CHAPTER 5 - MORE ABOUT THE HEAT EQUATION

I write in blue color probably the most important part of the material.

Contents

In this chapter we present some qualitative properties of the heat equation and more particularly we present several results on the self-similar behavior of the solutions in large time. These results are deduced from several functional inequalities, among them the Nash inequality, the Poincaré inequality and the Log-Sobolev inequality.
Let us emphasize that the approach lies on an interplay between evolution PDEs and functional inequalities and, although we only deal with (simple) linear situations, these methods are robust enough to be generalized to (some) nonlinear situations.

1. The heat equation

1.1. A first glance over the heat equation. The section is devoted to the heat equation

$$\frac{\partial f}{\partial t} = \frac{1}{2}\Delta f \quad \text{in } (0,\infty)\times\mathbb{R}^d, \qquad f(0,\cdot) = f_0 \quad \text{in } \mathbb{R}^d. \tag{1.1}$$

We start with formally observing several qualitative properties of the solutions to the heat equation. On the one hand, we have

$$\frac{d}{dt}\int_{\mathbb{R}^d} f(t,x)\,dx = \frac{1}{2}\int_{\mathbb{R}^d}\Delta f\,dx = 0,$$

so that the mass is conserved (by the flow of the heat equation)

$$\langle f(t,\cdot)\rangle := \int_{\mathbb{R}^d} f(t,x)\,dx = \int_{\mathbb{R}^d} f_0\,dx = \langle f_0\rangle, \quad \forall\, t\ge 0.$$

The dispersion/diffusion effect of the heat equation can be revealed through the decay of L^p norms, for instance

$$\frac{d}{dt}\int_{\mathbb{R}^d} f^2\,dx = \int_{\mathbb{R}^d} f\Delta f\,dx = -\int |\nabla f|^2 \le 0, \quad \forall\, t\ge 0. \tag{1.2}$$

The same computation gives

$$\frac{d}{dt}\int_{\mathbb{R}^d} f_+^2\,dx = \int_{\mathbb{R}^d} f_+\Delta f\,dx = -\int |\nabla f_+|^2 \le 0,$$

so that

$$\int_{\mathbb{R}^d} (f_+(t,\cdot))^2\,dx = 0, \quad \forall\, t\ge 0, \quad \text{if} \quad \int_{\mathbb{R}^d} (f_{0+})^2\,dx = 0.$$

Equivalently, that means

$$f(t,\cdot)\ge 0, \ \ \forall\, t\ge 0, \quad \text{if} \quad f_0\ge 0,$$

and the equation preservers the positivity. More generally, for any convex function β, we similarly have

$$\frac{d}{dt}\int_{\mathbb{R}^d} \beta(f)\,dx = \frac{1}{2}\int_{\mathbb{R}^d} \beta'(f)\Delta f\,dx = -\frac{1}{2}\int_{\mathbb{R}^d} \beta''(f)|\nabla f|^2\,dx \le 0, \ \ \forall\, t\ge 0,$$

and we thus obtain a large family of Lyapunov functional. In particular, the L^p-norm, for any $p\in[1,\infty]$, falls in this family, and thus

$$\|f(t,\cdot)\|_{L^p} \le \|f_0\|_{L^p}, \quad \forall\, t\ge 0. \tag{1.3}$$

Finally, for a positive solution, the dispersion/diffusion effect of the heat equation can also be brought out through the increasing of moments: we have indeed

$$\frac{d}{dt}\int_{\mathbb{R}^d} f(t,x)\langle x\rangle^k\,dx = \frac{1}{2}\int_{\mathbb{R}^d} f\Delta\langle x\rangle^k\,dx \ge 0, \quad \forall\, t\ge 0,$$

for $k+d-2\ge 0$ and $\langle x\rangle^2 := 1+|x|^2$ (since $\Delta\langle x\rangle^k = k\langle x\rangle^{k-4}[(k+d-2)|x|^2+d]\ge 0$).

In the very particular case of the $\mathbb{R}^d$ framework as considered here, solutions to the heat equation are given through the representation formula

$$f(t,.) = \gamma_t * f_0, \quad \gamma_t(x) := \frac{1}{(2\pi t)^{d/2}}\exp\Big(-\frac{|x|^2}{2t}\Big). \tag{1.4}$$

An alternative way for building solutions is the J.-L. Lions theory presented in a previous chapter. That last approach is a bit more involved but much more robust since it generalizes to many parabolic equations.

1.2. **Nash inequality and heat equation.** Thanks to the representation formula (1.4) and the Hölder inequality, one can classically prove that $f(t,.)\to 0$ as $t\to\infty$. More precisely, for any $p\in(1,\infty]$ and a constant $C_{p,d}$, the following rate of decay holds:

$$\|f(t,.)\|_{L^p} \le \frac{C_{p,d}}{t^{\frac{d}{2}(1-\frac{1}{p})}}\,\|f_0\|_{L^1} \qquad \forall\, t>0. \tag{1.5}$$

We aim to give a second proof of (1.5) which is not based on the above representation formula, which is clearly longer and more complicated, but which is also more robust in the sense that it applies to more general equations, even sometimes nonlinear. We start with the case $p=2$ which is the key argument and which is based on the so-called Nash inequality together with a nonlinear ODE estimate.

Nash inequality. There exists a constant C_d such that for any $f\in L^1(\mathbb{R}^d)\cap H^1(\mathbb{R}^d)$, there holds

$$\|f\|_{L^2}^{1+2/d} \le C_d\,\|f\|_{L^1}^{2/d}\,\|\nabla f\|_{L^2}. \tag{1.6}$$

Proof of Nash inequality. We write for any $R > 0$

$$\begin{aligned} \|f\|^2_{L^2} &= \|\hat f\|^2_{L^2} = \int_{|\xi|\le R} |\hat f|^2 + \int_{|\xi|\ge R} |\hat f|^2 \\ &\le c_d\, R^d\, \|\hat f\|^2_{L^\infty} + \frac{1}{R^2} \int_{|\xi|\ge R} |\xi|^2\, |\hat f|^2 \\ &\le c_d\, R^d\, \|f\|^2_{L^1} + \frac{1}{R^2}\, \|\nabla f\|^2_{L^2}, \end{aligned}$$

and we take the optimal choice for R by setting $R := (\|\nabla f\|^2_{L^2}/c_d\|f\|^2_{L^1})^{\frac{1}{d+2}}$ so that the two terms at the RHS pf the last line are equal. □

Alternative proofs of the Nash inequality (1.6) are presented in **Exercise 6.1** and **Exercise 6.2**.

We consider now a solution f to the heat equation (1.1) and we recall that

$$\frac{d}{dt} \int_{\mathbb{R}^d} f(t,x)^2\, dx = - \int_{\mathbb{R}^d} |\nabla f|^2\, dx, \quad \forall\, t \ge 0,$$

and

$$\|f(t,\cdot)\|_{L^1} \le \|f_0\|_{L^1}, \quad \forall\, t \ge 0,$$

from (1.2) and (1.3) with $p = 1$. Putting together that two last equations and the Nash inequality, we obtain the following ordinary differential inequality

$$\frac{d}{dt} \int_{\mathbb{R}^d} f(t,x)^2\, dx \le -K \left(\int_{\mathbb{R}^d} f(t,x)^2\, dx \right)^{\frac{d+2}{d}}, \quad K = C_d\, \|f_0\|^{-4/d}_{L^1}.$$

We last observe that for any solution u of the ordinary differential inequality

$$u' \le -K\, u^{1+\alpha}, \quad \alpha = 2/d > 0,$$

some elementary computations (as already performed in the first chapter about Gronwall lemma) lead to the inequality

$$u^{-\alpha}(t) \ge \alpha\, K\, t + u_0^\alpha \ge \alpha\, K\, t,$$

from which we conclude that

$$\int_{\mathbb{R}^d} f^2(t,x)\, dx \le C\, \frac{\left(\|f_0\|^{4/d}_{L^1}\right)^{d/2}}{t^{d/2}} = C\, \frac{\|f_0\|^2_{L^1}}{t^{d/2}}. \tag{1.7}$$

That is nothing but the announced estimate (1.5) for $p = 2$.

In order to prove the estimate for the full range of exponent $p \in (1, \infty]$ we use a duality and an interpolation arguments as follow. We introduce the heat semigroup $S(f) f_0 = f(t)$ associated to the heat equation as well as the dual semigroup $S^*(t)$. We clearly have $S^* = S$ because the Laplacian operator is symmetric in $L^2(\mathbb{R}^d)$. As a consequence, thanks to (1.7) and for any $f_0 \in L^2(\mathbb{R}^d)$, there holds

$$\begin{aligned} \|S(t) f_0\|_{L^\infty} &= \sup_{\phi \in B_{L^1}} \langle S(t) f_0, \phi \rangle = \sup_{\phi \in B_{L^1}} \langle f_0, S(t) \phi \rangle \\ &\le \sup_{\phi \in B_{L^1}} \|f_0\|_{L^2}\, \|S(t)\phi\|_{L^2} \le \|f_0\|_{L^2}\, \frac{C}{t^{d/4}}, \end{aligned}$$

which exactly means that $S(t) : L^2 \to L^\infty$ for positive times with norm bounded by $C\, t^{-d/4}$. We deduce

$$\|S(t)\|_{L^1 \to L^\infty} \le \|S(t/2)\|_{L^2 \to L^\infty}\, \|S(t/2)\|_{L^1 \to L^2} \le \frac{C}{t^{d/2}},$$

which establishes (1.5) for $p = \infty$. Finally, for any $p \in (1, \infty)$ and using the interpolation inequality

$$\|S(t) f_0\|_{L^p} \le \|S(t) f_0\|^\theta_{L^1}\, \|S(t) f_0\|^{1-\theta}_{L^\infty} \le \|S(t)\|^{1-\theta}_{L^1 \to L^\infty}\, \|f_0\|_{L^1} \quad \forall\, t > 0,$$

with $\theta = 1/p$, we have established (1.5) in the general case.

It is worth emphasizing that by differentiating the heat equation, we can easily establish some estimates on its smoothing effect. For example, for $f_0 \in H^1(\mathbb{R}^d)$, the associated solution to the heat equation satisfies

$$\partial_t f = \frac{1}{2} \Delta f \quad \text{and} \quad \partial_t \nabla f = \frac{1}{2} \Delta \nabla f$$

from what we deduce

$$\frac{d}{dt}\|f\|_{L^2}^2 = -\|\nabla f\|_{L^2}^2 \quad \text{and} \quad \frac{d}{dt}\|\nabla f\|_{L^2}^2 = -\|D^2 f\|_{L^2}^2$$

and then

$$\frac{d}{dt}\{\|f\|_{L^2}^2 + t\|\nabla f\|_{L^2}^2\} = -t\|D^2 f\|_{L^2}^2 \le 0, \quad \forall\, t > 0.$$

Integrating in time this differential inequality, we readily obtain that the solution to the heat equation satisfies

$$\|\nabla f(t)\|_{L^2} \le \frac{1}{t^{1/2}}\,\|f_0\|_{L^2}, \quad \forall\, t > 0.$$

1.3. **Self-similar solutions and the Fokker-Planck equation.** It is in fact possible to describe in a more accurate way that the mere estimate (1.5) how the heat equation solution $f(t,.)$ converges to 0 as time goes on.
In order to do so, the first step consists in looking for particular solutions to the heat equation that we will discover by identifying some good scaling. We thus look for a self-similar solution to (1.5), namely for a solution F with particular form

$$F(t,x) = t^\alpha\, G(t^\beta x),$$

for some $\alpha, \beta \in \mathbb{R}$ and a *"self-similar profile"* G. As F must be mass conserving, we have

$$\int_{\mathbb{R}^d} F(t,x)\,dx = \int_{\mathbb{R}^d} F(0,x)\,dx = t^\alpha \int_{\mathbb{R}^d} G(t^\beta x)\,dx = t^{\alpha-\beta d}\int_{\mathbb{R}^d} G(y)\,dy,$$

and we get from that the first equation $\alpha = \beta\, d$. On the other hand, we easily compute

$$\partial_t F = \alpha\, t^{\alpha-1} G(t^\beta\, x) + \beta\, t^{\alpha-1}\,(t^\beta\, x)\cdot(\nabla G)(t^\beta\, x), \quad \Delta F = t^\alpha\, t^{2\beta}\,(\Delta G)(t^\beta\, x).$$

In order that (1.1) is satisfied, we need thus to take $2\,\beta + 1 = 0$. We conclude with

$$F(t,x) = t^{-d/2}\, G(t^{-1/2}\, x), \qquad \frac{1}{2}\Delta G + \frac{1}{2}\mathrm{div}(x\, G) = 0. \tag{1.8}$$

Under the mild regularity assumption $G \in W^{1,1}(\mathbb{R}^d) \cap L^1_1(\mathbb{R}^d)$ on a solution to the second equation (profile equation) in (1.8), this one satisfies $\nabla G + x\, G = 0$ (see **Exercise 6.3**). Under the additional assumption $G \in \mathbf{P}(\mathbb{R}^d)$, we observe (and that is not a surprise!) that the profile G is unique and given by

$$G(x) := c_0\, e^{-|x|^2/2}, \quad c_0^{-1} = (2\,\pi)^{d/2} \quad \text{(normalized Gaussian function)}.$$

To sum up, we have proved that F is our favorite solution to the heat equation: that is the fundamental solution to the heat equation.

Changing of point view, we may now consider G as a stationary solution to the harmonic Fokker-Planck equation (sometimes also called the Ornstein-Uhlenbeck equation)

$$\frac{\partial}{\partial t} g = \frac{1}{2}\,\mathcal{L}\, g = \frac{1}{2}\,\nabla\cdot(\nabla g + g\, x) \qquad \text{in } (0,\infty)\times\mathbb{R}^d. \tag{1.9}$$

The link between the heat equation (1.1) and the Fokker-Planck equation (1.9) is as follows. If g is a solution to the Fokker-Planck equation (1.9), some elementary computations permit to show that

$$f(t,x) = (1+t)^{-d/2}\, g(\log(1+t), (1+t)^{-1/2}\, x)$$

is a solution to the heat equation (1.1), with $f(0,x) = g(0,x)$. Reciprocally, if f is a solution to the heat equation (1.1) then

$$g(t,x) := e^{d\,t/2}\, f(e^t - 1, e^{t/2}\, x) \tag{1.10}$$

solves the Fokker-Planck equation (1.9). The last expression also gives the existence of a solution *in the sense of distributions* to the Fokker-Planck equation (1.9) for any initial datum $f_0 = \varphi \in L^1(\mathbb{R}^d)$ as soon as we know the existence of a solution to the heat equation for the same initial datum (what we get thanks to the usual representation formula for instance).

2. Fokker-Planck equation and Poincaré inequality

2.1. Long time asymptotic behaviour of the solutions to the Fokker-Planck equation. From now on in this chapter, we consider the Fokker-Planck equation

$$\frac{\partial}{\partial t} f = \mathcal{L} f = \Delta f + \nabla \cdot (f \nabla V) \quad \text{in } (0,\infty) \times \mathbb{R}^d \tag{2.1}$$

$$f(0,x) = f_0(x) \quad \text{on } \mathbb{R}^d, \tag{2.2}$$

and we assume that the *"confinement potential"* V is the harmonic potential

$$V(x) := \frac{|x|^2}{2} + V_0, \quad V_0 := \frac{d}{2} \log 2\pi.$$

We start observing that

$$\frac{d}{dt} \int_{\mathbb{R}^d} f(t,x)\, dx = \int_{\mathbb{R}^d} \nabla_x \cdot (\nabla_x f + f \nabla_x V)\, dx = 0,$$

so that the mass (of the solution) is conserved. We also have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt} \int_{\mathbb{R}^d} (f_+)^2\, dx &= \int_{\mathbb{R}^d} f_+ (\Delta f + \operatorname{div}(x f))\, dx \\ &= -\int_{\mathbb{R}^d} |\nabla f_+|^2 - \int_{\mathbb{R}^d} f_+ \, x \cdot \nabla f_+\, dx \le \frac{d}{2} \int_{\mathbb{R}^d} (f_+)^2\, dx, \end{aligned}$$

and thanks to the Gronwall lemma, we conclude that the maximum principle holds. Moreover, the function $G = e^{-V} \in L^1(\mathbb{R}^d) \cap \mathbf{P}(\mathbb{R}^d)$ is nothing but the normalized Gaussian function, and since $\nabla G = -G \nabla V$, it is a stationary solution to the Fokker-Planck equation (2.1).

Theorem 2.1. *Let us fix $f_0 \in L^p(\mathbb{R}^d)$, $1 \le p < \infty$.*
(1) There exists a unique global solution $f \in C([0,\infty); L^p(\mathbb{R}^d))$ to the Fokker-Planck equation (2.1). This solution is mass conservative

$$\langle f(t,.)\rangle := \int_{\mathbb{R}^d} f(t,x)\, dx = \int_{\mathbb{R}^d} f_0(x)\, dx =: \langle f_0 \rangle, \quad \text{if } f_0 \in L^1(\mathbb{R}^d), \tag{2.3}$$

and the following maximum principle holds

$$f_0 \ge 0 \quad \Rightarrow \quad f(t,.) \ge 0 \quad \forall t \ge 0.$$

(2) Asymptotically in large time the solution converges to the unique stationary solution with same mass, namely

$$\|f(t,.) - \langle f_0 \rangle G\|_E \le e^{-\lambda_P t} \|f_0 - \langle f_0\rangle G\|_E \quad \text{as} \quad t \to \infty, \tag{2.4}$$

where $\|\cdot\|_E$ stands for the norm of the Hilbert space $E := L^2(G^{-1})$ defined by

$$\|f\|_E^2 := \int_{\mathbb{R}^d} f^2 G^{-1}\, dx$$

and λ_P is the best (larger) constant in the Poincaré inequality.

More generally, for any weight function $m : \mathbb{R}^d \to \mathbb{R}_+$, we denote by $L^p(m)$ the Lebesgue space associated to the mesure $m(x)dx$ and by L^p_m the Lebesgue space associated to the norm $\|f\|_{L^p_m} := \|f\, m\|_{L^p}$. We will also write $L^p_k := L^p_m$, when $m := \langle x \rangle^k$.
For the proof of point (1) we refer to Chapter 2 as well as the final remark of Section 1. It is worth emphasizing that gathering (1.4) and (1.10), we also have the representation formula

$$f(t,x) = e^{d\,t} (\gamma_{e^t - 1} * f_0)(e^t x)$$

for the solution to the Fokker-Planck equation (2.1)–(2.2). We are going to give the main lines of the proof of point 2. Because the equation is linear, we may assume in the sequel that $\langle f_0 \rangle = 0$.
Using that $G\, G^{-1} = 1$, we deduce that $\nabla V = -G^{-1} \nabla G = G \cdot \nabla(G^{-1})$. We can then write the Fokker-Planck equation in the equivalent form

$$\begin{aligned} \frac{\partial}{\partial t} f &= \operatorname{div}_x \big(\nabla_x f + G\, f \,\nabla_x G^{-1}\big) \\ &= \operatorname{div}_x \big(G\, \nabla_x (f\, G^{-1})\big). \end{aligned} \tag{2.5}$$

We then compute

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int f^2\,G^{-1} &= \int_{\mathbb{R}^d} (\partial_t f)\, f\, G^{-1}\,dx = \int_{\mathbb{R}^d} \mathrm{div}_x \left(G\,\nabla_x \left(\frac{f}{G}\right)\right) \frac{f}{G}\,dx \\ &= -\int_{\mathbb{R}^d} G\left|\nabla_x \frac{f}{G}\right|^2 dx. \end{aligned} \tag{2.6}$$

Using the Poincaré inequality established in the next Theorem 2.2 with the choice of function $h := f(t,.)/G$ and observing that $\langle f/G\rangle_G = 0$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int f^2\,G^{-1} \le -\lambda_P \int_{\mathbb{R}^d} G\left(\frac{f}{G}\right)^2 dx = -\lambda_P\int_{\mathbb{R}^d} f^2\,G^{-1}\,dx,$$

and we conclude using the Gronwall lemma.

Theorem 2.2 (Poincaré inequality)**.** *There exists a constant $\lambda_P > 0$ (which only depends on the dimension) such that for any $h \in \mathcal{D}(\mathbb{R}^d)$, there holds*

$$\int_{\mathbb{R}^d} |\nabla h|^2\,G\,dx \ge \lambda_P \int_{\mathbb{R}^d} |h - \langle h\rangle_G|^2\,G\,dx, \tag{2.7}$$

where we have defined

$$\langle h\rangle_\mu := \int_{\mathbb{R}^d} h(x)\,\mu(dx)$$

for any given (probability) measure $\mu \in \mathbf{P}(\mathbb{R}^d)$ and any function $h \in L^1(\mu)$.

We present below three slightly different proofs of this important result.

2.2. **A first proof of the Poincaré inequality.** We split the proof into three steps.

2.2.1. *Poincaré-Wirtinger inequality (in an open and bounded set Ω).*

Lemma 2.3. *Let us denote $\Omega = B_R$ the ball of $\mathbb{R}^d$ with center 0 and radius $R > 0$, and let us consider $\nu \in \mathbf{P}(\Omega)$ a probability measure such that (abusing notations) $\nu, 1/\nu \in L^\infty(\Omega)$. There exists a constant $\kappa \in (0,\infty)$, such that for any (smooth) function f, there holds*

$$\kappa\int_\Omega |f - \langle f\rangle_\nu|^2\,\nu \le \int_\Omega |\nabla f|^2\,\nu, \qquad \langle f\rangle_\nu := \int_\Omega f\,\nu, \tag{2.8}$$

and therefore

$$\int_\Omega f^2\,\nu \le \langle f\rangle_\nu^2 + \frac{1}{\kappa}\int_\Omega |\nabla f|^2\,\nu. \tag{2.9}$$

Proof of Lemma 2.3. We start with

$$f(x) - f(y) = \int_0^1 \nabla f(z_t)\cdot(x-y)\,dt, \quad z_t = t\,x + (1-t)\,y.$$

Multiplying that identity by $\nu(y)$ and integrating in the variable $y \in \Omega$ the resulting equation, we get

$$f(x) - \langle f\rangle_\nu = \int_\Omega\int_0^1 \nabla f(z_t)\cdot(x-y)\,dt\,\nu(y)\,dy.$$

Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \int_\Omega (f(x) - \langle f\rangle_\nu)^2\,\nu(x)\,dx &\le \int_\Omega\int_\Omega\int_0^1 |\nabla f(z_t)|^2\,|x-y|^2\,dt\,\nu(y)\,\nu(x)dydx \\ &\le C_1\int_\Omega\int_\Omega\int_0^{1/2} |\nabla f(z_t)|^2\,dtdy\,\nu(x)dx + C_1\int_\Omega\int_\Omega\int_{1/2}^1 |\nabla f(z_t)|^2\,dtdx\,\nu(y)dy, \end{aligned}$$

with $C_1 := \|\nu\|_{L^\infty} \operatorname{diam}(\Omega)^2$. Performing the the changes of variables $(x,y) \mapsto (z,y)$ and $(x,y) \mapsto (x,z)$ and using the fact that $z_t \in [x,y] \subset \Omega$, we deduce

$$\begin{aligned}
&\int_\Omega (f(x) - \langle f \rangle_\nu)^2 \, \nu(x) \, dx \\
&\le C_1 \int_\Omega \int_0^{1/2} \int_\Omega |\nabla f(z)|^2 \frac{dz}{(1-t)^d} \, dt \, \nu(x) dx + C_1 \int_\Omega \int_{1/2}^1 \int_\Omega |\nabla f(z)|^2 \frac{dz}{t^d} \, dt \, \nu(y) dy \\
&\le 2 C_1 \int_\Omega |\nabla f(z)|^2 \, dz.
\end{aligned}$$

We have thus established that the Poincaré-Wirtinger inequality (2.8) holds with the constant $\kappa^{-1} := 2\, C_1 \, \|1/\nu\|_{L^\infty}$. □

2.2.2. *Weighted L^2 estimate through L^2 estimate on the derivative.*

Proposition 2.4. *There holds*

$$\frac{1}{4} \int_{\mathbb{R}^d} h^2 \, |x|^2 \, G dx \quad \le \quad \int_{\mathbb{R}^d} |\nabla h|^2 \, G dx + \frac{d}{2} \int_{\mathbb{R}^d} h^2 \, G dx,$$

for any $h \in C^1_b(\mathbb{R}^d)$.

Proof of Proposition 2.4. We define $\Phi := -\log G = |x|^2/2 + \log(2\pi)^{d/2}$. For a given function h, we denote $g = h G^{1/2}$, and we expand

$$\begin{aligned}
\int_{\mathbb{R}^d} |\nabla h|^2 \, G \, dx &= \int_{\mathbb{R}^d} \left| \nabla g \, G^{-1/2} + g \, \nabla G^{-1/2} \right|^2 G \, dx \\
&= \int_{\mathbb{R}^d} \left\{ |\nabla g|^2 + g \nabla g \nabla \Phi + \frac{1}{4} g^2 |\nabla \Phi|^2 \right\} dx,
\end{aligned}$$

because $\nabla G^{-1/2} = \frac{1}{2} \nabla \Phi \, G^{-1/2}$. Performing one integration by part, we get

$$\int_{\mathbb{R}^d} |\nabla h|^2 \, G \, dx = \int_{\mathbb{R}^d} |\nabla g|^2 \, dx + \int_{\mathbb{R}^d} h^2 \left(\frac{1}{4} |\nabla \Phi|^2 - \frac{1}{2} \Delta \Phi \right) G dx.$$

We conclude by neglecting the first term and computing the second term at the RHS. □

2.2.3. *End of the first proof of the Poincaré inequality.* We split the L^2 norm into two pieces

$$\int_{\mathbb{R}^d} h^2 \, G \, dx = \int_{B_R} h^2 \, G \, dx + \int_{B_R^c} h^2 \, G \, dx,$$

for some constant $R > 0$ to be choosen later. One the one hand, we have

$$\begin{aligned}
\int_{B_R} h^2 \, G \, dx &\le C_R \int_{B_R} |\nabla h|^2 \, G \, dx + \Big(\int_{B_R^c} h \, G \, dx \Big)^2 \\
&\le C_R \int |\nabla h|^2 \, G \, dx + \Big(\int_{B_R^c} G \, dx \Big) \int h^2 \, G \, dx,
\end{aligned}$$

where in the first line, we have used the Poincaré-Wirtinger inequality (2.9) in B_R with

$$\nu := G(B_R)^{-1} \, G_{|B_R}, \quad G(B_R) := \int_{B_R} G \, dx,$$

and the fact that $\langle hG \rangle = 0$, and in the second line, we have used the Cauchy-Schwarz inequality. One the other hand, we have

$$\begin{aligned}
\int_{B_R^c} h^2 \, G \, dx &\le \frac{1}{R^2} \int_{\mathbb{R}^d} h^2 \, |x|^2 \, G \, dx \\
&\le \frac{4}{R^2} \int_{\mathbb{R}^d} |\nabla h|^2 \, G dx + \frac{2d}{R^2} \int_{\mathbb{R}^d} h^2 \, G dx,
\end{aligned}$$

by using Proposition 2.4. All together, we get

$$\int_{\mathbb{R}^d} h^2 \, G \, dx \le \big(C_R + \frac{4}{R^2} \big) \int_{\mathbb{R}^d} |\nabla h|^2 \, G dx + \Big(\frac{2d}{R^2} + \int_{B_R^c} G \, dx \Big) \int h^2 \, G \, dx,$$

and we choose $R > 0$ large enough in such a way that the constant in front of the last term at the RHS is smaller than 1. $\square$

2.3. An second proof of the Poincaré inequality.

2.3.1. *A Lyapunov condition.* There exists a function W such that $W \ge 1$ and there exist some constants $\theta > 0$, $b, R \ge 0$ such that

$$(L^*W)(x) := \Delta W(x) - \nabla V \cdot \nabla W(x) \le -\theta\, W(x) + b\, \mathbf{1}_{B_R}(x), \qquad \forall\, x \in \mathbb{R}^d, \tag{2.10}$$

where again $B_R = B(0,R)$ denotes the centered ball of radius R. The proof is elementary. We look for W as $W(x) := e^{\gamma \langle x \rangle}$. We then compute

$$\nabla W = \gamma \, \frac{x}{\langle x \rangle} \, e^{\gamma \langle x \rangle} \quad \text{and} \quad \Delta W = \left(\gamma^2 + \gamma \, \frac{d-1}{\langle x \rangle} \right) e^{\gamma \langle x \rangle},$$

and thus

$$\begin{aligned} L^*W = \Delta W - x \cdot \nabla W &= \gamma \, \frac{d-1}{\langle x \rangle} \, W + \left(\gamma^2 - \gamma \, \frac{|x|^2}{\langle x \rangle} \right) W \\ &\le -\theta\, W + b\, \mathbf{1}_{B_R}, \end{aligned}$$

with the choice $\theta = \gamma = 1$ and then R and b large enough. $\square$

2.3.2. *End of second the proof of the Poincaré inequality.* We write (2.10) as

$$1 \le - \frac{L^*W(x)}{\theta\, W(x)} + \frac{b}{\theta\, W(x)} \, \mathbf{1}_{B_R}(x), \quad \forall x \in \mathbb{R}^d.$$

For any $f \in C^2_b(\mathbb{R}^d)$, we deduce

$$\int_{\mathbb{R}^d} f^2 \, G \;\; \le \;\; - \int_{\mathbb{R}^d} f^2 \, \frac{L^*W(x)}{\theta\, W(x)} \, G + \frac{b}{\theta} \int_{B_R} f^2 \, \frac{1}{W} \, G \quad =: \; T_1 + T_2.$$

On the one hand, we have

$$\begin{aligned} \theta\, T_1 &= \int \nabla W \cdot \left\{ \nabla \left(\frac{f^2}{W} \right) G + \frac{f^2}{W} \, \nabla G \right\} + \int \frac{f^2}{W} \, \nabla V \cdot \nabla W \, G \\ &= \int \nabla W \cdot \nabla \left(\frac{f^2}{W} \right) G \\ &= \int 2 \, \frac{f}{W} \, \nabla W \cdot \nabla f \, G - \int \frac{f^2}{W^2} \, |\nabla W|^2 \, G \\ &= \int |\nabla f|^2 \, G - \int \left| \frac{f}{W} \, \nabla W - \nabla f \right|^2 G \\ &\le \int |\nabla f|^2 \, G. \end{aligned}$$

On the other hand, using the Poincaré-Wirtinger inequality in B_R and the notation

$$G(B_R) := \int_{B_R} G \, dx, \quad \nu_R := G(B_R)^{-1} \, G_{|B_R}, \quad \langle f \rangle_R = \int_{B_R} f \, \nu_R,$$

we have

$$\begin{aligned} \frac{\theta}{b} \, T_2 &= \int_{B_R} f^2 \, \frac{1}{W} \, G \le G(B_R) \int_{B_R} f^2 \, \nu_R \\ &\le G(B_R) \Big(\langle f \rangle_R^2 + C_R \int_{B_R} |\nabla f|^2 \, \nu_R \Big). \end{aligned}$$

Gathering the two above estimates, we have shown

$$\int_{\mathbb{R}^d} f^2 \, G \le C \Big(\langle f \rangle_R^2 + \int_{\mathbb{R}^d} |\nabla f|^2 \, G \Big). \tag{2.11}$$

Consider now $h \in C^2_b$. We know that for any $c \in \mathbb{R}$, there holds

$$\int_{\mathbb{R}^d} (h - \langle h \rangle_G)^2 G \le \phi(c) := \int_{\mathbb{R}^d} (h - c)^2 \, G, \tag{2.12}$$

with $\langle h\rangle_G$ defined in (2.8), because ϕ is a polynomial function of second degree which reaches is minimum value in $c_h := \langle h\rangle_G$. More precisely, by mere expantion, we have

$$\phi(c) = \int_{\mathbb{R}^d} (h - \langle h\rangle_G)^2 G\,dx + (c - \langle h\rangle_G)^2.$$

We last define $f := h - \langle h\rangle_R$, so that $\langle f\rangle_R = 0$, $\nabla f = \nabla h$. Using first (2.12) and next (2.11), we obtain

$$\begin{aligned}\int_{\mathbb{R}^d} (h - \langle h\rangle_G)^2\, G &\le \int_{\mathbb{R}^d} (h - \langle h\rangle_R)^2\, G = \int_{\mathbb{R}^d} f^2\, G \\ &\le C\Big(\langle f\rangle_R^2 + \int_{\mathbb{R}^d} |\nabla f|^2\, G\Big) = C\int_{\mathbb{R}^d} |\nabla h|^2\, G.\end{aligned}$$

That ends the proof of the Poincaré inequality (2.7). □

2.4. **A third proof of the Poincaré inequality.** From (2.5), introducing the unknown $h := f/G$, we have

$$\begin{aligned}\partial_t h &= G^{-1}\mathrm{div}(G\nabla h) \\ &= \Delta h - x\cdot\nabla h =: Lh.\end{aligned}$$

On the one hand, we have

$$h(Lh) = L(h^2/2) - |\nabla h|^2,$$

L is self-adjoint in $L^2(G)$ and $L^*1 = 0$. We then recover the identity (2.6), namely

$$\frac{1}{2}\frac{d}{dt}\int h^2 G\,dx = -\int |\nabla h|^2 G\,dx. \tag{2.13}$$

We fix $h_0 \in L^2(G)$ with $\langle h_0 G\rangle = 0$. We accept that $h_T \to 0$ in $L^2(G)$ as $T \to \infty$, what it has been already established during the proofs 1 and 2 or can be established without rate using softer argument (as it will be explained in the chapter about Lyapunov techniques). By time integration of (2.13), we thus have

$$\|h_0\|^2 = -\lim_{T\to\infty}\Big[\|h_t\|^2\Big]_0^T = \lim_{T\to\infty}\int_0^T 2\|\nabla h_t\|^2\,dt,$$

where here and below $\|\cdot\|$ denotes the $L^2(G)$ norm, and therefore

$$\|h_0\|^2 = \int_0^\infty 2\|\nabla h_t\|^2\,dt. \tag{2.14}$$

On the other hand, we compute

$$\begin{aligned}\nabla h\cdot\nabla Lh &= \nabla h\cdot\Delta\nabla h - \nabla h\cdot\nabla(x\cdot\nabla h) \\ &= \Delta(|\nabla h|^2/2) - |D^2h|^2 - |\nabla h|^2 - xDh : D^2h \\ &= L(|\nabla h|^2/2) - |D^2h|^2 - |\nabla h|^2.\end{aligned}$$

We deduce

$$\frac{1}{2}\frac{d}{dt}\int |\nabla h|^2 G\,dx = -\int |D^2h|^2 G\,dx - \int |\nabla h|^2 G\,dx \le -\int |\nabla h|^2 G\,dx.$$

Similarly, as above, we have

$$\|\nabla h_0\|^2 - \|\nabla h_T\|^2 = -\int_0^T \frac{d}{dt}\|\nabla h_t\|^2\,dt \ge \int_0^T \|\nabla h_t\|^2\,dt,$$

and therefore

$$\|\nabla h_0\|^2 \ge \int_0^\infty 2\|\nabla h_t\|^2\,dt. \tag{2.15}$$

Gathering (2.14) and (2.15), we conclude with the following Poincaré inequality with optimal constant (see **Exercise 6.5**).

Proposition 2.5 (Poincaré inequality with optimal constant)**.** *For any $h \in \mathcal{D}(\mathbb{R}^d)$ with $\langle hG\rangle = 0$,*

$$\|\nabla h\|_{L^2(G)} \ge \|h\|_{L^2(G)}.$$

We deduce from the above Poincaré inequality with optimal constant, the identity (2.13) and the Gronwall lemma, the following optimal decay estimate

$$\|h_t\|_{L^2(G)} \le e^{-t}\|h_0\|_{L^2(G)}, \quad \forall\, t \ge 0,$$

for any $h_0 \in L^2(G)$ such that $\langle h_0 G\rangle = 0$.

3. Fokker-Planck equation and Log Sobolev inequality.

The estimate (2.4) gives a satisfactory (optimal) answer to the convergence to the equilibrium issue for the Fokker-Planck equation (2.1). However, we may formulate two criticisms. The proof is "completely linear" (in the sense that it can not be generalized to a nonlinear equation) and the considered initial data are very confined/localized (in the sense that they belong to the strong weighted space E, and again that it is not always compatible with the well posedness theory for nonlinear equations).

We present now a series of results which apply to more general initial data but, above all, which can be adapted to nonlinear equations. On the way, we will establish several functional inequalities of their own interest, among them the famous Log-Sobolev (or logarithmic Sobolev) inequality.

3.1. **Fisher information.** We are still interested in the harmonic Fokker-Planck equation (2.1)-(2.2). We define

$$D := \left\{ f \in L^1(\mathbb{R}^d); \quad f \ge 0, \quad \int f = 1, \quad \int f\, x = 0, \quad \int f\, |x|^2 = d \right\}$$

and

$$D_{\le} := \left\{ f \in L^1(\mathbb{R}^d); \quad f \ge 0, \quad \int f = 1, \quad \int f\, x = 0, \quad \int f\, |x|^2 \le d \right\}.$$

We observe that D (and $D_{\le}$) are invariant set for the flow of Fokker-Planck equation (2.1). We also observe that G is the unique stationary solution which belongs to D. Indeed, the equations for the first moments are

$$\partial_t\langle f\rangle = 0, \quad \partial_t\langle f x\rangle = -\langle f x\rangle, \quad \partial_t\langle f|x|^2\rangle = 2d\langle f\rangle - 2\langle f|x|^2\rangle. \tag{3.1}$$

It is therefore quite natural to think that any solution to the Fokker-Planck equation (2.1)-(2.2) with initial datum $f_0 \in D$ converges to G. It is what we will establish in the next paragraphs.

We define the Fisher information (or Linnik functional) $I(f)$ by

$$I(f) = \int \frac{|\nabla f|^2}{f} = 4\int |\nabla\sqrt{f}|^2 = \int f |\nabla \log f|^2$$

and the relative Fisher information $I(f|G)$ by

$$I(f|G) = I(f) - I(G) = I(f) - d.$$

Lemma 3.1. *For any $f \in D_{\le}$, there holds*

$$I(f|G) \ge 0, \tag{3.2}$$

with equality if, and only if, $f = G$.

Proof of Lemma 3.1. We define $V := \{f \in D_{\le} \text{ and } \nabla\sqrt{f} \in L^2\}$. We start with the proof of (3.2). For any $f \in V$, we have

$$\begin{aligned} 0 \le J(f) &:= \int \left|2\nabla\sqrt{f} + x\,\sqrt{f}\right|^2 dx \\ &= \int \left(4\,|\nabla\sqrt{f}|^2 + 2\,x\cdot\nabla f + |x|^2 f\right) dx = I(f) + \langle f|x|^2\rangle - 2\,d \\ &\le I(f) - d = I(f) - I(G) = I(f|G). \end{aligned}$$

We consider now the case of equality. If $I(f|G) = 0$ then $J(f) = 0$ and $2\nabla\sqrt{f} + x\,\sqrt{f} = 0$ a.e.. By a bootstrap argument, using Sobolev inequality, we deduce that $\sqrt{f} \in C^0$. Consider $x_0 \in \mathbb{R}^d$ such that $f(x_0) > 0$ (which exists because $f \in V$) and then $\mathcal{O}$ the open and connected to x_0 component of the set $\{f > 0\}$. We deduce from the preceding identity that $\nabla(\log\sqrt{f} + |x|^2/4) = 0$ in $\mathcal{O}$ and then $f(x) = e^{C-|x|^2/2}$ on $\mathcal{O}$ for some constant $C \in \mathbb{R}$. By continuity of f, we deduce that

$\mathcal{O} = \mathbb{R}^d$, and then $C = -\log(2\,\pi)^{d/2}$ (because of the normalized condition imposed by the fact that $f \in V$). □

In some sense (see below) the relative Fisher information measure the distance to the steady state G. We also observe that

$$\frac{d}{dt} I(f) = I'(f) \cdot \mathcal{L} f, \tag{3.3}$$

with

$$I'(f) \cdot h = 2 \int \frac{\nabla f}{f} \nabla h - \int \frac{|\nabla f|^2}{f^2}\, h, \tag{3.4}$$

and we wish to establish that $I(f)$ decreases and converges to 0 with exponential decay.

Lemma 3.2. *For any smooth probability measure f, we have*

$$\frac{1}{2} I'(f) \cdot \Delta f = -\sum_{ij} \int \Big(\frac{1}{f^2}\, \partial_i f\, \partial_j f - \frac{1}{f} \partial_{ij} f \Big)^2 f, \tag{3.5}$$

$$\frac{1}{2} I'(f) \cdot (\nabla \cdot (f\, x)) = I(f), \tag{3.6}$$

$$\frac{1}{2} I'(f) \cdot \mathcal{L}\, f = -\sum_{ij} \int \Big(\frac{1}{f^2}\, \partial_i f\, \partial_j f - \frac{1}{f} \partial_{ij} f - \delta_{ij} \Big)^2 f - I(f|G). \tag{3.7}$$

As a consequence, there holds

$$\frac{1}{2} I'(f) \cdot \mathcal{L}(f) \le -I(f|G) \le 0.$$

Proof of Lemma 3.2. Proof of (3.5). Starting from (3.4) and integrating by part with respect to the x_i variable, we have

$$\begin{aligned} \frac{1}{2} I'(f) \cdot \Delta f &= \int \frac{1}{f} \partial_j f\, \partial_{iij} f - \int \frac{1}{2\, f^2} (\partial_j f)^2 \partial_{ii} f \\ &= \int \Big(\frac{\partial_i f}{f^2} \partial_j f\, \partial_{ij} f - \frac{1}{f} \partial_{ij} f\, \partial_{ij} f \Big) + \int \Big(\frac{1}{f^2} \partial_i f\, \partial_j f\, \partial_{ij} f - \frac{\partial_i f}{f^3} \partial_i f\, (\partial_j f)^2 \Big) \\ &= -\sum_{ij} \int \Big(\frac{1}{f^2}\, \partial_i f\, \partial_j f - \frac{1}{f} \partial_{ij} f \Big)^2 f. \end{aligned}$$

Proof of (3.6). We write

$$\frac{1}{2} I'(f) \cdot (\nabla \cdot (f\, x)) = \int \frac{\partial_j f}{f} \partial_{ij} (f\, x_i) - \frac{(\partial_j f)^2}{2\, f^2}\, \partial_i (f\, x_i).$$

We observe that

$$\begin{aligned} \partial_{ij}(f\, x_i) - \frac{(\partial_j f)}{2\, f}\, \partial_i (f\, x_i) &= (\partial_{ij} f)\, x_i + d\, \partial_j f + \delta_{ij}\, \partial_j f - \partial_i f\, \partial_j f\, \frac{x_i}{2\, f} - \frac{d}{2} \partial_j f \\ &= (\partial_{ij} f)\, x_i + (\frac{d}{2} + 1)\, \partial_j f - \partial_i f\, \partial_j f\, \frac{x_i}{2\, f}. \end{aligned}$$

Gathering the two preceding equalities, we obtain

$$\frac{1}{2} I'(f) \cdot (\nabla \cdot (f\, x)) = (\frac{d}{2} + 1)\, I(f) + \int \frac{\partial_j f}{f}\, \partial_{ij} f\, x_i - \int \frac{\partial_j f}{f}\, \partial_i f\, \partial_j f\, \frac{x_i}{2\, f}.$$

Last, we remark that thank to an integration by parts

$$-\frac{d}{2}\, I(f) = \frac{1}{2} \int \partial_i \left(\frac{(\partial_j f)^2}{f} \right) x_i = \int \frac{\partial_j f\, \partial_{ij} f}{f}\, x_i - \frac{1}{2} \frac{(\partial_j f)^2}{f^2}\, \partial_i f\, x_i,$$

and we then conclude

$$\frac{1}{2} I'(f) \cdot (\nabla \cdot (f\, x)) = I(f).$$

Proof of (3.7). Developing the expression below and using (3.5), we have

$$\begin{aligned} 0 &\le \sum_{ij} \int \Big(\frac{1}{f^2}\, \partial_i f\, \partial_j f - \frac{1}{f} \partial_{ij} f - \delta_{ij} \Big)^2 f \\ &= -\frac{1}{2} I'(f) \cdot \Delta f + 2 \sum_i \int \Big(\partial_{ii} f - \frac{1}{f}\, (\partial_i f)^2 \Big) + d \int f. \end{aligned}$$

From $\int f = 1$, $\int \partial_{ii} f = 0$ and (3.6), we then deduce

$$0 \le -\frac{1}{2} I'(f) \cdot \Delta f - 2 I(f) + d = -\frac{1}{2} I'(f) \cdot \mathcal{L} f + d - I(f),$$

which ends the proof of (3.7). □

Theorem 3.3. *The Fisher information I is decreasing along the flow of the Fokker-Planck equation, i.e. I is a Lyapunov functional, and more precisely*

$$I(f(t,.)|G) \le e^{-2\,t}\, I(f_0|G). \tag{3.8}$$

That implies the convergence in large time to G of any solution to the Fokker-Planck equation associated to any initial condition $f_0 \in D \cap V$. More precisely,

$$\forall\, f_0 \in D \cap V \qquad f(t,.) \to G \quad \text{in} \quad L^p \cap L^1_2 \quad \text{as} \quad t \to \infty, \tag{3.9}$$

for any $p \in [1, 2^\sharp)$ where $2^\sharp = \infty$ when $d = 1, 2$ and $2^\sharp = d/(d-2)$ when $d \ge 3$.

During the proof of Theorem 3.3, we will need the following result (see **Excercise 6.8**).

Lemma 3.4. *A sequence (f_n) which is bounded in $L^1_2 \cap L^q$, $q > 1$, and is such that $f_n \to g$ a.e. in $\mathbb{R}^d$, also satisfies*

$$f_n \to g \quad \text{in} \quad L^p \cap L^1_k, \quad \forall\, k \in [0, 2),\ \forall\, p \in [1, q).$$

If furthemore, $\|f_n\|_{L^1_2} = \|g\|_{L^1_2}$ for any $n \ge 1$, then $f_n \to g$ in L^1_2.

Proof of Theorem 3.3. We only consider the case $d \ge 3$. On the one hand, thanks to (3.7), we have

$$\frac{d}{dt} I(f|G) \le -2\, I(f|G), \tag{3.10}$$

and we conclude to (3.8) thanks to the Gronwall lemma. On the other hand, thanks to the Sobolev inequality, we have

$$\|f\|_{L^{2^*/2}} = \|\sqrt{f}\|^2_{L^{2^*}} \le C\, \|\nabla \sqrt{f}\|^2_{L^2} = C\, I(f) \le C\, I(f_0).$$

Consider now an increasing sequence (t_n) which converges to $+\infty$. Thanks to estimate (3.8) and the Rellich Theorem, we may extract a subsequence still denoted as (t_n) such that $\sqrt{f(t_n)}$ converges a.e. and weakly in $\dot H^1$ to a limit denoted by $\sqrt{g}$. As a consequence, $f(t_n) \to g$ a.e. and $(f(t_n))$ is bounded in $L^{2^*/2} \cap L^1_2$, so that $f_n \to g$ in $L^p \cap L^1_k$, $\forall\, k \in [0, 2)$, $\forall\, p \in [1, q)$, thanks to Lemma 3.6. From the lower semicontinuity of the norms, we have g is bounded in $L^{2^*/2} \cap L^1_2$, $\langle |v|^2 g\rangle \le \liminf \langle |v|^2 f(t_n)\rangle = d$ and $I(g) \le \liminf I(f(t_n)) < \infty$, so that $g \in D_\le \cap V$. Finally, since $2\nabla\sqrt{f(t_n)} - x\sqrt{f(t_n)} \to 2\nabla\sqrt{g} - x\sqrt{g}$ weakly in L^2_{loc} (for instance) and (3.8), we have

$$0 \le J(g) \le \liminf_{k\to\infty} J(f(t_n,.) = \liminf_{k\to\infty} I(f(t_n,.)|G) = 0.$$

From $J(g) = 0$ and $g \in V \cap D_\le$, we get $g = G$ as a consequence of Lemma 3.1, and it is then the all family $(f(t))_{t\ge 0}$ which converges to G as $t \to \infty$. The L^1_2 convergence is a consequence of the fact that $\langle f(t)\, |v|^2\rangle = \langle G\, |v|^2\rangle$ for any time $t \ge 0$ together with Lemma 3.6. □

3.2. **Entropy and Log-Sobolev inequality.** For a function $f \in D$, we define the entropy $H(f) \in \mathbb{R} \cup \{+\infty\}$ and the relative entropy $H(f|G) \in \mathbb{R} \cup \{+\infty\}$ by

$$H(f) = \int_{\mathbb{R}^d} f \, \log f \, dx, \quad H(f|G) = H(f) - H(G) = \int_{\mathbb{R}^d} j(f/G) \, G \, dx,$$

where $j : \mathbb{R}+ \to \mathbb{R}_+$, $j(s) := s \, \log s - s + 1$. It is worth emphasizing that the last integral is always defined in $\mathbb{R} \cup \{+\infty\}$ because $j(f) \ge 0$ and that for establishing the last equality we use that

$$\int_{\mathbb{R}^d} f \, \log G \, dx = \int_{\mathbb{R}^d} G \, \log G \, dx$$

because $f \in D$.
We start observing that for $f \in \mathbf{P}(\mathbb{R}^d) \cap \mathcal{S}(\mathbb{R}^d)$, there holds

$$\begin{aligned} H'(f) \cdot \mathcal{L} f &:= \int_{\mathbb{R}^d} (1 + \log f) \, [\Delta f + \nabla \cdot (x \, f)] \\ &= -\int_{\mathbb{R}^d} \nabla f \cdot \nabla \log f - \int_{\mathbb{R}^d} x \, f \cdot \nabla \log f \\ &= -I(f) + d \, \langle f \rangle = -I(f|G). \end{aligned}$$

As a consequence, the entropy is a Lyapunov functional for the Fokker-Planck equation and more precisely

$$\frac{d}{dt} H(f) = -I(f|G) \le 0. \tag{3.11}$$

Theorem 3.5. (Logarithmic Sobolev inequality). *For any $f \in D \cap V$, the following Log-Sobolev inequality holds*

$$H(f|G) \le \frac{1}{2} I(f|G). \tag{3.12}$$

That one also writes equivalently as

$$\int_{\mathbb{R}^d} f \ln f - \int_{\mathbb{R}^d} G \ln G \le \frac{1}{2} \Big(\int_{\mathbb{R}^d} \frac{|\nabla f|^2}{f} - d \Big)$$

or also as

$$\int_{\mathbb{R}^d} u^2 \, \log(u^2) \, G(dx) \le 2 \int_{\mathbb{R}^d} |\nabla u|^2 \, G(dx).$$

For some applications, it is worth emphasizing that the Log-Sobolev inequality depends on a nicer way of the dimension than the Poincaré inequality.

During the proof of Theorem 3.5, we will need the following result (see **Excercise 6.10**).

Lemma 3.6. *Consider a sequence (f_n) such that $0 \le f_n \to f$ in $L^q \cap L^1_k$, $q > 1$, $k > 0$, then $H(f_n) \to H(f)$.*

Proof of Theorem 3.5. We denote by f_t the solution to the Fokker-Planck equation (2.1) associated to the initial datum $f_0 := f$. On the one hand, from (3.9), Lemma 3.6 and (3.11), we get

$$\begin{aligned} H(f) - H(G) &= \lim_{T \to \infty} [H(f) - H(f_T)] = \lim_{T \to \infty} \int_0^T \Big[-\frac{d}{dt} H(f) \Big] \, dt \\ &= \lim_{T \to \infty} \int_0^T [I(f|G)] \, dt. \end{aligned}$$

From that identity and (3.10), we deduce

$$\begin{aligned} H(f) - H(G) &\le \lim_{T \to \infty} \int_0^T \Big[-\frac{1}{2} \frac{d}{dt} I(f|G) \Big] \, dt \\ &= \lim_{T \to \infty} \frac{1}{2} [I(f|G) - I(f_T|G)] = \frac{1}{2} I(f|G), \end{aligned}$$

thanks to (3.8). ☐

Lemma 3.7. (Csiszár-Kullback inequality). *Consider probability measure μ and a nonnegative measurable function g such that $g\,\mu$ is also a probability measure. Then*

$$\|g-1\|^2_{L^1(d\mu)} \le 2 \int g \log g \, d\mu. \tag{3.13}$$

Proof of Lemma 3.7. First proof. Thanks to the Taylor-Laplace expansion formula, there holds

$$\begin{aligned} j(g) &:= g\log g - g + 1 = j(1) + (g-1)\,j'(1) + (g-1)^2 \int_0^1 j''(1+s\,(g-1))\,(1-s)\,ds \\ &= (g-1)^2 \int_0^1 \frac{1-s}{1+s\,(g-1)}\,ds. \end{aligned}$$

Using Fubini theorem, we get

$$H(g) := \int (g\log g - g + 1)\,d\mu = \int_0^1 (1-s) \int \frac{(g-1)^2}{1+s\,(g-1)} d\mu\,ds.$$

For any $s \in [0,1]$, we use the Cauchy-Schwarz inequality and the fact that both μ and $g\,\mu$ are probability measures in order to deduce

$$\left(\int |g-1|\,d\mu\right)^2 \le \left(\int \frac{(g-1)^2}{1+s\,(g-1)} d\mu\right)\left(\int [1+s\,(g-1)] d\mu\right) = \int \frac{(g-1)^2}{1+s\,(g-1)} d\mu.$$

As a conclusion, we obtain

$$H(g) \ge \int_0^1 \left(\int |g-1|\,d\mu\right)^2 (1-s)\,ds = \frac{1}{2}\left(\int |g-1|\,d\mu\right)^2,$$

which ends the proof of the Csiszár-Kullback inequality.

Second proof. We give a shorter but probably more obscure proof. One easily checks (by differentiating three times both functions) that

$$\forall\, u \ge 0 \qquad 3\,(u-1)^2 \le (2\,u+4)\,(u\log u - u + 1).$$

Thanks to the Cauchy-Schwarz inequality one deduces

$$\int |g-1|\,d\mu \le \sqrt{\frac{1}{3}\int (2\,g+4)\,d\mu}\sqrt{\int (g\log g - g + 1)\,d\mu} = \sqrt{2\int g\log g\,d\mu},$$

which is nothing but the Csiszár-Kullback inequality again. □

Putting together (3.11), (3.12) and (3.13) with $G := \mu$ and $g := f/G$, we immediately obtain the following convergence result.

Theorem 3.8. *For any $f_0 \in D$ such that $H(f_0) < \infty$, the associated solution f to the Fokker-Planck equation* (2.1)-(2.2) *satisfies*

$$H(f|G) \le e^{-2t}\,H(f_0|G),$$

and then

$$\|f-G\|_{L^1} \le \sqrt{2}\,e^{-t}\,H(f_0|G)^{1/2}.$$

3.3. **From log-Sobolev to Poincaré.** The next result makes a possible connection between the log-Sobolev inequality and the Poincaré inequality.

Lemma 3.9. *If the log-Sobolev inequality*

$$\lambda\,H(f|G) \le \frac{1}{2} I(f|G), \quad \forall\, f \in D,$$

holds for some constant $\lambda > 0$, then the Poincaré inequality

$$(\lambda + d)\,\|g\|^2_{L^2(G^{-1})} \le \int |\nabla g|^2\,G^{-1}, \quad \forall\, g \in \mathcal{D}(\mathbb{R}^d),\ \langle g[1,x,|x|^2]\rangle = 0,$$

also holds (for the same constant $\lambda > 0$).

That lemma gives an alternative proof of the Poincaré inequality. Of course that proof is not very "cheap" in the sense that one needs to prove first the log-Sobolev inequality which is somewhat more difficult to prove than the Poincaré inequality. Moreover, the log-Sobolev inequality is known to be true under more restrictive assumption on the confinement potential than the Poincaré inequality. However, that allows to compare the constants involved in the two inequalities and the proof is robust enough so that it can be adapted to nonlinear situations.

Proof of Lemma 3.9. Consider $g \in \mathcal{D}(\mathbb{R}^d)$ such that $\int g(x)\,[1,x,|x|^2]\,dv = [0,0,0]$. Applying the Log-Sobolev inequality to the function $f = G + \varepsilon\, g \in D$ for $\varepsilon > 0$ small enough, we have

$$\lambda\, \frac{H(G+\varepsilon\, g) - H(G)}{\varepsilon^2} = \frac{\lambda}{\varepsilon^2}\, H(f|G) \le \frac{1}{2\,\varepsilon^2} I(f|G) = \frac{I(G+\varepsilon\, g) - I(G)}{2\,\varepsilon^2}.$$

Expending up to order 2 the two functionals, we have

$$f \log f = G \log G + \varepsilon\, g\,(1+\log G) + \frac{\varepsilon^2}{2}\,\frac{g^2}{G} + \mathcal{O}(\varepsilon^3),$$

$$\frac{|\nabla f|^2}{f} = \frac{|\nabla G|^2}{G} + \varepsilon\,\left\{2\,\frac{\nabla G}{G}\cdot\nabla g - \frac{|\nabla G|^2}{G^2}\,g\right\} + \frac{\varepsilon^2}{2}\,\left\{\frac{|\nabla g|^2}{G} - 2\,g\,\frac{\nabla G}{G^2}\cdot\nabla g + \frac{|\nabla G|^2}{G^3}\,g^2\right\} + \mathcal{O}(\varepsilon^3).$$

Passing now to the limit $\varepsilon \to 0$ in the first inequality and using that the zero and first order terms vanish because (performing one integration by parts)

$$H'(G)\cdot g = \int_{\mathbb{R}^d} (\log G + 1)\, g = 0,$$

$$I'(G)\cdot g = \int_{\mathbb{R}^d} \left\{\frac{|\nabla G|^2}{G^2} - 2\,\frac{\Delta G}{G}\right\} g = 0,$$

we get

$$\lambda\, H''(G)\cdot(g,g) \le I''(G)\cdot(g,g).$$

More explicitly, we have

$$\lambda \int \frac{g^2}{G} \le \int \left\{\frac{|\nabla g|^2}{G} + \nabla\Big(\frac{\nabla G}{G^2}\Big)\, g^2 + \frac{|\nabla G|^2}{G^3}\, g^2\right\},$$

and then

$$(\lambda + d) \int \frac{g^2}{G} = \int \frac{g^2}{G}\,\left\{\lambda - \frac{\Delta G}{G} + \frac{|\nabla G|^2}{G^2}\right\} \le \int \frac{|\nabla g|^2}{G},$$

which is nothing but the Poincaré inequality. □

4. Weighted L^1 decay through semigroups factorization technique

In this section, we establish the following weighted L^1 decay through a semigroups factorization technique and the already known weighted L^2 decay (consequence and equivalent to the Poincaré inequality).

Theorem 4.1. *For any $a \in (-\lambda_P, 0)$ and for any $k > k^* := \lambda_P$ there exists $C_{k,a}$ such that for any $\varphi \in L^1_k$, the associated solution f to the Fokker-Planck equation* (2.1)-(2.2) *satisfies*

$$\|f - \langle\varphi\rangle\, G\|_{L^1_k} \le C_{k,a}\, e^{a\,t}\, \|\varphi - \langle\varphi\rangle\, G\|_{L^1_k}. \tag{4.1}$$

A refined version of the proof below shows that the same estimate holds with $a := -\lambda_P$.

Proof of Theorem 4.1. In order to simplify a bit the presentation, we only present the proof in the case of the dimension $d \le 3$, but the same arguments can be generalized to any dimension $d \ge 1$.

Step 1. The splitting. We introduce the splitting $\mathcal{L} = \mathcal{A} + \mathcal{B}$ with

$$\mathcal{B} f := \Delta f + \nabla\cdot(f\,x) - M\,f\,\chi_R, \quad \mathcal{A} f := M\,f\,\chi_R,$$

where $\chi_R(x) = \chi(x/R)$, $\chi \in \mathcal{D}(\mathbb{R}^d)$, $\mathbf{1}_{B_1} \le \chi \le \mathbf{1}_{B_2}$, and where $R, M > 0$ are two real constants to be chosen later. We define, in any Banach space $\mathcal{X}$ such that $G \in \mathcal{X} \subset L^1$, the projection operator

$$\Pi f := \langle f\rangle\, G,$$

which thus satisfies $\Pi^2 = \Pi$ and $\Pi \in \mathcal{B}(\mathcal{X})$. When $S_\mathcal{L}$ is well defined as a semigroup in $\mathcal{X}$, we have

$$S_\mathcal{L}(I-\Pi) = (I-\Pi)\,S_\mathcal{L} = (I-\Pi)\,S_\mathcal{L}(I-\Pi) \tag{4.2}$$

as a consequence of the projection property $(I-\Pi)^2 = (I-\Pi)^2$, of the facts that G is a stationary solution to the Fokker-Planck equation and that the mass is preserved by the associated flow. Now, iteraration the Duhamel formula

$$S_\mathcal{L} = S_\mathcal{B} + S_\mathcal{L} * \mathcal{A}S_\mathcal{B},$$

we have

$$S_\mathcal{L} = S_\mathcal{B} + S_\mathcal{B} * (\mathcal{A}S_\mathcal{B}) + S_\mathcal{L} * (\mathcal{A}S_\mathcal{B}) * (\mathcal{A}S_\mathcal{B}). \tag{4.3}$$

The two identities (4.2) and (4.3) together and using the shorthand $\Pi^\perp = I - \Pi$, we have

$$S_\mathcal{L}\Pi^\perp \;=\; \Pi^\perp S_\mathcal{B}\Pi^\perp + \Pi^\perp S_\mathcal{B} * (\mathcal{A}S_\mathcal{B})\Pi^\perp + S_\mathcal{L}\Pi^\perp * (\mathcal{A}S_\mathcal{B}) * (\mathcal{A}S_\mathcal{B}\Pi^\perp) =: \sum_{i=1}^{3} \mathcal{T}_i(t).$$

In order to get (4.1), we will establish that

$$S_\mathcal{B}(t) : L^1_k \to L^1_k, \text{ with bound } \mathcal{O}(e^{a't}), \ \forall\, t \ge 0,\ \forall\, a' > a^*,\ \forall\, k > k^*, \tag{4.4}$$

$$S_\mathcal{B}(t) : L^1_K \to L^2_K, \text{ with bound } \mathcal{O}(\frac{e^{a't}}{t^{3/4}}), \ \forall\, t > 0,\ \forall\, a' > a^*,\ \forall\, K > K^*, \tag{4.5}$$

$$\mathcal{A} : L^1_k \to L^1_K,\ \mathcal{A} : L^2_K \to L^2(G^{-1}),\ \forall\, K > k^*,\ \forall\, k > k^*, \tag{4.6}$$

with $K^* := \lambda_P + d/2$. We also recall that

$$S_\mathcal{L}(t)\Pi^\perp : L^2(G^{-1}) \to L^2(G^{-1}), \text{ with bound } \mathcal{O}(e^{-\lambda_P t}), \ \forall\, t \ge 0, \tag{4.7}$$

which is nothing but (2.4). We finally observe that

$$u * w(t) = \mathcal{O}(e^{at}) \ \text{ and } u * v * w(t) = \mathcal{O}(e^{at}), \quad \forall\, t \ge 0,\ \forall\, a > a^*, \tag{4.8}$$

if

$$u(t) = \mathcal{O}(e^{a't}),\quad v(t) = \mathcal{O}(\frac{e^{a't}}{t^{3/4}}),\quad w(t) = \mathcal{O}(e^{a't}),\quad \forall\, t > 0,\ \forall\, a' > a^*. \tag{4.9}$$

The first estimate in (4.8) is obtained by writing

$$u * w(t) = \int_0^t u(s) w(t-s)\, ds \lesssim \int_0^t e^{a's} e^{a'(t-s)}\, ds \lesssim t e^{a't} \lesssim e^{at},$$

for any $t \ge 0$ and any $a > a' > a^*$. For the second estimate in (4.8), we first write

$$v * w(t) = \int_0^t v(s) w(t-s)\, ds \lesssim \int_0^t \frac{e^{a''s}}{s^{3/4}} e^{a''(t-s)}\, ds \lesssim t^{1/4} e^{a''t} \lesssim e^{a't},$$

for any $t \ge 0$ and any $a' > a'' > a^*$, and we conclude by combining that estimate with the first estimate in (4.8).

Step 2. The conclusion. With the help of the estimates stated in step 1, we are in position to prove (4.1) or equivalently that

$$\|\mathcal{T}_i(t)\|_{L^1_k \to L^1_k} \lesssim e^{at},\ \forall\, t \ge 0,\ \forall\, a > a^*,\ \forall\, k > k^*, \tag{4.10}$$

for any $i = 1, 2, 3$. For $i = 1$, (4.10) is nothing but (4.4) together with $\Pi^\perp \in \mathcal{B}(L^1_k)$. For proving (4.10) when $i = 2$, we use the first estimate in (4.8) with

$$u(t) := \|\Pi^\perp S_\mathcal{B}(t)\|_{L^1_k \to L^1_k},\quad w(t) := \|\mathcal{A}S_\mathcal{B}(t)\Pi^\perp\|_{L^1_k \to L^1_k},$$

where both functions satisfy the hypothesizes of (4.9) because of $\Pi^\perp \in \mathcal{B}(L^1_k)$, of the first estimate on $\mathcal{A}$ with $K = k$ in (4.6) and of the estimate (4.5) on $S_\mathcal{B}(t)$ in L^1_k. For proving (4.10) when $i = 3$, we use the second estimate in (4.8) with

$$u(t) := \|S_\mathcal{L}(t)\Pi^\perp\|_{L^2(G^{-1}) \to L^1_k},\quad v(t) := \|\mathcal{A}S_\mathcal{B}(t)\|_{L^1_K \to L^2(G^{-1})},\quad w(t) := \|\mathcal{A}S_\mathcal{B}(t)\Pi^\perp\|_{L^1_k \to L^1_K},$$

where the three functions satisfy the hypothesizes of (4.9). To check the estimate on u, we use (4.7) and $L^2(G^{-1}) \subset L^1_k$. For the estimate on v, we use (4.5) and the second estimate on $\mathcal{A}$ in (4.6). Finally, to check the estimate on w, we use the first estimate on $\mathcal{A}$ in (4.6), the estimate (4.4) on $S_\mathcal{B}(t)$ in L^1_k and $\Pi^\perp \in \mathcal{B}(L^1_k)$.

In order to conclude the proof of Theorem 4.1, we thus need to establish (4.4), (4.5) and (4.6). That is done in the three following steps.

Step 3. Proof of (4.6). The operator $\mathcal{A}$ is clearly bounded in any Lebesgue space and more precisely

$$\|\mathcal{A} f\|_{L^p(m)} \le C_{R,M} \, \|f\|_{L^p_\ell}, \quad \forall\, f \in L^p_\ell, \; \forall\, p = 1, 2,$$

for $m := \langle x \rangle^K$ or $m := G^{-1}$ and with

$$C_{R,M} := M \| \frac{m}{\langle \cdot \rangle^{p\ell}} \|^{1/p}_{L^\infty(B_{2R})}.$$

Step 4. Proof of (4.4). For any $k, \varepsilon > 0$ and for any $M, R > 0$ large enough (which may depend on k and ε) the operator $\mathcal{B}$ is dissipative in L^1_k in the sense that

$$\forall\, f \in \mathcal{D}(\mathbb{R}^d), \qquad \int_{\mathbb{R}^d} (\mathcal{B} f) \, (\mathrm{sign} f) \, \langle x \rangle^k \le (\varepsilon - k) \, \|f\|_{L^1_k}. \tag{4.11}$$

We immediately deduce (4.4) from (4.11) and the Gronwall lemma. In oder to establish (4.11), we set $\beta(s) = |s|$ (and more rigorously we must take a smooth version of that function) and $m = \langle x \rangle^k$, and we compute

$$\begin{aligned}
\int (\mathcal{L} f) \, \beta'(f) \, m &= \int (\Delta f + d\, f + x \cdot \nabla f) \, \beta'(f) \, m \\
&= \int \{ -\nabla f \, \nabla(\beta'(f) m) + d \, |f| \, m + m \, x \cdot \nabla |f| \} \\
&= -\int |\nabla f|^2 \, \beta''(f) \, m + \int |f| \, \{ \Delta m + d - \nabla(x\, m) \} \\
&\le \int |f| \, \{ \Delta m - x \cdot \nabla m \},
\end{aligned}$$

where we have used that β is a convex function. Defining

$$\begin{aligned}
\psi &:= \Delta m - x \cdot \nabla m - M \chi_R \, m \\
&= (k^2 \, |x|^2 \, \langle x \rangle^{-4} - k \, |x|^2 \, \langle x \rangle^{-2} - M \, \chi_R) \, m
\end{aligned}$$

we easily see that we can choose $M, R > 0$ large enough such that $\psi \le (\varepsilon - k) \, m$ and then (4.11) follows.

Step 5. Proof of (4.5). Fix now $K > K^*$ and $a > -\lambda_P$. There holds

$$\|S_{\mathcal{B}}(t) \varphi\|_{L^2_K} \le \frac{C_{a,K}}{t^{d/4}} \, e^{a\, t} \, \|\varphi\|_{L^1_K}, \quad \forall\, \varphi \in L^1_K, \tag{4.12}$$

which immediately implies (4.5) since we are restricted to the case of a dimension $d \le 3$. We set $m = \langle x \rangle^K$. A similar computation as in step 4 gives

$$\begin{aligned}
\int (\mathcal{B} f) \, f \, m^2 &= -\int |\nabla (f\, m)|^2 + \int |f|^2 \, \Big\{ \frac{|\nabla m|^2}{m^2} + \frac{d}{2} - x \cdot \nabla m - M \, \chi_R \Big\} m^2 \\
&= -\int |\nabla (f\, m)|^2 + (\frac{d}{2} + \varepsilon - K) \int |f|^2 \, m^2,
\end{aligned}$$

for $M, R > 0$ chosen large enough. Denoting by $f(t) = S_{\mathcal{B}}(t) \varphi$ the solution to the evolution PDE

$$\partial_t f = \mathcal{B} f, \qquad f(0) = \varphi,$$

we (formally) have

$$\frac{1}{2} \frac{d}{dt} \int f^2 \, m^2 = \int (\mathcal{B} f) \, f \, m^2 \le -\int |\nabla (f\, m)|^2 + a \int |f|^2 \, m^2.$$

On the one hand, throwing away the last (negative) term at the RHS of the above differential inequality and using the same Nash trick as in the proof of estimate (1.5) in section 1.2, we get

$$\|f(t) \, m\|_{L^2} \le \frac{C}{t^{d/4}} \, \|f(0) \, m\|_{L^1}, \quad \forall\, t > 0. \tag{4.13}$$

On the other hand, throwing away the first (negative) term at the RHS of the above differential inequality and using the Gronwall lemma exactly as in step 4, we get

$$\|f(t) \, m\|_{L^2} \le C e^{a(t - t_0)} \, \|f(t_0) \, m\|_{L^2}, \quad \forall\, t \ge t_0 \ge 0. \tag{4.14}$$

Using (4.13) for $t \in (0,1]$ and (4.14) for $t \ge 1$, we deduce (4.12). □

5. Coming back to local in time estimates

We consider a smooth, positive and fast decaying initial datum f_0, the solution f to the associated heat equation, and for a given $\alpha \in \mathbb{R}^d$, we define $g := f\, e^{\psi}$, $\psi(x) := \alpha \cdot x$. The equation satisfied by g is

$$\begin{aligned}\partial_t g &= \frac{1}{2} e^{\psi} \Delta(g\, e^{-\psi}) = \frac{1}{2}\Delta g - \nabla\psi \cdot \nabla g + \frac{1}{2}|\nabla\psi|^2 g \\ &= \frac{1}{2}\Delta g - \alpha \cdot \nabla g + \frac{1}{2}|\alpha|^2 g.\end{aligned}$$

For the L^1 norm, we have

$$\frac{d}{dt}\|g\|_{L^1} = \frac{1}{2}\alpha^2\, \|g\|_{L^1},$$

and then $\|g(t,.)\|_{L^1} = e^{\alpha^2 t/2}\, \|g_0\|_{L^1}$ for any $t \ge 0$. For the L^2 norm and thanks to the Nash inequality (1.6), we have

$$\begin{aligned}\frac{d}{dt}\|g\|^2_{L^2} &= -\|\nabla g\|^2_{L^2} + \alpha^2\, \|g\|^2_{L^2} \\ &\le -K_0\, e^{-2\alpha^2 t/d}\, \|g\|^{2(1+2/d)}_{L^2} + \alpha^2\, \|g\|^2_{L^2},\end{aligned}$$

with $K_0 := C_N\, \|g_0\|^{-4/d}_{L^1}$. We see that the function $u(t) := e^{-\alpha^2 t}\, \|g(t)\|^2_{L^2}$ satisfies the differential inequality

$$u' \le -K_0\, u^{1+2/d},$$

from what, exactly as in the Section 1.2, we deduce

$$\|g(t)\|^2_{L^2}\, e^{-\alpha^2 t} \le \frac{\|g_0\|^2_{L^1}}{(2/d\, C_N\, t)^{d/2}}, \quad \forall\, t > 0.$$

Denoting by $T(t)$ the semigroup associated to the parabolic equation satisfies by g, the above estimate writes

$$\|T(t) g_0\|_{L^2} \le \frac{C\, e^{\alpha^2 t/2}}{t^{d/4}}\, \|g_0\|_{L^1}, \quad \forall\, t > 0.$$

Because the equation associated to the dual operator is

$$\partial_t h = \frac{1}{2}\Delta h + \alpha \cdot \nabla h + \frac{1}{2}|\alpha|^2 h, \quad h(0) = h_0,$$

the same estimate holds on $T^*(t) h_0 = h(t)$, and we thus deduce

$$\|T(t) g_0\|_{L^\infty} \le \frac{C\, e^{\alpha^2 t/2}}{t^{d/4}}\, \|g_0\|_{L^2}, \quad \forall\, t > 0.$$

Using the trick $T(t) = T(t/2)T(t/2)$, both estimates together give an accurate time depend estimate on the mapping $T(t) : L^1 \to L^\infty$ for any $t > 0$. More precisely and in other words, we have proved that the heat semigroup S satisfies

$$\|(S(t) f_0)\, e^{\psi}\|_{L^\infty} \le \frac{C}{t^{d/2}}\, e^{\alpha^2 t/2}\, \|f_0\, e^{\psi}\|_{L^1}, \quad \forall\, t > 0.$$

Denoting $F(t,x,y) := (S(t)\delta_x)(y)$ the fundamental solution associated to the heat equation when starting from the Dirac function in $x \in \mathbb{R}^d$, the above estimate rewrites as

$$F(t,x,y) \le \frac{C}{t^{d/2}}\, e^{\alpha\cdot(x-y) - \alpha^2 t/2}, \quad \forall\, t > 0, \forall\, x, y, \alpha \in \mathbb{R}^d.$$

Choosing $\alpha := (x-y)/t$, we end with

$$F(t,x,y) \le \frac{C}{t^{d/2}}\, e^{-\frac{|x-y|^2}{2t}}, \quad \forall\, t > 0, \forall\, x, y \in \mathbb{R}^d.$$

6. Exercises and Complements

Exercise 6.1. *1. Give another proof of the Nash inequality by using the Sobolev inequality in dimension $d \ge 3$. (Hint. Write the interpolation estimate*

$$\|f\|_{L^2} \le \|f\|_{L^1}^{\theta} \, \|f\|_{L^{2^*}}^{1-\theta}$$

and then use the Sobolev inequality associated to the Lebesgue exponent $p = 2$).
2. Give another proof of the Nash inequality by using the Sobolev inequality in dimension $d = 2$. (Hint. Prove the interpolation estimate

$$\|f\|_{L^2} \le \|f\|_{L^1}^{1/4} \, \|f^{3/2}\|_{L^2}^{1/2},$$

then use the Sobolev inequality associated to the Lebesgue exponent $p = 1$ and $p^* := 2$ and finally the Cauchy-Schwartz inequality in order to bound the second term).
3. Give another proof of the Nash inequality by using the Sobolev inequality in dimension $d = 1$. (Hint. Prove the interpolation estimate

$$\|f\|_{L^2} \le \|f\|_{L^1}^{1/2} \, \|f^{3/2}\|_{L^\infty}^{1/3},$$

then use the Sobolev inequality associated to the Lebesgue exponent $p = 1$ and $p^* := \infty$ and finally the Cauchy-Schwartz inequality in order to bound the second term).

We propose now a third proof based on the Poincaré-Wirtinger inequality.

Exercise 6.2. *Prove that for any $f \in H^1(\mathbb{R}^d)$, there holds*

$$\|\rho_\varepsilon * f - f\|_{L^2} \le C\,\varepsilon\,\|\nabla f\|_{L^2},$$

for a constant $C > 0$ which only depends on the function $\rho \in \mathbf{P}(\mathbb{R}^d) \cap \mathcal{D}(\mathbb{R}^d)$ used in the definition of the mollifier (ρ_ε). Deduce the Nash inequality. (Hint. Write $f = f - \rho_\varepsilon * f + \rho_\varepsilon * f$).

We present the proof in the case when ρ is the characteristic function of a ball. We write

$$\|f\|_{L^2}^2 = (f, f - f_r) + (f, f_r), \quad \text{with} \quad f_r(x) := \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\,dy.$$

We have

$$\|f\|_{L^2}^2 \le \|f\|_{L^2}\,\|f - f_r\|_{L^2} + \|f\|_{L^1}\,\|f_r\|_{L^\infty}.$$

On the one hand,

$$\|f_r\|_{L^\infty} \le \frac{C}{r^d}\,\|f\|_{L^1}.$$

On the other hand,

$$\begin{aligned}
\|f - f_r\|_{L^2}^2 &= \int_{\mathbb{R}^d} \Big| \frac{C_d}{r^d} \int_{B(x,r)} (f(y) - f(x))\,dy \Big|^2 dy \\
&\le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathbf{1}_{|y-x|\le r} |f(y) - f(x)|^2\,dxdy \\
&\le r^2 \int_0^1 \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathbf{1}_{|y-x|\le r} |\nabla f((1-t)x + ty)|^2\,dxdydt \\
&\le r^2 \int_0^{1/2} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathbf{1}_{|y-x|\le r} |\nabla f((1-t)x + ty)|^2\,dxdydt \\
&\quad + r^2 \int_{1/2}^1 \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathbf{1}_{|y-x|\le r} |\nabla f((1-t)x + ty)|^2\,dxdydt \\
&\le r^2 \int_0^{1/2} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathbf{1}_{|y-x|\le r} |\nabla f(z)|^2\,dzdydt \\
&\quad + r^2 \int_{1/2}^1 \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathbf{1}_{|y-x|\le r} |\nabla f(z)|^2\,dxdzdt \\
&\le r^2\, C \int_{\mathbb{R}^d} |\nabla f(z)|^2\,dz.
\end{aligned}$$

All together, we get

$$\begin{aligned}\|f\|_{L^2}^2 &\le C_1\, r\, \|f\|_{L^2}\, \|\nabla f\|_{L^2} + C_2\, r^{-d}\, \|f\|_{L^1}^2\\ &\le \frac{1}{2}\|f\|_{L^2}^2 + \frac{C_1}{2}\, r^2\, \|\nabla f\|_{L^2}^2 + C_2\, r^{-d}\, \|f\|_{L^1}^2\end{aligned}$$

and we obtain the Nash inequality by choosing $r := (\|f\|_{L^1}^2/\|\nabla f\|_{L^2}^2)^{1/(d+2)}$.

Exercise 6.3. *Establish that $E \in L^1(\mathbb{R}^d)$ and $\mathrm{div} E = 0$ imply $E = 0$. (Hint. First observe that*

$$\forall\, i = 1, \dots, d,\ \forall\, \varphi \in C^1_b(\mathbb{R}^d),\ \varphi(x) = \varphi(x_i), \quad \int_{\mathbb{R}^d} E_i \varphi'\, dx = 0.$$

Next establish that $u' = 0$ in $\mathcal{D}'(\mathbb{R})$ implies that u is a constant. Conclude).

Exercise 6.4. *Establish that for any $\lambda < \lambda_P$, there exists $\varepsilon > 0$ so that the following stronger version*

$$\begin{aligned}\int_{\mathbb{R}^d} \left|\nabla\left(\frac{f}{G}\right)\right|^2 G dx \ &\ge\ \lambda \int_{\mathbb{R}^d} f^2\, G^{-1} dx\\ &\quad + \varepsilon \int_{\mathbb{R}^d} \left(f^2\, |x|^2 + |\nabla f|^2\right)\, G^{-1} dx\end{aligned} \tag{6.1}$$

holds for any $f \in \mathcal{D}(\mathbb{R}^d)$ with $\langle f \rangle = 0$. (Hint: Proceed along the lines of the proof of Proposition 2.4).

Proof of (6.1). We define $\Phi := -\log G = |x|^2/2 + \log(2\pi)^{d/2}$. On the one hand, by developing the LHS term, we find

$$T := \int_{\mathbb{R}^d} \left|\nabla\left(\frac{f}{G}\right)\right|^2 G dx = \int_{\mathbb{R}^d} |\nabla f|^2\, G^{-1} dx - \int_{\mathbb{R}^d} f^2\, (\Delta\Phi)\, G^{-1} dx.$$

On the other hand, a similar computation leads to the following identity

$$\begin{aligned}T &= \int_{\mathbb{R}^d} \left|\nabla(fG^{-1/2})\, G^{1/2} + (fG^{-1/2})\, \nabla G^{1/2}\right|^2 G dx\\ &= \int_{\mathbb{R}^d} \left|\nabla(f\, G^{-1/2})\right|^2 dx + \int_{\mathbb{R}^d} f^2 \left(\frac{1}{4}|\nabla\Phi|^2 - \frac{1}{2}\Delta\Phi\right) G^{-1} dx.\end{aligned}$$

The two above identities together with (2.7) imply that for any $\theta \in (0,1)$

$$\begin{aligned}T \ &\ge\ (1-\theta)\lambda_P \int_{\mathbb{R}^d} f^2\, G^{-1} dx + \theta \int_{\mathbb{R}^d} f^2 \left(\frac{1}{16}|\nabla\Phi|^2 - \frac{3}{4}\Delta\Phi\right) G^{-1} dx\\ &\quad + \frac{\theta}{16} \int_{\mathbb{R}^d} f^2\, |\nabla\Phi|^2\, G^{-1} dx + \frac{\theta}{2} \int_{\mathbb{R}^d} |\nabla f|^2\, G^{-1} dx.\end{aligned}$$

Observe that $|\nabla\Phi|^2 - 12\Delta\Phi \ge 0$ for x large enough, and we can choose $\theta > 0$ small enough to conclude the proof. □

Exercise 6.5. *Observe that the function $H := x_k$ satisfies*

$$H \in L^2(G), \quad \langle HG \rangle = 0, \quad LH = -H, \quad \|\nabla H\|_{L^2(G)}^2 = \|H\|_{L^2(G)}^2.$$

Conclude that the constant $\lambda_P = 1$ in the Poincaré inequality established in Proposition 2.5 is optimal.

Exercise 6.6. *Establish* (2.10) *in the following situations:*

(i) *$V(x) := \langle x \rangle^\alpha$ with $\alpha \ge 1$;*

(ii) *there exist $\alpha > 0$ and $R \ge 0$ such that*

$$x \cdot \nabla V(x) \ge \alpha \qquad \forall\, x \notin B_R;$$

(iii) *there exist $a \in (0,1)$, $c > 0$ and $R \ge 0$ such that*

$$a\, |\nabla V(x)|^2 - \Delta V(x) \ge c \qquad \forall\, x \notin B_R;$$

(iv) *V is convex (or it is a compact supported perturbation of a convex function) and satisfies $e^{-V} \in L^1(\mathbb{R}^d)$.*

Exercise 6.7. *Generalize the Poincaré inequality to a general superlinear potential* $V(x) = \langle x\rangle^\alpha/\alpha + V_0$, $\alpha \ge 1$, *in the following strong (weighted) formulation*

$$\int |\nabla g|^2 \, \mathcal{G} \ge \kappa \int |g - \langle g\rangle_{\mathcal{G}}|^2 \, (1 + |\nabla V|^2) \, \mathcal{G} \qquad \forall \, g \in \mathcal{D}(\mathbb{R}^d),$$

where we have defined $\mathcal{G} := e^{-V} \in \mathbf{P}(\mathbb{R}^d)$ *(for an appropriate choice of* $V_0 \in \mathbb{R}$*).*

Exercise 6.8. *Establish Lemma 3.6.*

A possible solution of Exercise 6.8. We first establish that $f_n \to g$ strongly in L^1. We write

$$\|f_n - g\|_{L^1} \quad \le \quad \int_{B_R} |f_n - g| \wedge M + 2R^{-2} \sup_n \int_{B_R^c} |f_n||x|^2 + 2M^{1-q} \sup_n \int_{B_R} |f_n|^q,$$

for any $R, M > 0$ and any $k \ge 1$, and by using the dominated convergence theorem of Lebesgue for the first term. Thanks to the interpolation inequalities

$$\|h\|_{L^p} \le \|h\|_{L^1}^{1-\alpha} \|h\|_{L^q}^\alpha \quad \text{and} \quad \|h\|_{L^1_k} \le \|h\|_{L^1_2}^{k/2} \|h\|_{L^1}^{1-k/2},$$

with $1/p = 1 - \alpha + \alpha/q$, we next get that $f_n \to g$ strongly in $L^p \cap L^1_k$, for any $p \in [1, q)$, $k \in [0, 2)$. When we furthermore assume $f_n, g \ge 0$ and $\langle g|x|^2\rangle = \langle f_n|x|^2\rangle = d$ for any $n \ge 1$, from Fatou lemma, we may first deduce

$$\begin{aligned}\limsup_n \int_{B_R^c} f_n|x|^2 \quad &= \quad d - \liminf_n \int_{B_R} f_n|x|^2 \\ &\le \quad d - \int_{B_R} f|x|^2 = \int_{B_R^c} f|x|^2,\end{aligned}$$

for any $R > 0$. On the other hand, we have

$$\int |f_n - f| \, |x|^2 \, dx \le \int_{B_R} |f_n - f| \, |x|^2 \, dx + \int_{B_R^c} f_n \, |x|^2 \, dx + \int_{B_R^c} f \, |x|^2 \, dx.$$

From the two above informations together with the convergence $f_n \to f$ in L^1, we deduce

$$\limsup_n \|f - f_n\|_{L^1_2} \le 2 \int_{B_R^c} f \, |x|^2 \, dx,$$

for any $R > 0$, and thus the conclusion by letting $R \to \infty$. □

Exercise 6.9. *Prove the convergence* (3.8) *for any* $f_0 \in \mathbf{P}(\mathbb{R}^d) \cap L^1_2(\mathbb{R}^d)$ *such that* $I(f_0) < \infty$. *(Hint. Compute the equations for the moments of order* 1 *and* 2 *and introduce the relative Fisher information* $I(f|M_{1,u,\theta})$ *associatred to a normalized Gaussian with mean velocity* $u \in \mathbb{R}^d$ *and temperature* $\theta > 0$*).*

Exercise 6.10. *Prove that* $0 \le f_n \to f$ *in* $L^q \cap L^1_k$, $q > 1$, $k > 0$, *implies that* $H(f_n) \to H(f)$. *(Hint. Use the splitting*

$$s \,|\log s| \le \sqrt{s}\, \mathbf{1}_{0 \le s \le e^{-|x|^k}} + s\, |x|^k \, \mathbf{1}_{e^{-|x|^k} \le s \le 1} + s(\log s)_+ \, \mathbf{1}_{s \ge 1} \quad \forall \, s \ge 0$$

and the dominated convergence theorem).

Exercise 6.11. *Generalize Theorem 3.8 to the case when* $f_0 \in \mathbf{P}(\mathbb{R}^d) \cap L^1_q(\mathbb{R}^d)$, $q = 2$ *or* $q > 1$, *such that* $H(f_0) < \infty$. *(Hint. Proceed along the same line as in Exercise 6.9).*

Exercise 6.12. *Generalize Theorem 3.5 and Theorem 3.8 to the case of a super-harmonic potential* $V(x) = \langle x\rangle^\alpha/\alpha$, $\alpha \ge 2$, *and to an initial datum* $\varphi \in \mathbf{P}(\mathbb{R}^d) \cap L^1_2(\mathbb{R}^d)$ *such that* $H(\varphi) < \infty$.

Exercise 6.13. *Establish Theorem 4.1 in any dimension* $d \ge 1$.

7. Bibliographic discussion

The Nash inequality and its application to the heat equation are due to Nash [12]. The Poincaré inequality for gaussian measure can be proved thanks to the help to Hermit polynomial and it is quite hold (see again [12] for instance). The proof we present here is based on the use of Lyapunov fiunction and it is picked up from [2]. The strong version of the Poincaré inequality belongs to folklore. The logarithmic Sobolev inequality is due to Stam [15], Blachman [4] and rediscoved by Gross [8]. It is related to the hypercontractivity property of Nelson [13] and the Γ_2 calculus of Bakry and Emery [3]. We follow here the presentation given by Toscani [16]. The Csiszár-Kullback inequality is due to Kullback [10], Pinsker [14] and Csiszár [5]. The proofs we present here are picked up (first proof) from [1] and (second proof) from some notes I read from C. Villani. The fact that the log-Sobolev inequality implies the Poincaré inequality (as stated in Lemma 3.9) is due to Gross [8]. The weighted L^1 convergence presented in Section 4 are taken from recent results due to Gualdani, Mouhot and myself [9, 11]. See also [7] for related previous results. The third proof of the Nash inequality presented in Section 6 is due to Diaconis and Saloff-Coste [6].

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