CHAPTER 4 - EVOLUTION EQUATION AND SEMIGROUP - COMPLEMENTS -

Contents

1. Transport equation

Proposition 1.1. *Under the standard assumptions on the vector field $a = a(x)$ and the usual definition on the associated flow Φ_t, for any $f_0 \in L^p(\mathbb{R}^d)$, the function*

$$\bar f(t,x) := f_0(\Phi_{-t}(x)) \tag{1.1}$$

is the unique weak solution in $C([0,T);L^p(\mathbb{R}^d))$ when $p \in [1,\infty)$ (resp. in $C([0,T);L^1_{\rm loc}(\mathbb{R}^d)) \cap L^\infty(0,T;L^\infty(\mathbb{R}^d))$ when $p=\infty$) to the transport equation

$$\partial_t f + a\cdot\nabla_x f = 0, \quad f(0) = f_0.$$

Proof of Proposition 1.1. From the above definition and the group property of the flow, for a.e. $y \in \mathbb{R}^d$ and for any $t \in (0,\infty)$, we observe that

$$\bar f(t+s,\Phi_s(y)) = \bar f(t,y), \quad \forall\, s \ge 0. \tag{1.2}$$

Recalling Liouville theorem, we know that the Jacobian function $J := {\rm det}D\Phi_t(y)$ satisfies the ODE

$$\frac{d}{dt}J = ({\rm div}a(t,\Phi_t(y)))J, \quad J(0,y) = 1,$$

so that

$$ {\rm det}D\Phi_t(y) = e^{\int_0^t ({\rm div}a(s,\Phi_s(y)))ds}. \tag{1.3}$$

Let us then fix $\varphi \in \mathcal{D}((0,T)\times\mathbb{R}^d)$. We compute

$$\begin{aligned}
0 &= \frac{d}{ds}\int_0^T\int_{\mathbb{R}^d} \bar f(t,y)\varphi(t,y)\,dydt \\
&= \frac{d}{ds}\int_0^T\int_{\mathbb{R}^d} \bar f(t+s,\Phi_s(y))\varphi(t,y)\,dydt \\
&= \frac{d}{ds}\int_0^T\int_{\mathbb{R}^d} \bar f(t,x)\varphi(t-s,\Phi_{-s}(x))e^{-\int_0^s ({\rm div}a)(\Phi_\tau(x))d\tau}\,dxdt \\
&= \int_0^T\int_{\mathbb{R}^d} \bar f(t,x)\frac{d}{ds}[\varphi(t-s,\Phi_{-s}(x))e^{-\int_0^s {\rm div}a(x_\tau(x))ds}]\,dxdt \\
&= \int_0^T\int_{\mathbb{R}^d} \bar g(t,x)[-\partial_t\varphi - a\cdot\nabla\varphi - {\rm div}a\varphi](t-s,\Phi_{-s}(x))\,dxdt,
\end{aligned}$$

where we have used the relation (1.2) in the second line and the change of variables $x = \Phi_s(y)$ together with the Liouville theorem (1.3) in the third line. Taking $s = 0$, we get

$$0 = \int_0^T\int_{\mathbb{R}^d} \bar g(t,x)[-\partial_t\varphi - a\cdot\nabla\varphi - ({\rm div}a)\varphi](t,x)\,dxdt,$$

which exactly means that $\bar g$ is a solution to equation (1.1) in the distributional sense. □

Proposition 1.2. *Under the same assumptions as in the above Proposition and defining the semigroup S on $L^p(\mathbb{R}^d)$ by*

$$(S_t f_0)(x) := f_0(\Phi_{-t}(x)), \quad \forall\, f_0 \in L^p(\mathbb{R}^d),$$

the generator $\mathcal{L}$ of S has domain

$$D(\mathcal{L}) = W_p := \{f \in L^p(\mathbb{R}^d);\ a \cdot \nabla f \in L^p(\mathbb{R}^d)\}$$

and is defined by

$$\mathcal{L} f = -a \cdot \nabla f, \quad \forall\, f \in D(\mathcal{L}).$$

Proof of Proposition 1.2. For $f_0 \in L^p(\mathbb{R}^d)$ and $\varphi \in C^1_c(\mathbb{R}^d)$, we compute

$$\begin{aligned}\langle \frac{S_t f_0 - f_0}{t}, \varphi \rangle &= \frac{1}{t} \int_{\mathbb{R}^d} [f_0(\Phi_{-t}(x)) - f_0(x)] \varphi(x)\, dx \\ &= \frac{1}{t} \int_{\mathbb{R}^d} f_0(x) [\varphi(\Phi_t(x)) e^{\int_0^t (\mathrm{div} a)(\Phi_s(x))\, ds} - \varphi(x)]\, dx.\end{aligned}$$

We may pass to the limit in the RHS term, and we get

$$\lim_{t\to 0} \langle \frac{S_t f_0 - f_0}{t}, \varphi \rangle = \int_{\mathbb{R}^d} f_0(x) [a \cdot \nabla \varphi + (\mathrm{div} a) \varphi]\, dx.$$

If now, $f_0 \in D(\mathcal{L})$, we also have for any $\varphi \in C^1_c(\mathbb{R}^d)$, We may pass to the limit in the RHS term, and we get

$$\lim_{t\to 0} \langle \frac{S_t f_0 - f_0}{t}, \varphi \rangle = \int_{\mathbb{R}^d} (\mathcal{L} f_0) \varphi\, dx,$$

and thus

$$\int_{\mathbb{R}^d} f_0(x) [a \cdot \nabla \varphi + (\mathrm{div} a) \varphi]\, dx = \int_{\mathbb{R}^d} (\mathcal{L} f_0) \varphi\, dx, \quad \forall\, \varphi \in C^1_c(\mathbb{R}^d).$$

In particular,

$$\Big| \int_{\mathbb{R}^d} f_0(x) [\mathrm{div}(a\varphi)]\, dx \Big| \le \|\mathcal{L} f_0\|_{L^p} \, \|\varphi\|_{L^{p'}} \quad \forall\, \varphi \in C^1_c(\mathbb{R}^d),$$

which exactly means that $a \cdot \nabla f_0 \in L^p$ and thus $f_0 \in W_p$. We have established that $D(\mathcal{L}) \subset W_p$ and $\mathcal{L} f_0 = -a \cdot \nabla f_0$.

On the other way round, let us consider $f_0 \in C^1_c(\mathbb{R}^d)$. The Taylor expansion formula writes

$$f_0(\Phi_{-t}(x)) = f_0(x) + \int_0^t (a \cdot \nabla f_0)(\Phi_{-s}(x))\, ds.$$

Using a duality argument, this formula extends to a W_p framework: for any fixed $f_0 \in W^p$, there holds

$$f_0(\Phi_t(x)) = f_0(x) + \int_0^t (a \cdot \nabla f_0)(\Phi_{-s}(x))\, ds, \quad \text{for a.e. } x \in \mathbb{R}^d,\ \forall\, t > 0.$$

We deduce

$$\frac{S_t f_0 - f_0}{t} + a \cdot \nabla f_0 = \frac{1}{t} \int_0^t [(a \cdot \nabla f_0)(\Phi_{-s}(x)) + (a \cdot \nabla f_0)(x)]\, ds,$$

and we have to prove that the last term tends to 0 in L^p norm as $t \to 0$. □