T)$ *there holds*

$$\text{(A.5)}\quad \int_{\mathbb{R}^d} g(t)\varphi(t)\,dx = \int_{\mathbb{R}^d} g_0\varphi(0)\,dx + \int_0^t\int_{\mathbb{R}^d}(G\varphi + g\partial_t\varphi)\,dxds$$
$$+\int_0^t\int_{\mathbb{R}^d}\{(b_i\,\partial_i f + cf)\,\varphi - a_{ij}\,\partial_j f\partial_i\varphi\}\,dxds + \int_0^t\int_{\mathbb{R}^d\times\mathbb{R}^d} k(t,x,y)\,f(t,y)\,\varphi(s,x)\,dxdyds$$

(Hint. Define

$$a_i := \frac{n}{T}\int_{t_{i-1}}^{t_i} a(t,\cdot)\,dt, \quad i = 1,\dots,n, \quad t_i := iT/n,$$

and a similar way b_i, c_i, k_i*, and prove that there exists a unique variational solution* $g_i \in X_{T/n}$ *to equation* (1.1)-(1.2) *associated to the* a_i, b_i, c_i, k_i *and the initial condition* g_0 *when* $i = 1$*,* $g_{i-1}(T/n)$ *when* $i \ge 2$*. Build next a solution* $g^n \in X_T$ *to the equation* (A.3) *associated to the piecewise constant functions* $a^n(t) = a_i$ *if* $t \in [t_i, t_{i+1})$*,* $i = 0,\dots,n-1$*, and* b^n, c^n, k^n *defined similarly. Conclude by passing to the limit* $n\to\infty$*).*

Exercise A.6. *We consider the nonlinear McKean-Vlasov equation*

$$\text{(A.6)}\qquad \partial_t f = \Lambda[f] := \Delta f + div(F[f]f), \quad f(0) = f_0,$$

with

$$F[f] := a * f, \quad a \in L^\infty(\mathbb{R}^d)^d.$$

1) Prove the a priori estimates

$$\|f(t)\|_{L^1} = \|f_0\|_{L^1} \quad \forall\, t\ge 0, \quad \|f(t)\|_{L^2_k} \le e^{Ct}\,\|f_0\|_{L^2_k} \quad \forall\, t \ge 0,$$

for any $k > 0$ *and a constant* $C := C(k, \|a\|_{L^\infty}, \|f_0\|_{L^1})$*, where we define the weighted Lebesgue space* L^2_k *by its norm* $\|f\|_{L^2_k} := \|f\langle x\rangle^k\|_{L^2}$*,* $\langle x\rangle := (1+|x|^2)^{1/2}$*.*
2) We set $H := L^2_k$*,* $k > d/2$*, and* $V := H^1_k$*, where we define the weighted Sobolev space* H^1_k *by its norm* $\|f\|^2_{H^1_k} := \|f\|^2_{L^2_k} + \|\nabla f\|^2_{L^2_k}$*. Observe that for any* $f \in V$ *the distribution* $\Lambda[f]$ *is well defined in* V' *thanks to the identity*

$$\langle\Lambda[f], g\rangle := -\int_{\mathbb{R}^d}(\nabla f + (a*f)f)\cdot\nabla(g\langle x\rangle^{2k})\,dx \quad \forall\, g\in V.$$

(Hint. Prove that $L^2_k \subset L^1$*). Write the variational formulation associated to the nonlinear McKean-Vlasov equation. Establish that if moreover the variational solution to the nonlinear McKean-Vlasov equation is nonnegative then it is mass preserving, that is* $\|f(t)\|_{L^1} = \|f_0\|_{L^1}$ *for any* $t \ge 0$*. (Hint. Take* $\chi_M\,\langle x\rangle^{-2k}$ *as a test function in the variational formulation, with* $\chi_M(x) := \chi(x/M)$*,* $\chi \in \mathcal{D}(\mathbb{R}^d)$*,* $\mathbf{1}_{B(0,1)} \le \chi \le \mathbf{1}_{B(0,2)}$*).*
3) Prove that for any $0 \le f_0 \in H$ *and* $g \in C([0,T];H)$ *there exists a unique mass preserving variational solution* $0 \le f \in X_T$ *to the linear McKean-Vlasov equation*

$$\partial_t f = \Delta f + div(F[g]f), \quad f(0) = f_0.$$

Prove that the mapping $g \mapsto f$ *is a contraction in* $C([0,T];H)$ *for* $T > 0$ *small enough. Conclude to the existence and uniqueness of a global (in time) variational solution to the nonlinear McKean-Vlasov equation.*

Exercise A.7. *For* $a \in W^{1,\infty}(\mathbb{R}^d)$*,* $c \in L^\infty(\mathbb{R}^d)$*,* $f_0 \in L^p(\mathbb{R}^d)$*,* $1 \le p \le \infty$*, we consider the linear parabolic equation*

$$\text{(A.7)}\qquad \partial_t f = \Lambda f := \Delta f + a\cdot\nabla f + cf, \quad f(0) = f_0.$$

We introduce the usual notations $H := L^2$*,* $V := H^1$ *and* X_T *the associated space for some given* $T > 0$*.*

1) Prove that for $\gamma \in C^1(\mathbb{R})$, $\gamma(0) = 0$, $\gamma' \in L^\infty$, there holds $\gamma(f) \in H$ for any $f \in H$ and $\gamma(f) \in V$ for any $f \in V$.

2) Prove that $f \in X_T$ is a variational solution to (A.7) *if and only if*

$$\frac{d}{dt} f = \Lambda f \text{ in } V' \text{ a.e. on } (0,T).$$

3) On the other hand, prove that for any $f \in X_T$ and any function $\beta \in C^2(\mathbb{R})$, $\beta(0) = \beta'(0) = 0$, $\beta'' \in L^\infty$, there holds

$$\frac{d}{dt} \int_{\mathbb{R}^d} \beta(f) = \langle \frac{d}{dt} f, \beta'(f) \rangle_{V',V} \text{ a.e. on } (0,T).$$

(Hint. Consider $f_\varepsilon = f *_t \rho_\varepsilon \in C^1([0,T]; H^1)$ and pass to the limit $\varepsilon \to 0$).

4) Consider a convex function $\beta \in C^2(\mathbb{R})$ such that $\beta(0) = \beta'(0) = 0$ and $\beta'' \in L^\infty$. Prove that any variational solution $f \in X_T$ to the above linear parabolic equation satisfies

$$\int_{\mathbb{R}^d} \beta(f_t)\, dx \le \int_{\mathbb{R}^d} \beta(f_0)\, dx + \int_0^t \int_{\mathbb{R}^d} \{c\, f\, \beta'(f) - (div\, a)\, \beta(f)\}\, dxds,$$

for any $t \ge 0$.

5) Assuming moreover that there exists a constant $K \in (0,\infty)$ such that $0 \le s\,\beta'(s) \le K \beta(s)$ for any $s \in \mathbb{R}$, deduce that for some constant $C := C(a,c,K)$, there holds

$$\int_{\mathbb{R}^d} \beta(f_t)\, dx \le e^{Ct} \int_{\mathbb{R}^d} \beta(f_0)\, dx, \quad \forall\, t \ge 0.$$

6) Prove that for any $p \in [1,2]$, for some constant $C := C(a,c)$ and for any $f_0 \in L^2 \cap L^p$, there holds

$$\|f(t)\|_{L^p} \le e^{Ct} \|f_0\|_{L^p}, \quad \forall\, t \ge 0.$$

(Hint. Define β on $\mathbb{R}_+$ and extend it to $\mathbb{R}$ by symmetry. More precisely, define $\beta''_\alpha(s) = 2\theta \mathbf{1}_{s\le\alpha} + p(p-1)s^{p-2}\mathbf{1}_{s>\alpha}$, with $2\theta = p(p-1)\alpha^{p-2}$ and then the primitives which vanish at the origin, which are thus defined by $\beta'_\alpha(s) = 2\theta s \mathbf{1}_{s\le\alpha} + (ps^{p-1} + p(p-2)\alpha^{p-1})\mathbf{1}_{s>\alpha}$, $\beta_\alpha(s) = \theta s^2 \mathbf{1}_{s\le\alpha} + (s^p + p(p-2)\alpha^{p-1}s + A\alpha^p)\mathbf{1}_{s>\alpha}$, $A := p(p-1)/2 - 1 - p(p-2)$. Observe that $s\beta'_\alpha(s) \le 2\beta_\alpha(s)$ because $s\beta''_\alpha(s) \le \beta'_\alpha(s)$ and $\beta_\alpha(s) \le \beta(s)$ because $\beta''_\alpha(s) \le \beta''(s)$).

7) Prove that for any $p \in [2,\infty]$ and for some constant $C := C(a,c,p)$ there holds

$$\|f(t)\|_{L^p} \le e^{Ct} \|f_0\|_{L^p}, \quad \forall\, t \ge 0.$$

(Hint. Define $\beta''_R(s) = p(p-1)s^{p-2}\mathbf{1}_{s\le R} + 2\theta \mathbf{1}_{s>R}$, with $2\theta = p(p-1)R^{p-2}$, and then the primitives which vanish in the origin and which are thus defined by $\beta'_R(s) = ps^{p-1}\mathbf{1}_{s\le R} + (pR^{p-1} + 2\theta(s-R))\mathbf{1}_{s>R}$, $\beta_R(s) = s^p \mathbf{1}_{s\le R} + (R^p + pR^{p-1}(s-R) + \theta(s-R)^2)\mathbf{1}_{s>R}$. Observe that $s\beta'_R(s) \le p\beta_R(s)$ because $s\beta''_R(s) \le (p-1)\beta'_R(s)$ and $\beta_R(s) \le \beta(s)$ because $\beta''_R(s) \le \beta''(s)$. Pass to the limit $p \to \infty$ in order to deal with the case $p = \infty$).

8) Prove that for any $f_0 \in L^p(\mathbb{R}^d)$, $1 \le p \le \infty$, there exists at least one weak (in the sense of distributions) solution to the linear parabolic equation (A.7)*. (Hint: Consider $f_{0,n} \in L^1 \cap L^\infty$ such that $f_{0,n} \to f_0$ in L^p, $1 \le p < \infty$, and prove that the associate variational solution $f_n \in X_T$ is a Cauchy sequence in $C([0,T]; L^p)$. Conclude the proof by passing to the limit $p \to \infty$).*

9) Establish the L^p estimates with "optimal" constant C (that is the one given by the formal computations).

10) Extend the above result to an equation with an integral term and/or a source term.

11) Prove the existence of a weak solution to the McKean-Vlasov equation (A.6) *for any initial datum $f_0 \in L^1(\mathbb{R}^d)$.*

12) Recover the positivity result of exercise A.2. (Hint. Choose first $\beta(s) := s_-^p$ with $p > 1$ and let next $p \to 1$).

Appendix B. The Bochner integral

In this section we present several definitions and results (without proof) about the Bochner integral which generalizes the Lebesgue integral for functions taking values in a general Banach space. We refer to [3, sections V.3, V.4] for details.

We consider $\mathcal{X}$ a Banach space endowed with the strong topology (associated to its norm) and $(\Omega, \mathscr{T}, \mu)$ a measured space.

We say that $f : \Omega \to \mathcal{X}$ is a simple function if

$$f = \sum_{i \in I} a_i \, \mathbf{1}_{A_i},$$

for some finite set I, some measurable sets A_i and some $a_i \in \mathcal{X}$. For a simple function, we define the integral by

$$\int_\Omega f \, d\mu := \sum_{i \in I} a_i \, \mu(A_i) \in \mathcal{X}.$$

We say that a function $f : \Omega \to \mathcal{X}$ is *measurable* if there exists a sequence (f_n) of simple functions such that

$$\text{a.e. on} \quad \Omega, \quad f_n \to f.$$

Similarly and for later reference, we say a function $f : \Omega \to \mathcal{X}$ is *weakly measurable* if there exists a sequence (f_n) of simple functions such that

$$\text{a.e. on} \quad \Omega, \quad f_n \rightharpoonup f \text{ weakly}.$$

Here, weakly has to be understood in the sense of the weak $\sigma(\mathcal{X}, \mathcal{X}')$ topology or of the weak$*$ $\sigma(\mathcal{Y}', \mathcal{Y})$ when $\mathcal{X} = \mathcal{Y}'$ for some Banach space $\mathcal{Y}$. In both cases, we have $\|f\| \le \liminf \|f_n\|$. Observe that if f is (weakly) measurable then $\|f\|$ is measurable (because the norm is lsc for the weak topologies). We say that a measurable function f is integrable (in the sense of Bochner) if there exists a sequence (f_n) of simple functions such that

$$f_n \to f \ \text{ a.e. on } \Omega \quad \text{and} \quad \int_\Omega \|f_n - f\| \, d\mu \to 0.$$

Equivalently, a measurable function $f : \Omega \to \mathcal{X}$ is integrable if $\|f\|$ is integrable. For a integrable function, we define the (Bochner) integral by

$$\int_\Omega f \, d\mu := \lim_{n \to \infty} \int_\Omega f_n \, d\mu.$$

We have then

$$\Big\| \int_\Omega f \, d\mu \Big\| \le \int_\Omega \|f\| \, d\mu.$$

In the sequel, we do not distinguish between two measurable functions f, g such that $f = g$ a.e. on Ω, and we just write $f = g$ in that case (which as usually means that f and g are in the same class of functions for the equivalence relation which is the a.e. equality). We define the Lebesgue spaces

$$L^p(\Omega; \mathcal{X}) := \{ f : \Omega \to \mathcal{X} \text{ measurable}; \ \|f\|_{L^p} < \infty \}$$

with

$$\|f\|_{L^p} := \Big(\int_\Omega \|f\|^p \, d\mu \Big)^{1/p}, \ p \in [1, \infty), \quad \|f\|_{L^\infty} := \inf\{\lambda \ge 0; \ \|f\| \le \lambda \text{ a.e.}\}.$$

The fundamental example is the following. We consider two measurable spaces $(E, \mathcal{A}, \mu)$ and $(F, \mathcal{B}, \nu)$ as well as a measurable function $f : E \times F \to \mathbb{R}$. Abusing notations, for $p, q \in [1, \infty)$, we have the equivalence between $f \in L^p(E; L^q(F))$ and

$$\int_E \Big(\int_F |f(x,y)|^q d\nu(y) \Big)^{p/q} d\mu(x) < \infty,$$

thanks to the Fubini-Tonelli theorem, and a similar equivalence holds when p or $q = \infty$. Because of this observation, when $\mathcal{X} = L^q(F)$, most of the results presented below can alternatively be proved using the usual Lebesgue integration theory and Fubini-Tonelli type results.
At this point, it is worth mentioning that another alternative to the Bochner integral is to define the integral by weak$*$ duality.

Fact B.1. *Consider a function $f : \Omega \to \mathcal{Y}'$ which is weakly∗ measurable and such that $\|f\|$ is integrable. Then, there exists a unique $\xi \in \mathcal{Y}'$ such that*

$$\forall\, y \in \mathcal{Y}, \quad \langle \xi, y\rangle_{\mathcal{Y}',\mathcal{Y}} = \int_\Omega \langle f(x), y\rangle_{\mathcal{Y}',\mathcal{Y}} d\mu(x).$$

We say that ξ is the weak∗ integral of f, and we write

$$\xi = \int_\Omega f(x)\, d\mu(x).$$

Of course, when $\mathcal{X}$ is reflexive the weak∗ integral coincides with the Bochner integral. We do not pursue this point of view in the sequel.

Fact B.2. *The Lebesgue spaces $L^p(\Omega; \mathcal{X})$ are Banach spaces for any $1 \le p \le \infty$, and $C_c(\Omega; \mathcal{X})$ is dense in $L^p(\Omega; \mathcal{X})$ when $1 \le p < \infty$ and $(\Omega, \mathscr{T})$ is the borelian σ-algebra associated to a locally compact and σ-compact topologic space Ω (for instance when $\Omega \subset \mathbb{R}$ endowed with the usual topology).*

Fact B.3. If $\Omega = (0, T)$ and $\mathscr{T}$ is the Lebesgue σ-algebra, any weakly $\sigma(\mathcal{X}, \mathcal{X}')$ continuous function $f : \Omega \to \mathcal{X}$ is measurable. If (f_n) is a sequence of measurable functions such that $f_n \rightharpoonup f$ a.e. in $\sigma(\mathcal{X}, \mathcal{X}')$, then f is measurable.

The two preceding facts are consequences of a more general and fundamental result.

Fact B.4. (Pettis) A function $f : \Omega \to \mathcal{X}$ is measurable iff it is weakly measurable (for the $\sigma(\mathcal{X}, \mathcal{X}')$ topology) and separately valued, which means that its range $\{f(x),\, x \in \Omega\}$ is a separable set.

Fact B.5. (Lebesgue dominated convergence Theorem) If (f_n) is a sequence of integrable functions and $g : \Omega \to \mathbb{R}_+$ is an integrable function such that

$$f_n \to f \text{ a.e.} \quad \text{and} \quad \|f_n\| \le g \text{ a.e.},$$

then f is integrable and $f_n \to f$ in $L^1(\Omega, \mathcal{X})$.

Fact B.6. Consider X, Y two Banach spaces, $1 \le p \le \infty$ and $A \in \mathscr{B}(X, Y)$. If $f \in L^p(\Omega; X)$ then $Af \in L^p(\Omega; Y)$. If furthermore $p = 1$ then

$$A \int f\, d\mu = \int (Af)\, d\mu.$$

In particular, we have

$$\Big\langle \xi, \int f(x) d\mu(x) \Big\rangle = \int \langle \xi, f(x)\rangle d\mu(x), \quad \forall\, \xi \in \mathcal{X}'.$$

If furthermore, $X \subset Y$, then $f \in L^1(\Omega; X)$ implies $f \in L^1(\Omega; Y)$ and the two integrals coincide.

Fact B.7. If $f \in L^p(\Omega; \mathcal{X})$, $g \in L^{p'}(\Omega, \mathbb{R})$, $h \in L^{p'}(\Omega, \mathcal{X}')$, we have $fg \in L^1(\Omega; \mathcal{X})$, $\langle f, h\rangle \in L^1(\Omega, \mathbb{R})$. In the particular case $\Omega = \mathbb{R}$, we may also define the convolution product $f * g \in L^1(\mathbb{R}; \mathcal{X})$ for any $f \in L^1(\mathbb{R}; \mathcal{X})$ and $g \in L^1(\mathbb{R}; \mathbb{R})$. For a sequence of mollifiers (ρ_ε) with $\rho_\varepsilon \in C^k_c(\mathbb{R})$, $\rho_\varepsilon \rightharpoonup \delta_0$, we then have $f * \rho_\varepsilon \in C^k(\mathbb{R}; \mathcal{X})$ and $f * \rho_\varepsilon \to f$ in $L^p(\mathbb{R}; \mathcal{X})$.

Fact B.8. If $\mathcal{X}$ is a reflexive space and $p \in (1, \infty)$, then $L^p(\Omega; \mathcal{X})$ is also a reflexive space, in particular if (f_n) is a bounded sequence in $L^p(\Omega; \mathcal{X})$, there exists $f \in L^p(\Omega; \mathcal{X})$ such that $f_n \rightharpoonup f$, which means

$$\int_\Omega \langle \xi(x), f_n(x)\rangle d\mu(x) \;\to \int_\Omega \langle \xi, f(x)\rangle d\mu(x),$$

for any $\xi \in L^{p'}(\Omega, \mathcal{X}')$. Similarly, if $\mathcal{X} = \mathcal{Y}'$ with $\mathcal{Y}$ separable, then $L^\infty(\Omega; \mathcal{X})$ is weakly∗ sequentially compact, which means that if (f_n) is a bounded sequence in $L^\infty(\Omega; \mathcal{X})$, there exists $f \in L^\infty(\Omega; \mathcal{X})$ such that $f_n \rightharpoonup f$ weakly∗, or in other words

$$\int_\Omega \langle f_n(x), u(x)\rangle d\mu(x) \;\to \int_\Omega \langle f(x), u(x)\rangle d\mu(x),$$

for any $u \in L^1(\Omega, \mathcal{Y})$.

Fact B.9. For $1 \le p \le \infty$, if (f_n) is a sequence of $L^p(\Omega; \mathcal{X})$, $f : \Omega \to \mathcal{X}$ is a function such that

$$\|f_n\|_{L^p(\Omega;\mathcal{X})} \le C, \quad f_n \rightharpoonup f \textit{ a.e.}$$

then $f \in L^p(\Omega; \mathcal{X})$ and $\|f\|_{L^p} \le \liminf \|f_n\|_{L^p}$.

Fact B.10. The space $\mathcal{D}((0,T);\mathcal{X})$ of smooth and with compact support functions with values in $\mathcal{X}$ is dense in $L^p(0,T;\mathcal{X})$ for any $1 \le p < \infty$, and weakly$*$ dense in $L^\infty(0,T;\mathcal{X})$.

Assume that Ω is an open set of $\mathbb{R}^d$. A distribution is a linear and continuous mapping $T : \mathcal{D}(\Omega) \to \mathcal{X}$, we note $T \in \mathcal{D}'(\Omega;\mathcal{X})$. For example, if $f \in L^1_{loc}(\Omega;\mathcal{X})$, we define

$$\langle T_f, \varphi\rangle := \int_\Omega f(x)\varphi(x)\,dx$$

and we observe that $T_f \in \mathcal{D}'(\Omega;\mathcal{X})$. We have $T_f = 0$ implies $f = 0$ a.e. For $T \in \mathcal{D}'(\Omega;\mathcal{X})$, we define $\partial T \in \mathcal{D}'(\Omega;\mathcal{X})$ by

$$\langle \partial T, \varphi\rangle := -\langle T, \partial\varphi\rangle, \quad \forall\, \varphi \in \mathcal{D}(\Omega).$$

For $T \in (0,\infty)$, we define

$$W^{1,p}((0,T);\mathcal{X}) := \{f \in L^p(0,T;\mathcal{X}),\ f' \in L^p(0,T;\mathcal{X})\}.$$

We have $W^{1,p}((0,T);\mathcal{X}) \subset C([0,T];\mathcal{X})$.

Appendix C. Further results: C_0-semigroup, evolution equation with source term and Duhamel formula

C.1. **C_0-semigroup.** We explain how we may associate a C_0-semigroup to the evolution equation (2.1), (2.2) as a mere consequence of the linearity of the equation and of the existence and uniqueness result.

Definition C.1. *Consider X a Banach space, and denote by $\mathscr{B}(X)$ the set of linear and bounded operators on X. We say that $S = (S_t)_{t\ge 0}$ is a strongly continuous semigroup of linear operators on X, or just a C_0-semigroup on X, we also write $S(t) = S_t$, if*

(i) $\forall t \ge 0,\quad S_t \in \mathscr{B}(X)$ (one parameter family of operators);

(ii) $\forall f \in X,\quad t \mapsto S_t\, f \in C([0,\infty), X)$ (continuous trajectories);

(iii) $S_0 = I;\quad \forall\, s,\, t \ge 0\quad S_{t+s} = S_t\, S_s$ (semigroup property).

Proposition C.2. *The operator Λ generates a semigroup on H defined in the following way. For any $g_0 \in H$, we set $S_t g := g(t)$ where $g(t)$ is the unique variational solution associated to g_0 and given by Theorem 3.2. We also denote $S_\Lambda(t) = e^{\Lambda t} = S_t$ for any $t \ge 0$.*

• *S satisfies (i).* By linearity of the equation and uniqueness of the solution, we clearly have

$$S_t(g_0 + \lambda f_0) = g(t) + \lambda f(t) = S_t g_0 + \lambda S_t f_0$$

for any $g_0, f_0 \in H$, $\lambda \in \mathbb{R}$ and $t \ge 0$. Thanks to estimate (2.3) we also have $|S_t g_0| \le e^{bt}\,|g_0|$ for any $g_0 \in H$ and $t \ge 0$. As a consequence, $S_t \in \mathscr{B}(H)$ for any $t \ge 0$.

• *S satisfies (ii).* Thanks to lemma 3.3 we have $t \mapsto S_t g_0 \in C(\mathbb{R}_+;H)$ for any $g_0 \in H$.

• *S satisfies (iii).* For $g_0 \in H$ and $t_1, t_2 \ge 0$ denote $g(t) = S_t g_0$ and $\tilde g(t) := g(t+t_1)$. Making the difference of the two equations (3.1) written for $t = t_1$ and $t = t_1 + t_2$, we see that $\tilde g$ satisfies

$$\begin{aligned}
(\tilde g(t_2), \tilde\varphi(t_2)) &= (g(t_1+t_2), \varphi(t_1+t_2)) \\
&= (g(t_1), \varphi(t_1)) + \int_{t_1}^{t_1+t_2} \big\{\langle \Lambda g(s), \varphi(s)\rangle + \langle \varphi'(s), g(s)\rangle\big\}\,ds \\
&= (\tilde g(0), \tilde\varphi(0)) + \int_0^{t_2} \big\{\langle \Lambda \tilde g(s), \tilde\varphi(s)\rangle + \langle \tilde\varphi'(s), \tilde g(s)\rangle\big\}\,ds,
\end{aligned}$$

for any $\varphi \in X_{t_1+t_2}$ with the notation $\tilde\varphi(t) := \varphi(t+t_1) \in X_{t_2}$. Since the equation on the functions $\tilde g$ and $\tilde\varphi$ is nothing but the variational formulation associated to the equation (2.1), (2.2) with initial datum $\tilde g(0)$, we obtain

$$S_{t_1+t_2} g_0 = g(t_1+t_2) = \tilde g(t_2) = S_{t_2}\tilde g(0) = S_{t_2} g(t_1) = S_{t_2} S_{t_1} g_0.$$

Exercise C.3. *We denote by S_t the semigroup in H generated by a coercive+dissipative operator $\Lambda : V \subset H \to V'$.*

1) Prove that for any $g_0 \in H$ and $\varphi \in V$ the function $t \mapsto (S_t g_0, \varphi)$ belongs to $H^1(0,T)$ and

$$\frac{d}{dt}(S_t g_0, \varphi) = (\Lambda S_t g_0, \varphi) \quad in \quad H^{-1}(0,T).$$

2) Prove that for any $G \in C([0,T];H)$ and $\varphi \in V$ there holds

$$\frac{d}{dt}\int_0^t (S_{t-s}G(s),\varphi)\,ds = (G(t),\varphi) + \int_0^t (\Lambda S_{t-s}G(s),\varphi)\,ds \quad in \quad H^{-1}(0,T).$$

3) Establish the Duhamel formula, namely that for $g_0 \in H$ and $G \in C([0,T];H)$, the function

$$g(t) := S_t g_0 + \int_0^t S_{t-s}G(s)\,ds$$

is a weak (make precise the sense) solution to the evolution equation with source term

$$\frac{dg}{dt} = \Lambda g + G \ \ on\ \ [0,\infty), \quad g(0) = g_0.$$

C.2. **Evolution equation with source term and Duhamel formula.** In that last section, we come back on Exercises A.3 and C.3.

• For $g_0 \in H$ and $G \in L^2(0,T;V')$ a function $g \in X_T$ is a variational solution to the evolution equation with source term

$$\frac{dg}{dt} = \Lambda g + G \ \text{ on } \ [0,T], \quad g(0) = g_0, \tag{C.1}$$

if that equation holds in V', namely if for any $\varphi \in V$ there holds

$$\frac{d}{dt}(g(t),\varphi) = \langle \Lambda g(t),\varphi\rangle + \langle G(t),\varphi\rangle \quad \text{in the sense of} \quad \mathcal{D}'(0,T;\mathbb{R}).$$

That is equivalent to

$$\langle \frac{d}{dt}g(t),\varphi\rangle = \langle \Lambda g(t),\varphi\rangle + \langle G(t),\varphi\rangle \quad a.e.\ t \in (0,T),$$

or more explicitly

$$\int_0^T (g(t),\varphi)\,\chi'\,dt - (g_0,\varphi)\,\chi(0) = \int_0^T \{\langle \Lambda g(t),\varphi\rangle + \langle G(t),\varphi\rangle\}\chi(t)\,dt,$$

for any $\chi \in C^1_c([0,T))$ and $\varphi \in V$. One can then deduce from the last formulation and by density of the separate variables functions $\mathcal{D}(0,T)\otimes V$ into X_T, or just by taking the next formulation as a definition of a variational solution, that for any $\varphi \in X_T$

$$[\langle g,\varphi\rangle]_0^T - \int_0^T \langle \frac{d}{dt}\varphi, g\rangle dt = \int_0^T \langle \Lambda g + G,\varphi\rangle dt. \tag{C.2}$$

• When $g_0 \in H$ and $G \in C([0,T];H)$ one can define thanks to Proposition C.2 the following function

$$g(t) := e^{\Lambda\, t}\, g_0 + \int_0^t e^{\Lambda\,(t-s)}\, G(s)\,ds. \tag{C.3}$$

We see that $g \in C([0,T];H)$, and using the estimate (2.3)

$$\int_0^T \|e^{\Lambda\, t} f\|_V^2\,dt \le \frac{e^{2bT}}{2\alpha}\,|f|_H^2,$$

we easily find

$$\int_0^T \|g(t)\|_V^2\,dt \le C_1(T)\,|g_0|_H^2 + C_2(T)\int_0^T |G(s)|_H^2\,ds.$$

Finally, since

$$t \mapsto e^{\Lambda\, t} f \in H^1(0,T;V'), \quad \|\partial_t(e^{\Lambda\, t} f)\|_{L^2(V')} \le C_3(T)\,|f|_H^2,$$

we deduce that $g \in H^1(0,T;V')$ with explicit estimates. We may then compute in $L^2(0,T;V')$

$$\partial_t g = \Lambda e^{\Lambda t} g_0 + G(t) + \int_0^t \Lambda e^{\Lambda\,(t-s)}\,G(s)\,ds = \Lambda g + G(t),$$

(see also Exercise C.3) and we obtain that $g(t)$ is a variational solution to the evolution equation with source term (C.1).

• When $g_0 \in H$ and $G \in L^2([0,T];V')$ the sense of the Duhamel formula is less clear. One can however prove the existence of a variational solution by just repeating the proof used to tackle the sourceless evolution equation (1.1). More precisely, we consider the following discrete scheme: we build (g_k) iteratively by setting

$$\frac{g_{k+1} - g_k}{\varepsilon} = \Lambda g_{k+1} + G_k, \quad G_k := \int_{t_k}^{t_{k+1}} G(s)\, ds.$$

We compute

$$\begin{aligned} |g_{k+1}|^2(1-\varepsilon b) + \varepsilon\,\alpha\,\|g_{k+1}\|_V^2 &\le |g_k|\,|g_{k+1}| + \varepsilon\,\|g_{k+1}\|_V\,\|G_k\|_{V'} \\ &\le \frac{1}{2}\,|g_k|^2 + \frac{1}{2}\,|g_{k+1}|^2 + \varepsilon\,\frac{\alpha}{2}\,\|g_{k+1}\|_V^2 + \frac{\varepsilon}{2\,\alpha}\,\|G_k\|_{V'}^2, \end{aligned}$$

and then

$$|g_{k+1}|^2(1-2\varepsilon b) + \varepsilon\,\alpha\,\|g_{k+1}\|_V^2 \le |g_k|^2 + \frac{\varepsilon}{\alpha}\,\|G_k\|_{V'}^2.$$

We get an estimate on $|g_{k+1}|^2$ which is uniform on k when $k\,\varepsilon \le T$, for $T > 0$ fixed, by using a discrete version of the Gronwall lemma. We conclude as in the proof of Theorem 3.2.

• We may argue in a different way. When $g_0 \in H$ and $G \in C([0,T];H)$ the Duhamel formula (C.3) gives a variational solution to the evolution equation with source term in the sense (C.2). Making the choice $\varphi = g$, we get

$$\begin{aligned} \frac{1}{2}[|g|^2]_0^T &= \int_0^T (\langle \Lambda g, g\rangle + \langle G, g\rangle)\, dt \\ &\le \int_0^T \{-\alpha\,\|g\|_V^2 + b\,|g|^2 + \|G\|_{V'}\,\|g\|_V\}\, dt \\ &\le -\frac{\alpha}{2}\int_0^T \|g\|_V^2\, dt + \frac{b^2}{2\alpha}\int_0^T \|G\|_{V'}^2\, dt, \end{aligned}$$

and thanks to the Gronwall lemma, we obtain

$$|g(T)|^2 + \alpha\int_0^T \|g\|_V^2\, dt \le e^{bT}|g_0|_H^2 + C_T\int_0^T \|G\|_{V'}^2\, dt.$$

We conclude to the existence by smoothing the source term G (what it is always possible in the explicit examples $H = L^2$, $V = H^1$) and by passing to the limit in the variational formulation.

Appendix D. References

Theorems 3.2 and A.5 are stated in [1, Théorème X.9] where the quoted reference for a proof is [2].

- [1] Brezis, H. *Analyse fonctionnelle.* Collection Mathématiques Appliquées pour la Maîtrise. [Collection of Applied Mathematics for the Master's Degree]. Masson, Paris, 1983. Théorie et applications. [Theory and applications].
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- [3] Yosida, K. *Functional analysis.* Classics in Mathematics, Springer-Verlag, Berlin, 1995.