CHAPTER 6 - MORE ABOUT LONGTIME ASYMPTOTIC : ENTROPY AND POSITIVITY TECHNIQUES

- STILL A DRAFT -

Contents

This chapter is twofold. On the one hand, it is an introduction to entropy (or Lyapunov) methods for general (possibly nonlinear) dynamical system and an illustration on some exemples of evolution PDEs (linear, positivity preserving and mass preserving), namely a general Fokker-Planck model, a scattering (or linear Boltzmann) equation. On the other hand it is an introduction of the analysis of stochastic semigroup following Harris-Meyn-Tweedie type approach. The aim is thus to develop some quite general tools which make possible to get a better understanding of the longtime asymptotic issue.

1. Dynamic system, equilibrium and entropy methods

1.1. Existence of steady states.

Definition 1.1. *We say that $(S_t)_{t\ge 0}$ is a dynamical system (or a continuous (possibly nonlinear) semigroup) on a metric space $(\mathcal{Z}, d)$ if*

(S1) $\forall t \ge 0$, $S_t \in C(\mathcal{Z}, \mathcal{Z})$ (continuously defined on $\mathcal{Z}$);

(S2) $\forall x \in \mathcal{Z}$, $t \mapsto S_t\, x \in C([0,\infty), \mathcal{Z})$ (trajectories are continuous);

(S3) $S_0 = I$; $\forall\, s, t \ge 0$, $S_{t+s} = S_t\, S_s$ (semigroup property).

We say that $\bar z \in \mathcal{Z}$ is invariant (or is a steady state, a stationary point) if $S_t \bar z = \bar z$ for any $t \ge 0$. We denote by $\mathcal{E}$ the set of all steady states,

$$\mathcal{E} := \{y \in \mathcal{Z};\ S_t y = y \ \forall\, t \ge 0\}.$$

We remark that $\mathcal{E}$ is closed by definition ($\mathcal{E} = \cap_{t \ge 0} (S_t - I)^{-1}(\{0\})$).

Theorem 1.2. (Dynamic system and steady state). *Consider a bounded and convex subset $\mathcal{Z}$ of a Banach space X which is sequentially compact when it is endowed with the metric associated to the norm $\|\cdot\|_X$ (strong topology), to the weak topology $\sigma(X, X')$ or to the weak-$\star$ topology $\sigma(X, Y)$, $Y' = X$ (see Section 8.1 for the precise requirement we need on X and $\mathcal{Z}$). Then any dynamical system $(S_t)_{t \ge 0}$ on $\mathcal{Z}$ admits at least one steady state, that is $\mathcal{E} \neq \emptyset$.*

Proof of Theorem 1.2. For any $t > 0$, there exists $z_t \in \mathcal{Z}$ such that $S_t z_t = z_t$ thanks to the Schauder or the Tychonoff point fixed Theorem (see Section 8.1). On the one hand, from the semigroup property *(S3)*

$$S_{i\, 2^{-m}} z_{2^{-n}} = z_{2^{-n}} \quad \text{for any} \quad i, n, m \in \mathbb{N}, \quad m \le n. \tag{1.1}$$

On the other hand, by compactness of $\mathcal{Z}$, we may extract a subsequence $(z_{2^{-n_k}})_k$ which converges weakly to a limit $\bar z \in \mathcal{Z}$. By the continuity assumption *(S1)* on S_t, we may pass to the limit $n_k \to \infty$ in (1.1) and we obtain $S_t\, \bar z = \bar z$ for any dyadic time $t \ge 0$. We conclude that $\bar z$ is a stationary point by the trajectorial continuity assumption *(S2)* on S_t and the density of the dyadic real numbers in the real line. □

1.2. **ω-limit set of trajectories compact dynamical system.** Consider a dynamical system $(S_t)_{t \ge 0}$ on a metric space $(\mathcal{Z}, d)$. For any given $z \in \mathcal{Z}$, we define the associated *omega-limit set* as

$$\omega(z) = \{y \in \mathcal{Z};\ \exists t_n \nearrow \infty \ \text{ et } \ S_{t_n} z \to y\},$$

or equivalently

$$\omega(z) := \bigcap_{T > 0} \omega_T(z), \quad \omega_T(z) := \overline{\{S_t z;\ t \ge T\}}. \tag{1.2}$$

We obviously have

$$\mathcal{E}_z := \{y \in \omega_0(z);\ S_t y = y \ \forall\, t \ge 0\} \subset \omega(z)$$

and $\mathcal{E}_z = \{\bar z\}$ if $S_t z \to \bar z$ when $t \to \infty$.

Theorem 1.3. (Dynamic system and ω-limit set). *Consider a dynamical system $(S_t)_{t \ge 0}$ on a metric space $(\mathcal{Z}, d)$ which trajectories are relatively compact. More precisely, we assume*

(S4) $\omega_0(z)$ is compact for some fixed $z \in \mathcal{Z}$.

Then there hold

(i) $S_t(\omega(z)) = \omega(z)$ $\forall t \ge 0$;

(ii) $\omega(z)$ is a nonempty connected and compact subset of $\mathcal{Z}$. More precisely, any $y \in \omega(z)$ belongs to an eternal trajectory in the sense that there exists $(y_t)_{t \in \mathbb{R}} \subset \omega(z)$ such that $y_0 = y$ and $y_{s+t} = S(s) y_t$ for any $t \in \mathbb{R}$ and any $s \ge 0$;

(iii) $d(S_t\, z, \omega(z)) \to 0$ as $t \to \infty$;

(iv) If furthermore $\omega(z)$ is a discrete set, then $\omega(z)$ is a singleton and $\omega(z) \subset \mathcal{E}_z$. More explicitly, there exists $\bar z \in \mathcal{Z}$ such that $\omega(z) = \{\bar z\} \subset \mathcal{E}_z$ or equivalently such that $S_t z \to \bar z$ as $t \to \infty$.

Proof of Theorem 1.3. (i) On the one hand, for any $y \in \omega(z)$, there exists (t_n) such that $S_{t_n} z \to y$, so that $S_{t_n + t} z \to S_t y$ and $S_t y \in \omega(z)$. That proves $S_t(\omega(z)) \subset \omega(z)$. On the other hand, given $y \in \omega(z)$ and $t_n \to \infty$ such that $S_{t_n} z \to y$, there exists $w \in \mathcal{Z}$ and a subsequence $(t_{n'})$ such that $S_{t_{n'} - t} z \to w$ because of assumption *(S4)*, and then $w \in \omega(z)$. We deduce

$$S_t w = S_t(\lim S_{t_{n'} - t} z) = \lim S_{t_{n'}} z = y.$$

That proves the reverse inclusion $\omega(z) \subset S_t(\omega(z))$.

(ii) For any $n \ge 0$, the set $\omega_n(z)$ is a nonempty connected and compact subset of $\mathcal{Z}$ by assumption *(S4)*. The sequence $(\omega_n(z))$ being decreasing, we have $\omega(z) = \lim \omega_n(z)$ which is nothing but (1.2) and thus (ii). More precisely, consider $y \in \omega(z)$ and (t_n) such that $S(t_n) z \to y$. For any $t \in \mathbb{R}_-$, we may extract a subsequence of $S(t_n + t) z$ which converges to a limit y_t. Better, thanks to Cantor's diagonal process, there exists one subsequence (t_{n_k}) such that for any $t \in \mathbb{Z}_-$ there holds $S(t_{n_k} + t)z \to y_t$ and next, for any $t \in \mathbb{R}_-$,

$$S(t_{n_k} + t)z = S(-[t] + 1 + t)S(t_{n_k} + [t] - 1)z \to S(-[t] + 1 - t)y_{[t]-1} =: y_t.$$

As a consequence, $y_t \in \omega(z)$, $y_0 = y$ and $y_{t+s} = \lim S(t_{n_k} + s + t)z = \lim S(s)S(t_{n_k} + t)z = S(s)y_t$ for any $t \in \mathbb{R}$ and $s \in \mathbb{R}_-$.

(iii) We argue by contradiction. Assume that there exist a sequence $t_n \to \infty$ and a real number $\epsilon > 0$ such that $d(S_{t_n} z, \omega(z)) \ge \epsilon$. From assumption *(S4)*, there exists a subsequence $(t_{n'})$ such that $S_{t_{n'}} z \to w \in \omega(z)$ and then $d(S_{t_{n'}} z, \omega(z)) \to 0$, which is absurd.

(iv) First, $\omega(z)$ is a singleton as a discrete and connected nonempty set, we then have $\omega = \{\bar z\}$. Next, by uniqueness of the possible limits, we deduce $S_t z \to \bar z$ as $t \to \infty$. □

1.3. **Dissipation of entropy method.** Consider a dynamical system $(S_t)_{t\ge 0}$ on a metric space $(\mathcal{Z}, d)$. We say that a functional $\mathcal{H} : \mathcal{Z} \to \mathbb{R}$ is an entropy if there exists a *dissipation of entropy* functional $\mathcal{D} : \mathcal{Z} \to \mathbb{R}_+$ such that for any $z \in \mathcal{Z}$ there holds

$$\frac{d}{dt}\mathcal{H}(S_t z) = -\mathcal{D}(S_t z) \le 0 \quad \forall\, t > 0,$$

or equivalently

$$\mathcal{H}(S_t z) + \int_0^t \mathcal{D}(S_s z)\, ds = \mathcal{H}(z). \tag{1.3}$$

As a consequence $t \mapsto \mathcal{H}(S_t z)$ is a decreasing function, and more importantly here, under the additional lower bound assumption

$$\mathcal{H}_z > -\infty, \quad \mathcal{H}_z := \inf_{y \in \omega_0(z)} \mathcal{H}(y), \tag{1.4}$$

there holds

$$\int_0^\infty \mathcal{D}(S_s z)\, ds \le \mathcal{H}(z) - \mathcal{H}_z < \infty. \tag{1.5}$$

We define

$$\omega_{\mathcal{D}}(z) := \{y \in \omega_0(z);\ \mathcal{D}(S_t y) = 0\ \forall\, t \ge 0\},$$

and we observe that $\mathcal{E}_z \subset \omega_{\mathcal{D}}(z)$ at least when (1.3) holds. (not clear ?)

Theorem 1.4. (Dissipation of entropy method - weak version). *Consider a dynamical system $(S_t)_{t\ge0}$ on a metric space $(\mathcal{Z}, d)$ and $z \in \mathcal{Z}$. We assume*

(S4′) $(S_t z)_{t\ge0}$ is "locally uniformly compact" in the sense that $(\mathcal{S}^{z,T}_t)_{t\ge0}$ is relatively compact in $C([0,T]; \mathcal{Z})$ for any fixed time $T \in \mathbb{R}_+$, where we have defined $s \mapsto \mathcal{S}^{z,T}_t(s) := S_{t+s} z$;

(H1) there exists a lsc dissipation of entropy functional $\mathcal{D}$ on $\mathcal{Z}$ such that $t \mapsto \mathcal{D}(S_t z) \in L^1$.

Then, we have $\omega(z) \subset \omega_{\mathcal{D}}(z)$, and therefore $d(S_t z, \omega_{\mathcal{D}}(z)) \to 0$ as $t \to \infty$.

Proof of Theorem 1.4. We define $z^t := \mathcal{S}^{z,T}_t \in C([0,T]; \mathcal{Z})$, $T > 0$, and we observe that

$$\int_0^T \mathcal{D}(z^t(s))\, ds = \int_t^{t+T} \mathcal{D}(S_s z)\, ds \le \int_t^\infty \mathcal{D}(S_s z)\, ds.$$

Consider $y \in \omega(z)$ and a sequence $t_n \to \infty$ such that $S_{t_n} z \to y$ as $n \to \infty$. From the compactness assumption *(S4′)* and a diagonal Cantor procedure, there exist a subsequence $(t_{n'})$ and a function $z^* \in C([0,\infty); \mathcal{Z})$ such that $z^{t_{n'}} \to z^*$ in $C([0,T]; \mathcal{Z})$ for any $T > 0$ and obviously $z^*(s) = S_s y$ for any $s \ge 0$. From the assumptions *(H1)* made on the dissipation of entropy and the above inequality, we then deduce

$$\int_0^T \mathcal{D}(z^*(s))\, ds \le \liminf_{n'\to\infty} \int_{t_{n'}}^\infty \mathcal{D}(S_s z)\, ds = 0.$$

As a consequence $\mathcal{D}(z^*(s)) = 0$ for any $s \ge 0$ and then $y \in \omega_{\mathcal{D}}(z)$. We conclude thanks to (iii) in Theorem 1.3. □

Exercise 1.5. *Assume furthermore that $\omega_{\mathcal{D}}(z) = \{\bar z\}$. Taking up again the proof of Theorem 1.4, prove directly (without using Theorem 1.3 nor Theorem 1.6) that $\omega(z) = \{\bar z\} \subset \mathcal{E}$.*

Theorem 1.6. (Dissipation of entropy method - strong version). *We assume furthermore that*

$$\omega_{\mathcal{D}}(z) \quad \text{is discrete.} \tag{1.6}$$

Then, $\omega(z)$ is a singleton and $\omega(z) \subset \mathcal{E}_z$. More explicitly, we have $\omega(z) = \{z^*\} \subset \mathcal{E}_z \cap \omega_{\mathcal{D}}(z)$ for some $z^* \in \mathcal{Z}$ or equivalently $S_t z \to z^*$ as $t \to \infty$.

Proof of Theorem 1.6. From Theorem 1.4 we have $\omega(z) \subset \omega_{\mathcal{D}}(z)$ which is assumed to be discrete. We conclude thanks to (iv) in Theorem 1.3. □

1.4. **Lyapunov functional and La Salle invariance principle.**

Definition 1.7. *Consider a dynamical system $(S_t)_{t\ge 0}$ on a metric space $(\mathcal{Z}, d)$.*
- We say that $\mathcal{H}$ is a Lyapunov functional if $\mathcal{H} \in C(\mathcal{Z}, \mathbf{R})$ and $t \mapsto \mathcal{H}(S_t z)$ is decreasing.
- For a given $z \in \mathcal{Z}$ we recall that $\mathcal{H}_z$ is defined in (1.4) and we define

$$\omega_{\mathcal{H}}(z) := \{y \in \omega_0(z);\ \mathcal{H}(S_t y) = \mathcal{H}_z \ \forall\, t \ge 0\}.$$

Theorem 1.8. (La Salle invariance principle). *Consider a dynamical system $(S_t)_{t\ge 0}$ on a metric space $(\mathcal{Z}, d)$ and $z \in \mathcal{Z}$. Assuming that*

(S4) $(S_t z)_{t\ge 0}$ is relatively compact;

(H2) $\mathcal{H}$ is a Lyapunov functional;

there holds $\omega(z) \subset \omega_{\mathcal{H}}(z)$, and more precisely

$$\mathcal{H}_z \in \mathbb{R}, \quad \mathcal{H}(S_t z) \searrow \mathcal{H}_z \text{ as } t \to \infty \quad \text{and} \quad d(S_t z, \omega_{\mathcal{H}}(z)) \to 0 \text{ as } t \to \infty.$$

Proof of Theorem 1.8. On the one hand, $\mathcal{H}(S_t z)$ is decreasing so that $\lim \mathcal{H}(S_t z) = \mathcal{H}_z$ and bounded (because the trajectories are relatively compact) so that $\mathcal{H}_z \in \mathbb{R}$. On the other hand, for any $y \in \omega(z)$ there exists $t_n \to \infty$ such that $S_{t_n} z \to y$ which in turns implies $\mathcal{H}_z = \lim \mathcal{H}(S_{t_n+s} z) = \mathcal{H}(\lim S_{t_n+s} z) = \mathcal{H}(S_s y)$ for any $s \ge 0$. In other words, we have $\omega(z) \subset \omega_{\mathcal{H}}(z)$ and the second convergence result is a consequence of (iii) in Theorem 1.3. □

We immediately deduce

Theorem 1.9. (Lyapunov method). *Assuming furthermore that*

$$\omega_{\mathcal{H}}(z) \quad \text{is discrete,}$$

there holds $\omega(z) = \{z^*\}$ for some $z^* \in \mathcal{E}_z$, or equivalently $S_t\, z \to z^*$ as $t \to \infty$.

Proof of Theorem 1.9. Since then $\omega(z) \subset \omega_{\mathcal{H}}(z)$ is discrete, we may use (iv) in Theorem 1.3 and conclude. □

1.5. **Discussions on the entropy methods.** For the sake of simplicity, consider here the situation when the semigroup (S_t) is (formally) associated to an (abstract) evolution equation

$$\frac{d}{dt} z_t = \mathcal{Q}(z_t) \text{ on } (0, \infty), \quad z_0 \in \mathcal{Z}. \tag{1.7}$$

More precisely, we assume that for any $z_0 \in \mathcal{Z}$ there exists a unique solution $z_t \in C([0,\infty); \mathcal{Z})$ to the equation (1.7), and for any $z \in \mathcal{Z}$ we set $S_t z = z_t$ where z_t is the solution to (1.7) associated to the initial datum $z_0 = z$. We may observe that

$$\mathcal{E}_z = \{y \in \omega(z);\ Q(y) = 0\}.$$

For any function $\mathcal{H} : \mathcal{Z} \to \mathbb{R}$, we have (formally)

$$\frac{d}{dt} \mathcal{H}(S_t z) = \mathcal{H}'(z_t) \cdot \frac{d}{dt} z_t = \mathcal{H}'(z_t) \cdot \mathcal{Q}(z_t).$$

The condition

$$\forall\, z \in \mathcal{Z} \quad \mathcal{D}(z) := -\mathcal{H}'(z) \cdot \mathcal{Q}(z) \ge 0$$

then (formally) guaranties that the functional $\mathcal{H}$ is an entropy (decreases along trajectories) and $\mathcal{D}$ is a *dissipation of entropy functional.*

In the two entropy methods and for a given metric space $(\mathcal{Z}, d)$, the compactness condition $(S4')$ is clearly stronger than the condition $(S4)$. It is however not difficult to deduce $(S4')$ from $(S4)$ for an evolution equation in the applications we have in mind.

The first main difference between the two entropy methods lies on the fact that we assume that
- $\mathcal{D}$ is lower semicontinuous in the first method;
- $\mathcal{H}$ is continuous in the second method.

In many applications, the lower semicontinuity condition on $\mathcal{D}$ is easier to prove than the continuity condition on $\mathcal{H}$.

More importantly, the decreasing condition on $\mathcal{H}$ is obtained by writing the identity (1.3) while the integrability condition (1.5) is a consequence of the mere inequality

$$\mathcal{H}(S_t z) + \int_0^t \mathcal{D}(S_s z)\, ds \le \mathcal{H}(z) \quad \forall t \ge 0. \tag{1.8}$$

Again, that last inequality is easier to obtain than the identity (1.3): in many cases it can be proved by an approximation procedure and using the fact that both $\mathcal{H}$ and $\mathcal{D}$ are lower semicontinuous.

Let us then discuss the accuracy of the two methods. For that purpose we introduce the subsets

$$\mathcal{E}_{\mathcal{H}}(z) := \{y \in \omega_0(z);\ \mathcal{H}(y) = \mathcal{H}_z\}, \quad \mathcal{E}_{\mathcal{D}}(z) := \{y \in \omega_0(z);\ \mathcal{D}(y) = 0\}$$

which are defined through "a stationary formulation" (they are not related to the semigroup or the evolutionary problem). We easily check the following inclusions

$$\mathcal{E}_z \subset \omega(z) \subset \omega_{\mathcal{H}}(z) = \mathcal{E}_{\mathcal{H}}(z) \subset \omega_{\mathcal{D}}(z) \subset \mathcal{E}_{\mathcal{D}}(z)$$

from the convergence theorems proved before and thanks to the inequality (1.8). We deduce that the conclusion in Theorem 1.8 is a bit stronger than in Theorem 1.4 because the target set is in general smaller and because of the additional convergence of the entropy functional. However, in the case the identity (1.3) holds, we have $\omega_{\mathcal{H}}(z) = \omega_{\mathcal{D}}(z)$ and the target sets are the same in both theorems. In practice, in order to identify the possible limit set, we try to characterize the set $\omega_{\mathcal{D}}(z)$ or the set

$$\{y \in \omega_0(z);\ \mathcal{H}'(y) \perp \omega_0(z)\}$$

which clearly contains the set $\mathcal{E}_{\mathcal{H}}(z)$.

The above entropy methods are quite general and efficient. The shortcoming of the method is that it does not give any rate of convergence to the stationary state. In order to overcome that lack of convergence, the usual strategy is to try to prove some functional inequality of the kind

$$\mathcal{D}(y) \ge \Theta(\mathcal{H}(y|\bar z)), \quad \mathcal{H}(y|\bar z) := \mathcal{H}(y) - \mathcal{H}(\bar z), \quad \mathcal{H}(\bar z) = \mathcal{H}_z,$$

for some function $\Theta \in C^1(\mathbb{R};\mathbb{R})$, $\Theta(s) > 0$ for $s \neq 0$, $\Theta(0) = 0$. But that is another story ... .

2. Relative entropy for linear and positive PDE

We consider the general evolution PDE

$$\partial_t f = \Delta f - a\cdot\nabla f + cf + \int b\, f_*, \quad \int b\, f_* := \int b(x,x_*) f(x_*)\, dx_*, \quad b \ge 0. \tag{2.1}$$

If $g > 0$ is a solution

$$\partial_t g = \Delta g - a\cdot\nabla g + cg + \int b\, g_*$$

and if $\phi \ge 0$ is a solution to the dual evolution problem

$$-\partial_t \phi = \Delta\phi + \mathrm{div}(a\,\phi) + c\,\phi + \int b_*\,\phi_*, \quad \int b_*\,\phi_* := \int b(x_*,x)\,\phi(x_*)\, dx_*,$$

we can exhibit a family of entropies associated to the evolution PDE (2.1). More precisely, we establish the following result (and in fact a bit more accurate formulation of it).

Theorem 2.1. *For any real values convex function H, the functional*

$$f \mapsto \mathcal{H}(f) := \int_{\mathbb{R}^d} H(f/g)\, g\,\phi,$$

is an entropy for the evolution PDE (2.1).

Step 1. First order PDE. We assume that

$$\begin{aligned} \partial_t f &= -a\cdot\nabla f + cf \\ \partial_t g &= -a\cdot\nabla g + cg \\ -\partial_t \phi &= \mathrm{div}(a\,\phi) + c\,\phi, \end{aligned}$$

and we show that

$$\partial_t(H(X)g\phi) + \mathrm{div}(aH(X)g\phi) = 0, \quad X = f/g.$$

We compute

$$\begin{aligned} &\partial_t(H(X)g\phi) + \mathrm{div}(aH(X)g\phi) \\ &= H'(X)g\phi\,[\partial_t X + a\nabla X] + H(x)\,[\partial_t(g\phi) + \mathrm{div}(ag\phi)] \end{aligned}$$

The first term vanishes because

$$\partial_t X + a\nabla X = \frac{1}{g}\,(\partial_t f + a\nabla f) - \frac{f}{g^2}\,(\partial_t g + a\nabla g) = \frac{1}{g}\,(cf) - \frac{f}{g^2}\,(cg) = 0.$$

The second term also vanishes because

$$\partial_t(g\phi) + \mathrm{div}(ag\phi) = \phi\,[\partial_t g + a\nabla g] + g\,[\partial_t\phi + \mathrm{div}(a\phi)] = \phi\,[-cg] + g\,[+c\phi] = 0.$$

Step 2. Second order PDE. We assume that

$$\begin{aligned} \partial_t f &= \Delta f + cf \\ \partial_t g &= \Delta g + cg \\ -\partial_t \phi &= \Delta\phi + c\,\phi, \end{aligned}$$

and we show

$$\partial_t(H(X)g\phi) - \mathrm{div}(\phi\nabla(H(X)g)) + \mathrm{div}(gH(X)\nabla\phi) = -H''(X)g\phi|\nabla X|^2.$$

We first observe that

$$\begin{aligned} \Delta X &= \mathrm{div}\Big(\frac{\nabla f}{g} - f\,\frac{1}{g^2}\,\nabla g\Big) \\ &= \frac{\Delta f}{g} - 2\nabla f\,\frac{\nabla g}{g^2} + 2\,f\,\frac{|\nabla g|^2}{g^3} - \frac{f}{g^2}\,\Delta g \\ &= \frac{\Delta f}{g} - \frac{f\,\Delta g}{g^2} - 2\,\frac{\nabla g}{g}\cdot\nabla X, \end{aligned}$$

which in turn implies

$$\partial_t X - \Delta X = 2\,\frac{\nabla g}{g}\cdot\nabla X.$$

We then compute

$$\begin{aligned} &\partial_t(H(X)g\phi) - \mathrm{div}(\phi\nabla(H(X)g)) + \mathrm{div}(gH(X)\nabla\phi) = \\ &= (\partial_t H(X))\,g\phi + H(X)\,\partial_t(g\phi) - \phi\,\mathrm{div}[gH'(X)\nabla X + H(X)\nabla g] + gH(X)\Delta\phi \\ &= H'(X)g\phi\,\big\{\partial_t X - \Delta X - 2\frac{\nabla g}{g}\cdot\nabla X\big\} - g\phi\,H''(X)\,|\nabla X|^2 + H(X)\,[\partial_t(g\phi) - \phi\Delta g + g\Delta\phi] \\ &= -g\phi\,H''(X)\,|\nabla X|^2, \end{aligned}$$

since the first term and the last term independently vanish.

Step 3. Integral equation. We assume that

$$\begin{aligned} \partial_t f &= cf + \int bf_* \\ \partial_t g &= cg + \int bg_* \\ -\partial_t \phi &= c\,\phi + \int b_*\phi_*, \end{aligned}$$

with the notations

$$\int b\psi_* := \int b(x,x_*)\,\psi(x_*)\,dx_*, \qquad \int b_*\psi_* := \int b(x_*,x)\,\psi(x_*)\,dx_*,$$

and we show

$$\partial_t(H(X)g\phi) + \int H(X)gb_*\phi_* - \int bH(X_*)g_*\phi = -\int bg_*\phi\Big\{H(X_*) - H(X) - H'(X)(X_* - X)\Big\}$$

We compute indeed

$$\begin{aligned}\partial_t(g\phi H(X)) &= H(X)g\partial_t\phi + H(X)\phi\partial_t g + H'(X)\phi(\partial_t f - X\partial_t g)\\ &= -\int H(X)gb_*\phi_* + \int bH(X_*)g_*\phi\\ &\quad + \int bg_*\phi\Big\{-H(X_*) + H(X) + H'(X)X_* - H'(X)X\Big\}\end{aligned}$$

Step 4. Conclusion. For any solutions (f, g, ϕ) to the system of (full) equations, we have summing up the three computations

$$\begin{aligned}&\partial_t(g\phi H(X)) +\\ &+\mathrm{div}(aH(X)g\phi) - \mathrm{div}(\phi\nabla(H(X)g)) + \mathrm{div}(gH(X)\nabla\phi) + \int bH(X_*)g_*\phi - \int H(X)gb_*\phi_*\\ &= -g\phi\, H''(X)\,|\nabla X|^2 - \int bg_*\phi\Big\{H(X_*) - H(X) - H'(X)(X_* - X)\Big\}.\end{aligned}$$

Since when we integrate in the x variable the term on the second line vanishes, we find out

$$\frac{d}{dt}\mathcal{H}(f) = -D_{\mathcal{H}}(f),$$

with

$$D_{\mathcal{H}}(f) := \int g\phi\, H''(X)\,|\nabla X|^2 + \iint bg_*\phi\Big\{H(X_*) - H(X) - H'(X)(X_* - X)\Big\} \ge 0.$$

Exercise 2.2. *We consider a semigroup $S_t = e^{tL}$ of linear and bounded operators on L^1 and we assume that*
(i) $S_t \ge 0$;
(ii) $\exists\, g > 0$ such that $Lg = 0$, or equivalently $S_t g = g$ for any $t \ge 0$;
(iii) $\exists\, \phi \ge 0$ such that $L^*\phi = 0$, or equivalently $\langle S_t h, \phi\rangle = \langle h, \phi\rangle$ for any $h \in L^1$ and $t \ge 0$.

Our aim is to generalize to that a bit more general (and abstract) framework the general relative entropy principle we have presented for the evolution PDE (2.1).

(a) Prove that for any real affine function ℓ, there holds $\ell[(S_t f)/g]g = S_t[\ell(f/g)g]$.

(b) Prove that for any convex function H and any f, there holds $H[(S_t f)/g]g \le S_t[H(f/g)g]$. (Hint. Use the fact that $H = \sup_{\ell \le H} \ell$).

(c) Deduce that

$$\int H[(S_t f)/g]g\phi \le \int H[f/g]g\phi, \quad \forall\, t \ge 0.$$

3. First example: a general Fokker-Planck equation

In this section we consider the Fokker-Planck equation

$$\partial_t f = \mathcal{L}f = \Delta f + \mathrm{div}(Ef), \tag{3.2}$$

on the density $f = f(t, x)$, $t \ge 0$, $x \in \mathbb{R}^d$, where the force field $E \in \mathbb{R}^d$ is a given fixed (exterior) vectors field or is a function of the density.

3.1. **Conservation, explicit steady states and self-adjointness property.** Any solution f to the Fokker-Planck equation (3.2) is mass conservative in the sense that

$$\frac{d}{dt}\int f\,dx = \int \mathrm{div}(\nabla f + Ef)\,dx = 0,$$

because of the divergence structure of the Fokker-Planck operator $\mathcal{L}$ and the Stokes formula.

In the case when

$$E = \nabla U + E_0, \quad \mathrm{div}(E_0\, e^{-U}) = 0, \tag{3.3}$$

for a confinement potential $U : \mathbb{R}^d \to \mathbb{R}$ and a non gradient force field perturbation $E_0 : \mathbb{R}^d \to \mathbb{R}^d$, we may observe that the positive function $e^{-U(x)+U_0}$ is a stationary state for any $U_0 \in \mathbb{R}$. When furthermore $e^{-U(x)} \in L^1(\mathbb{R}^d)$, we may fix $U_0 \in \mathbb{R}$ such that

$$G(x) := e^{-U(x)+U_0} \text{ is a stationary state and a probability.}$$

On the other way round, in the most general case we just assume there exists a steady state

$$G \in L^1(\mathbb{R}^d) \cap \mathbb{P}(\mathbb{R}^d), \quad \mathrm{div}(\nabla G + E\,G) = 0,$$

where $\mathbb{P}(\mathbb{R}^d)$ stands for the set of probability measures, we may observe that $G \in C^1(\mathbb{R}^d)$ thanks to a bootstrap regularization argument and $G > 0$ thanks to the strong maximum principle. Then we define $U := -\log G$ and $E_0 := E - \nabla U$, so that (3.3) holds again.

Consider a weight function $m : \mathbb{R}^d \to \mathbb{R}_+$ and the associated Lebesgue space $L^2(m)$ with $\|f\|_{L^2(m)} := \|fm\|_{L^2}$. For $f, g \in \mathcal{D}(\mathbb{R}^d)$, we compute

$$\begin{aligned} I &:= (\mathcal{L}f, g)_{L^2(m)} - (f, \mathcal{L}g)_{L^2(m)} \\ &= \int (\Delta f + \mathrm{div}(E\,f))\, g\, m^2 - \int (\Delta g + \mathrm{div}(E\,g))\, f\, m^2 \\ &= \int g\, m^2\, E \cdot \nabla f - g\, \nabla f \cdot \nabla m^2 + \int f\, \nabla g \cdot \nabla m^2 - f\, m^2\, E \cdot \nabla g \\ &= 2 \int g\, (m^2\, E - \nabla m^2) \cdot \nabla f + \int g f (\mathrm{div}(m^2 E) - \Delta m^2). \end{aligned} \tag{3.4}$$

In the one hand, if $I(f, g) = 0$ for any f, g, by choosing f as a constant function, we get

$$0 = \int g(\Delta m^2 - \mathrm{div}(m^2 E))$$

for any g, and then

$$\Delta m^2 - \mathrm{div}(m^2 E) = 0.$$

Plugging that information into (3.4), we get

$$I = 2 \int g\, (m^2\, E - \nabla m^2) \cdot \nabla f,$$

and the equation $I(f, g) = 0$ for any f, g, by choosing $f = x_i$, implies

$$\int g(\partial_i m^2 - m^2 E_i) = 0,$$

for any g. We deduce

$$\partial_i m^2 - m^2 E_i = 0$$

and then $E = \nabla U$ with $U := \log(m^2)$ or equivalently $m = e^{U/2}$. In other words, we just have proved that $\mathcal{L}$ is a self-adjoint operator in the Hilbert space $L^2(m)$ if and only if $E = \nabla U$ and $m = e^{U/2}$ for some confinement potential $U : \mathbb{R}^d \to \mathbb{R}$. In that case, $G = \exp(-U - U_0)$, $U_0 \in \mathbb{R}$, is the family of steady states.

3.2. **General a priori estimates and well-posedness issue.**

Lemma 3.3. *For any $f \in \mathcal{D}(\mathbb{R}^d)$ and any weight function $m : \mathbb{R}^d \to \mathbb{R}_+$, we have*

$$\int (\mathcal{L}f) f^{p-1} m^p = -(p-1) \int |\nabla f|^2 f^{p-2} m^p + \int f^p m^p \psi_1$$

with

$$\psi_1 := (p-1) \frac{|\nabla m|^2}{m^2} + \frac{\Delta m}{m} + \left(1 - \frac{1}{p}\right) div\, E - E \cdot \frac{\nabla m}{m}.$$

Proof of Lemma 3.3. It is a good exercise! Just perform two integrations by part: one on the term which involves the Laplacian, another on the term which involves the $E \cdot \nabla f$ function. □

Observe that (at least formally):

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^d} |f|^p m^p &= \frac{p}{2} \int_{\mathbb{R}^d} (|f|^2)^{p/2-1} \partial_t (f \bar{f}) m^p \\ &= \frac{p}{2} \int_{\mathbb{R}^d} |f|^{p-2} (\mathcal{L}f \, \bar{f} + f \, \bar{\mathcal{L}} f) m^p, \end{aligned}$$

so that defining $f^* := \|f\|^{2-p}_{L^p(m)} \, \bar{f} \, |f|^{p-2}$, we get

$$\begin{aligned} \frac{d}{dt} \|f\|^2_{L^p(m)} &= \frac{2}{p} (\|f\|^p_{L^p(m)})^{2/p-1} \frac{d}{dt} \|f\|^p_{L^p(m)} = \int_{\mathbb{R}^d} (\mathcal{L}f \, f^* + \bar{f}^* \, \mathcal{L}f) m^p \\ &= 2 \, \Re e \langle \mathcal{L}f, f^* \rangle. \end{aligned} \tag{3.5}$$

As a consequence, (3.5) together with Lemma 3.3 lead to some differential inequality on the L^p-norm which provides an a priori estimate on a solution of (3.2) when the function ψ_1 in Lemma 3.3 is uniformly bounded above.

Exercise 3.4. *(a) Generalize Lemma 3.3 to the case of a complex valued function $f \in \mathcal{D}(\mathbb{R}^d; \mathbb{C})$.*
(b) For any $f > 0$, prove that (at least formally)

$$\int (\mathcal{L}f) \log f = -4 \int |\nabla \sqrt{f}|^2 + \int (div\, E) \, f.$$

As a consequence of the previous identity we obtain several existence results. In the sequel we assume either

$$E = E(t,x) \in L^\infty(0,T; W^{1,\infty}(\mathbb{R}^d)), \tag{3.6}$$

or that $E = E(x) \in W^{1,\infty}_{loc}$ and, for some $\gamma \geq 2$,

$$|E(x)| \leq K_1 \langle x \rangle^{\gamma-1}, \quad |\mathrm{div} E(x)| \leq K_2 \langle x \rangle^{\gamma-2}, \quad E(x) \cdot x \geq |x|^\gamma \quad \forall x \in \mathbb{R}^d. \tag{3.7}$$

We define

$$H := L^2(m), \quad V := H^1(m) \cap L^2(m_1) \tag{3.8}$$

with $m = m_1 = \langle x \rangle^k$, $k \geq 0$, in the first case, and with $m := e^{\kappa \langle x \rangle^\gamma}$, $m_1 := \langle x \rangle^{\gamma-1} e^{\kappa \langle x \rangle^\gamma}$, $\kappa := \gamma/4$, in the second case. We next define

$$X_T := C([0,T]; H) \cap L^2(0,T; V).$$

Proposition 3.5. *For any $f_0 \in H$, there exists a unique variational solution $f \in X_T$ to the Fokker-Planck equation (3.2). Moreover, if $f_0 \geq 0$ then $f(t) \geq 0$ for any $t \geq 0$; if $f_0 \in L^1$ then $f(t) \in L^1$ and $\langle f(t) \rangle = \langle f_0 \rangle$ for any $t \geq 0$.*

Proof of Proposition 3.5. We observe that the (possibly time dependent) bilinear form

$$\begin{aligned} a(t,f,g) &:= -\int \mathcal{L}(t) f \, g \, m^2 \\ &= \int \{\nabla f \cdot \nabla g \, m^2 - \nabla f \cdot (\nabla m^2 + E m^2) \, g - \mathrm{div} E \, f \, g \, m^2\} \, dx \end{aligned}$$

is continuous in V. Moreover, thanks to Lemma 3.3, it satisfies the following coercivity lower bound

$$a(t,f,f) = \int |\nabla f|^2 \, m^2 + \int |f|^2 \; m^2 \, \psi_1$$

with

$$\psi_1 = -\frac{k^2}{\langle x\rangle^2} - \frac{k(k-1)}{\langle x\rangle^2} - \frac{1}{2}\mathrm{div}E + k\,E\cdot\frac{x}{\langle x\rangle^2} \ge C,$$

$C \in \mathbb{R}$, in the first case, and

$$\begin{aligned}\psi_1 &= -\frac{1}{16}|x|^2\langle x\rangle^{2\gamma-4} - \frac{d}{4}\langle x\rangle^{\gamma-2} - \frac{\gamma-2}{4}|x|^2\langle x\rangle^{\gamma-4} - \frac{1}{16}|x|^2\langle x\rangle^{2\gamma-4}\\ &\quad -\frac{1}{2}\mathrm{div}E + \frac{1}{4}E\cdot x\,\langle x\rangle^{\gamma-2} \ge \frac{1}{8}\langle x\rangle^{2\gamma-2} + C,\end{aligned}$$

$C \in \mathbb{R}$, in the second case. We conclude to the existence and the uniqueness of a variational solution $f \in X_T$ by applying Lions' Theorem 3.2 in Chapter 1. □

Proposition 3.6. *Assume $a \in W^{1,\infty}\cap L^2$. For any $f_0 \in L^2_k$, $k > d/2$, there exists a unique solution*

$$f \in C([0,T); L^2_k) \cap L^2(0,T; H^1_k), \quad \forall T > 0,$$

to the nonlinear Fokker-Planck equation

$$\partial_t f = \Delta f + div((a * f)f). \tag{3.9}$$

Proof of Proposition 3.6. Step 1. A priori bounds. On the one hand, we clearly have

$$\int_{\mathbb{R}^d} |f|\,dx \le \int_{\mathbb{R}^d} |f_0|\,dx,$$

and then

$$\begin{aligned}\frac{d}{dt}\int f^2\frac{\langle x\rangle^{2k}}{2} &= -\int |\nabla f|^2\langle x\rangle^{2k}\\ &\quad + \int f^2\langle x\rangle^{2k}\{\frac{k^2}{\langle x\rangle^2} + \frac{k(k-1)}{\langle x\rangle^2} + \frac{1}{2}(\mathrm{div}a)*f + k\,(a*f)\cdot\frac{x}{\langle x\rangle^2}\}\\ &\le -\int |\nabla f|^2\langle x\rangle^{2k}\\ &\quad + \{2k^2 + \frac{1}{2}\|\nabla a\|_{L^\infty}\|f_0\|_{L^1} + k\,\|a\|_{L^\infty}\|f_0\|_{L^1}\}\int f^2\langle x\rangle^{2k}.\end{aligned}$$

Step 2. Existence. To prove the existence we consider the mapping $g \mapsto f$ defined for $g \in C([0,T]; L^2_k)$, $k > d/2$, so that $L^2_k \subset L^1$, by solving the linear evolution PDE

$$\partial_t f = \Delta f + \mathrm{div}((a*g)f).$$

For the linear (and g dependent) problem, by repeating the same computations as in step 1, we also have

$$\sup_{[0,T]}\|f\|_{L^1} \le \|f_0\|_{L^1}, \quad \sup_{[0,T]}\|f\|_{L^2_k} \le \mathcal{A}_T,$$

where $\mathcal{A}_T$ only depends on $\|f_0\|_{L^2_k}$, k, a and T. We then define

$$\mathcal{C}_T := \{f \in C([0,T]; L^2_k),\ \|f(t)\|_{L^1} \le \|f_0\|_{L^1},\ \|f(t)\|_{L^2_k} \le \mathcal{A}_T\}$$

and we have $\Phi : \mathcal{C}_T \to \mathcal{C}_T$. We consider two solutions

$$\partial_t f_i = \Delta f_i + \mathrm{div}((a*g_i)f_i)$$

so that the differences $f = f_2 - f_1$ and $g := g_2 - g_1$ satisfy

$$\partial_t f = \Delta f + \mathrm{div}((a*g_1)f) + \mathrm{div}((a*g)f_2).$$

As a consequence, using the Young inequality, we have

$$\begin{aligned}\frac{d}{dt}\|f\|^2_{L^2} &= -2\int|\nabla f|^2 + \int(\nabla a*g_1)\,f^2 - 2\int(a*g)f_2\nabla f\\ &\le -\int|\nabla f|^2 + \|\nabla a*g_1\|_{L^\infty}\int f^2 + \|a*g\|^2_{L^\infty}\int f_2^2\\ &\le \|\nabla a\|_{L^\infty}\,\|f_0\|_{L^1}\|f\|^2_{L^2} + 2\,\|a\|_{L^2}\,\mathcal{A}_T^2\,\|g\|^2_{L^2},\end{aligned}$$

from which we deduce

$$\sup_{[0,T]}\|f\|^2_{L^2} \le \varepsilon_T \sup_{[0,T]}\|g\|_{L^2}$$

with $\varepsilon_T \to 0$ as $T \to 0$. We conclude to the existence by a Banach fixed point theorem. □

Exercise 3.7. *Prove that the assumption $a \in L^2$ can be removed by making the contraction argument with the L^2_k norm.*

3.3. **Long-time behaviour.** We briefly discuss the long-time asymptotic for the linear and non-linear Fokker-Planck equations (3.2) and (3.9).

• In the case $E = \nabla U$, $U = \langle x\rangle^\gamma/\gamma$, $\gamma \ge 2$, $\mathcal{L}$ is self-adjoint and dissipative in $L^2(G^{-1/2})$ and the resolvent $R_\mathcal{L}(b)$ is compact, because for $b > 0$ large enough, the bilinear form $a'(f, g) := a(f, g) + b(f, g)$, with a defined in the proof of Proposition 3.5, is coercive in the space $V := H^1(G^{-1/2}) \cap L^2(\langle x\rangle^{\gamma-1}G^{-1/2})$ and V is compactly embedded in $L^2(G^{1/2})$. More precisely, performing just one integration by part on the first (Laplacian) term, we have

$$\begin{aligned} a'(f,f) &= \int [-(\mathcal{L}f)f + b\,f^2]\,G^{-1} \\ &= \int [-\Delta f - \Delta U\, f - \nabla U\cdot\nabla f + bf)\, f e^U \\ &= \int \{|\nabla f|^2 + (b-\Delta U)\, f^2\}\, e^U. \end{aligned}$$

Introducing the notation $g := f\,G^{-1/2}$ and observing that

$$\begin{aligned} \int [-\Delta f]\, f\, e^U &= \int \nabla(g\,G^{1/2})\cdot\nabla(g\,G^{-1/2}) \\ &= \int |\nabla g|^2 + \int g^2\,\nabla G^{1/2}\cdot\nabla G^{-1/2} \\ &= \int |\nabla g|^2 - \frac14\int f^2\,G^{-1}\,|\nabla U|^2, \end{aligned}$$

as well as

$$-\int (\nabla U\cdot\nabla f)\, f e^U = \frac12\int (\Delta U + |\nabla U|^2)\, f^2\, G^{-1},$$

we also have

$$a'(f,f) = \int |\nabla g|^2 + \int f^2\,[b - \frac12\Delta U + \frac14|\nabla U|^2]\, e^U.$$

Gathering these two identities, we deduce that for any $f \in \mathcal{D}(\mathbb{R}^d)$

$$a'(f,f) \ge \frac12\int |\nabla f|^2\, e^U + \int \{b - \frac12\Delta U + \frac18\,|\nabla U|^2)\, f^2\}\, e^U.$$

Observing now that $|\nabla U|^2 \ge |x|^{2(\gamma-1)}$ and $|\nabla U| \le (d+\gamma-2)\,\langle x\rangle^{\gamma-2}$, we obtain by taking $b > 0$ large enough the following lower (coercivity) bound: for any $f \in \mathcal{D}(\mathbb{R}^d)$

$$a'(f,f) \ge \frac12\int |\nabla f|^2\, e^U + \int \{\frac{b}{2} + \frac{1}{16}\,|x|^{2(\gamma-1)})\, f^2\}\, e^U =: \|f\|_V^2.$$

On the other hand, thanks to the first identity in Proposition 3.5, we also have

$$a'(f,g) = \int \{\nabla f\cdot\nabla g - 2\nabla f\cdot\nabla U\, g + (b - \Delta U f\, g\}\, e^U\, dx,$$

and then $|a'(f,g)| \le C\,\|f\|_V\,\|g\|_V$ for any $f, g \in V$. Thanks to the Lax-Milgram Theorem, we deduce that $b \in \rho(\mathcal{L})$. Moreover, for any $g \in H := L^2(G^{-1/2})$, the (variational) solution $f := R_\mathcal{L}(b)f \in V$ to the equation

$$(\mathcal{L} - b)f = g$$

satisfies

$$\|f\|_V^2 \le ((b-\mathcal{L})f, f)_H = (-g, f)_H \le C\,\|g\|_H\,\|f\|_V.$$

As a consequence, $\|R_\mathcal{L}(b)g\|_V \le C\,\|g\|_H$. Because $V \subset H$ is compactly embedded, we get that $R_\mathcal{L}(b)$ is a compact operator. We may apply Theorem **??**: the $L^2(G^{-1/2})$-norm is a Lyapunov functional and any solution converges with exponential rate to the associated equilibrium (uniquely defined thanks to the mass conservation).

• In the case $E = a * f + x$ with $a = \nabla U$, U a convex function, we write

$$\partial_t f = \mathcal{L} f = \mathrm{div}[f \nabla(\log f + |x|^2/2 + U * f)].$$

We define

$$\mathcal{H}(f) := \int_{\mathbb{R}^d} f\{\log f + \frac{1}{2}|x|^2 + \frac{1}{2} U * f\}, \quad \mathcal{D}(f) := \int_{\mathbb{R}^d} f\,|\nabla(\log f + |x|^2/2 + U * f)|^2.$$

We may compute

$$\frac{d}{dt}\mathcal{H}(f) = \int_{\mathbb{R}^d} (\partial_t f)(1 + \log f + |x|^2/2 + U * f) = -\mathcal{D}(f).$$

The functional $\mathcal{H}$ is then an entropy. It is moreover a Lyapunov functional under some additional assumption on U. We accept the following result.

Lemma 3.8. *If U is a convex function then $\mathcal{H}$ is a convex functional and there exists a unique minimizer f_∞ to the minimizing problem*

$$\mathcal{H}(f_\infty) = \min\Big\{\mathcal{H}(f);\ 0 \le f \in L^1(\mathbb{R}^d),\ \int_{\mathbb{R}^d} f\,dx = 1\Big\}.$$

Moreover, f_∞ is smooth and positive, and we have $\mathcal{D}(f) = 0$ implies $f = f_\infty$.

Exercise 3.9. *Show the convergence of the solutions to the unique equilibrium f_∞ by applying Theorem 1.6 or Theorem 1.9.*

• In the case $E = \nabla U + E_0$, $U = \langle x\rangle^\gamma/\gamma$, $\gamma \ge 2$, $\mathrm{div}(E_0\, e^{-U}) = 0$, $E_0 \ne 0$, the operator $\mathcal{L}$ satisfies the same properties as the first case (when $E_0 = 0$) except that $\mathcal{L}$ is not self-adjoint anymore. Because $\mathcal{L}G = 0$ and $\mathcal{L}^*1 = 0$, we may apply the GRE method which readily implies

$$\frac{d}{dt}\mathcal{H}(f) = -\mathcal{D}(f),$$

with

$$\mathcal{H}(f) := \int (f - \langle f_0\rangle G)^2 G^{-1} \quad \text{and} \quad \mathcal{D}(f) = \int |\nabla(f/G)|^2 G.$$

The equation $D(f) = 0$ is equivalent to $f = \langle f\rangle G$ and by conservation of mass $f = \langle f_0\rangle G$. As a consequence, $\mathcal{H}$ is a entropy, $\mathcal{D}$ is a dissipation of entropy functional (it is lsc for the weak L^2 convergence), and then Theorem 1.6 says that $f(t) \rightharpoonup \langle f_0\rangle G$ weakly in L^2 as $t \to \infty$ for any $f_0 \in H := L^2(G^{-1/2})$ (take $\mathcal{Z} := \{g \in H;\ \|g\|_H \le \|f_0\|_H,\ \langle g\rangle = \langle f_0\rangle\}$). In order to enlarge of the class of initial data and to strengthen the sense of convergence we may argue as follows (we present the argument in dimension $d = 1$ for the sake of simplicity of the notation). By developing the term $\mathcal{D}(f)$ or just using Lemma 3.3, we have for any $K > 0$ and for some $K_0 = K_0(\mathcal{H}(f_0), K)$

$$\begin{aligned}\frac{d}{dt}\int f^2 G^{-1} &= -\int (\partial f)^2 G^{-1} + \int f^2 G^{-1}\psi_1 \\ &\le -\int (\partial f)^2 G^{-1} - K\Big(\int f^2 G^{-1}\Big)^2 + K_0,\end{aligned}$$

because $\psi_1 \le C$ and $\mathcal{H}(f) \le \mathcal{H}(f_0)$.
The equation satisfied by ∂f is

$$\partial_t \partial f = \Delta \partial f + \partial(\mathrm{div}E) f + (\mathrm{div}E)\partial f + \partial E \cdot \nabla f + E \cdot \nabla \partial f,$$

from which we deduce for some $\theta \in (0,1)$

$$\begin{aligned}\frac{d}{dt}\int (\partial f)^2 &\le -\int (\partial^2 f)^2 + \int \big\{|D^2 E|\,|f|\,|\nabla f| + \frac{3}{2}\,|\mathrm{div}E|\,|\nabla f|^2\big\} \\ &\le -\Big(\int f^2\Big)^{-1}\Big(\int (\partial f)^2\Big)^2 + \int (C\,f^2 + \theta^{-1}(\partial f)^2) G^{-1}.\end{aligned}$$

We define

$$u := \int f^2 G^{-1} + \theta \int (\partial f)^2,$$

which satisfies the differential ODE

$$\begin{aligned}\frac{du}{dt} &\le -K\Big(\int f^2 G^{-1}\Big)^2 + K_0 - \theta \|f_0\|_H^{-2} \Big(\int (\partial f)^2\Big)^2 + \theta C \, \|f_0\|_H^2 \\ &\le -\theta' u^2 + K_0',\end{aligned}$$

for some constants $\theta', K_0' > 0$, which only depend on $\|f_0\|_H$. Defining $K_1 = K_1(\|f_0\|_H) := K_0'/(2\theta')$ and the set

$$\mathcal{Z}_1 := \big\{ g \in H;\ \|g\|_H \le \|f_0\|_H,\ u[g] \le K_1 \big\},$$

we deduce that if $f_0 \in \mathcal{Z}_1$ then $f(t) \in \mathcal{Z}_1$ for any $t \ge 0$, and on the contrary, defining $\tau := \sup\{t > 0;\ u(t') > K_1\ \forall t' \in [0,t)\}$, we have

$$\frac{du}{dt} \le -\frac{\theta'}{2} u^2 \quad \text{on} \quad (0,\tau).$$

As a consequence, we get $u(t) \le 2/(\theta' t)$ on $(0,\tau)$, so that necessarily $u(t) \le K_1$ for some $t \le 2/(\theta' K) := T$. We have then proved that $\mathcal{Z}_1$ is invariant and attractive in the sense that $f(t) \in \mathcal{Z}_1$ for any $t \ge T$. Because $\mathcal{Z}_1 \subset L^1$ with compact embedding, we deduce from the previous (weak) convergence that $f(t) \to \langle f_0 \rangle G$ strongly in L^1. It is worth emphasizing that we get the same conclusion by using the La Salle invariance principle (Theorem 1.8) by observing that $\mathcal{H}$ is a Lyapunov functional in the set $\mathcal{Z}_1$.

For a general initial datum $f_0 \in L^1$, we use the splitting $f_0 = f_{0,n} + g_{0,n}$ with $f_{0,n} = \mathbf{1}_{B(0,n)}\, f_0 * \rho_n \in H$ for a mollifier sequence (ρ_n). We then have

$$\|f_n(t) - \langle f_{0,n} \rangle G\|_{L^1} \to 0 \text{ as } t \to \infty,\ \forall\, n \ge 0,$$

by the previous analysis, and

$$\sup_{t \ge 0} \|g_n(t)\|_{L^1} \le \|g_{0,n}\| \to 0 \text{ as } n \to \infty.$$

Putting together the two above estimates and the fact that $\langle f_{0,n} \rangle \to \langle f_0 \rangle$ as $n \to \infty$, we conclude to $f(t) \to \langle f_0 \rangle G$ in L^1 as $t \to \infty$.

• In the general case when E satisfies (3.7), we verify that $\mathcal{L} - b$ is dissipative in the space H defined in (3.8) for $b > 0$ large enough and again that $R_{\mathcal{L}}(b)$ is compact. Moreover $\mathcal{L}$ satisfies Kato's inequality and the strong maximum principle as stated and proved bellow. We may then apply the existence result Theorem 1.2 and we obtain that there exists of nonnegative and mass normalized steady state $f_1 \in H$. We conclude by applying the GRE method as in the previous case, and we get again $f(t) \rightharpoonup \langle f_0 \rangle f_1$ weakly in L^2 as $t \to \infty$ for any $f_0 \in H$. We can improve (enlarge and strengthen) the above convergence by following the same argument as in the previous case.

Proposition 3.10. *The operator $\mathcal{L}$ satisfies "Kato's inequalities" and the "strong maximum principle" in H*

Proof of Proposition 3.10. *Step 1. Kato's inequalities.* For a convex function $\beta : \mathbb{R} \to \mathbb{R}$ such that $\beta(s) = s\beta'(s)$, we clearly have

$$\mathcal{L}\beta(f) = \beta''(f)|\nabla f|^2 + \beta'(f)\mathcal{L}f \ge \beta'(f)\mathcal{L}f.$$

For the square of the modulus function $s \in \mathbb{C} \mapsto \theta(s) = |s| = \sqrt{s\bar{s}}$, we have on the one hand

$$\begin{aligned}\mathcal{L}f \cdot \theta'(f) &:= [(\mathcal{L}f)\,\bar{f} + (\mathcal{L}\bar{f})\,f]/(2|f|) \\ &= [(\Delta f)\,\bar{f} + (\Delta \bar{f})\,f]/(2|f|) + \mathrm{div}(E|f|).\end{aligned}$$

On the other hand, introducing the real part R and the imaginary part I in such a way that $f = R + iI$, $R, I \in \mathbb{R}$, we easily compute

$$\begin{aligned}\Delta|f| &= \mathrm{div}\Big(\frac{\nabla|f|^2}{2|f|}\Big) = \mathrm{div}\Big(\frac{\nabla f\,\bar{f} + \nabla\bar{f}\,f}{2|f|}\Big) \\ &= \frac{\Delta f\,\bar{f} + \Delta\bar{f}\,f}{2|f|} + \frac{|\nabla f|^2}{|f|} - \frac{1}{4}\frac{\big|\nabla|f|^2\big|^2}{|f|^3} \\ &= \frac{\Delta f\,\bar{f} + \Delta\bar{f}\,f}{2|f|} + \frac{(I\nabla R - R\nabla I)^2}{|f|}.\end{aligned} \tag{3.10}$$

The two identities together imply

$$\begin{aligned}\mathcal{L}|f| &= \Delta|f| + \mathrm{div}(E|f|) \\ &\geq [(\Delta f)\bar f + f\,(\Delta \bar f)]/(2|f|) + \mathrm{div}(E|f|) = \mathcal{L}f \cdot \theta'(f).\end{aligned} \tag{3.11}$$

It is worth emphasizing that (3.10) is clearly true for a $W^{2,d}_{loc}(\mathbb{R}^d)$ and not vanishing function f. For a function $f \in W^{2,d}_{loc}(\mathbb{R}^d)$ which may vanish, we introduce the quantity $|f|_\varepsilon := (\varepsilon^2 + |f|^2)^{1/2}$, and we similarly have

$$\Delta |f|_\varepsilon = \frac{\Delta f\,\bar f + \Delta \bar f\, f}{2|f|_\varepsilon} + \frac{|\nabla f|^2}{|f|_\varepsilon} - \frac{1}{4}\frac{\left|\nabla |f|^2\right|^2}{|f|^3_\varepsilon} \geq \frac{\Delta f\,\bar f + \Delta \bar f\, f}{2|f|_\varepsilon}.$$

By passing to the limit $\varepsilon \to 0$, we recover (3.11).

Step 2. Strong maximum principle for a real values function. Consider $f \in H\backslash\{0\}$ such that $\mathcal{L}f = 0$. By a bootstrap regularization argument, we classically have $f \in W^{2,d}_{loc}(\mathbb{R}^d) \subset C(\mathbb{R}^d)$. By assumption there exist then $x_0 \in \mathbb{R}^d$, $c, r > 0$, such that $|f(x)| \geq c$ on $B(x_0, r)$. From Lemma 3.3, we also have that $\mathcal{L} - a$ is -1-dissipative for $a \geq 0$ large enough, in the sense that

$$\forall\, h \in D(\mathcal{L}) \quad ((\mathcal{L} - a)h, h)_H \leq -\|h\|^2_H. \tag{3.12}$$

We next observe that for $\sigma > 0$ large enough, the function $g(x) := c\exp(\sigma r^\gamma - \sigma|x - x_0|^\gamma)$ satisfies $g = c$ on $\partial B(x_0, r)$ and

$$\begin{aligned}(-\mathcal{L} + a)g &= \big[-\sigma^2\gamma^2|x - x_0|^{2(\gamma-1)} + \sigma\gamma(d + \gamma - 2)|x - x_0|^{\gamma-2} \\ &\qquad -\mathrm{div}E + E\cdot(x - x_0)\gamma\sigma|x - x_0|^{\gamma-2} - a\big]\, g \leq 0 \quad \text{on} \quad B(x_0, r)^c.\end{aligned}$$

We define $h := (g - |f|)_+$ and $\Omega := \mathbb{R}^d\backslash B(x_0, r)$. We have $h \in H^1_0(\Omega, m dx)$ and

$$\begin{aligned}(\mathcal{L} - a)h &\geq \theta'(g - |f|)\,\mathcal{L}(g - |f|) - a\,h \\ &= \theta'(g - |f|)\,[(\mathcal{L} - a)g + a|f|] \geq 0,\end{aligned}$$

where we have used the notation $\theta(s) = s_+$. Thanks to a straightforward generalization of (3.12) to $H^1_0(\Omega, m)$, we deduce

$$0 \leq ((\mathcal{L} - a)h, h)_{L^2(\Omega,m)} \leq -\|h\|^2_{L^2(\Omega,m)},$$

and then $h = 0$. That implies $|f| \geq g$ on Ω, next $|f| > 0$ on $\mathbb{R}^d$ and then $f > 0$ or $f < 0$ because $f \in C(\mathbb{R}^d)$.

Step 3. Strong maximum principle for a complex values function. Consider a complex values function $f \in D(\mathcal{L})\backslash\{0\}$ such that

$$\mathcal{L}\theta(f) = \mathcal{L}f \cdot \theta'(f) = 0$$

for $\theta(s) = |s|$. The strong maximum principle for a real values function implies that $|f| > 0$ and we may assume that both R and I do not vanish in some open set $\mathcal{O}$. Using the case of equality in Kato's inequality, we deduce with the notation of Step 2 that

$$\frac{(I\nabla R - R\nabla I)^2}{|f|} = 0$$

which in turns implies

$$|\nabla \log R - \nabla \log I|^2 = 0.$$

We have then proved $R = CI$ for some constant $C \in \mathbb{R}$ in $\mathcal{O}$ and then in $\mathbb{R}^d$, which exactly means $f = u\,|f|$ for some $u \in \mathbb{C}^*$. $\square$

4. Second example: the scattering equation

The linear Boltzmann (or scattering) equation of the density function $f = f(t,v) \ge 0$, $t \ge 0$, $v \in \mathcal{V} \subset \mathbb{R}^d$, writes

$$\partial_t f = \mathcal{L}\, f := \int_{\mathcal{V}} (b_* f_* - b\, f)\, dv_*, \tag{4.1}$$

where $f = f(v)$, $f_* = f(v_*)$, $b = b(v,v_*)$ and $b_* = b(v_*,v)$, $b \ge 0$ is a given function (the rate of collisions), or more generally

$$\partial_t f = \mathcal{L}\, f := \int_{\mathcal{V}} b_* f_*\, dv_* - B(v)\, f, \tag{4.2}$$

and we assume that there exists a function $\phi > 0$ such that

$$\mathcal{L}^*\phi := \int_{\mathcal{V}} b\, \phi_*\, dv_* - B\, \phi = 0, \quad \text{in other words} \quad B(v) := \int_{\mathcal{V}} \frac{\phi_*}{\phi}\, b\, dv_*,$$

with again $\phi = \phi(v)$ and $\phi_* = \phi(v_*)$. The first equation (4.1) corresponds to the choice

$$B(v) = \int_{\mathcal{V}} b\, dv_*, \quad \phi \equiv 1,$$

in the second equation (4.2).

Example 1. We assume $\mathcal{V} \subset \mathbb{R}^d$, $b_* = k(v,v_*)\, F(v)$, for a symmetric function $k(v,v_*) = k(v_*,v) > 0$ and a given function $0 < F \in L^1(\mathcal{V}) \cap \mathbf{P}(\mathcal{V})$. The equation (4.1) becomes

$$\partial_t f = \mathcal{L}\, f := \int_{\mathcal{V}} k\, (F\, f_* - F_*\, f)\, dv_*. \tag{4.3}$$

It is worth noticing that $F = F(v)$ is a stationary solution to the equation (4.5) since

$$\partial_t F = 0 = \mathcal{L}\, F. \tag{4.4}$$

Example 2. We assume $\mathcal{V} = (0,\infty)$, $b_* = b_*\, \mathbf{1}_{v_*>v}$, $\phi(v) = v$, and then the equation (4.2) becomes the *fragmentation equation*

$$\partial_t f = \mathcal{L}\, f := \int_0^\infty b_*\, f_*\, dv_* - B(v)\, f(v), \quad B(v) := \int_0^v \frac{v_*}{v}\, b\, dv_*. \tag{4.5}$$

Conservation law. Without any additional assumption, we immediately deduce that the equation (4.2) has one law of conservation: any solution satisfies (at least formally)

$$\int_{\mathcal{V}} f(t,v)\, \phi(v)\, dv = \int_{\mathcal{V}} f(0,v)\, \phi(v)\, dv,$$

because

$$\frac{d}{dt} \int_{\mathcal{V}} f\, \phi\, dv = \int_{\mathcal{V}} (\mathcal{L} f)\, \phi\, dv = \int_{\mathcal{V}} f\, (\mathcal{L}^* \phi)\, dv = 0.$$

Lyapunov/entropy functional. We assume that there exists a function $0 < F \in L^1(\mathcal{V}) \cap \mathbf{P}(\mathcal{V})$ which is a stationary solution

$$\mathcal{L} F = \int_{\mathcal{V}} b_* F_*\, dv_* - \int_{\mathcal{V}} \frac{\phi_*}{\phi}\, b\, dv_*\, F = 0,$$

what it is the situation in Example 1. Then any solution f to the equation (4.2) satisfies (at least formally)

$$\frac{d}{dt} \int_{\mathcal{V}} f^2\, \frac{\phi}{F}\, dv = 2 \int_{\mathcal{V}} (\mathcal{L}\, f)\, \frac{f\, \phi}{F}\, dv = -D_2(f) \tag{4.6}$$

with

$$D_2(f) := \int_{\mathcal{V}} \int_{\mathcal{V}} b_*\, F\, \phi \left(\frac{f_*}{F_*} - \frac{f}{F} \right)^2 dv dv_*. \tag{4.7}$$

We then say that

$$\mathcal{H}_2(f) := \int_{\mathcal{V}} f^2\, \frac{\phi}{F}\, dv$$

is a Lyapunov (or generalized relative entropy) for the equation (4.2).

To prove (4.6) in the case $\phi = 1$, we perform the following computations

$$\begin{aligned}(\mathcal{L}f, f/F) &= \iint b_* F_* \frac{f_*}{F_*}\frac{f}{F} - \frac{1}{2}\iint b\, F\, \frac{f^2}{F^2} - \frac{1}{2}\iint b\, F\, \frac{f^2}{F^2} \\ &= \iint b_* F_* \frac{f_*}{F_*}\frac{f}{F} - \frac{1}{2}\iint b_*\, F_*\, \frac{(f_*)^2}{(F_*)^2} - \frac{1}{2}\iint b_*\, F_*\, \frac{f^2}{F^2} \\ &= -\frac{1}{2}\iint b_* F_* \left(\frac{f_*}{F_*} - \frac{f}{F}\right)^2,\end{aligned}$$

where in order to pass from the first to the second line we have just changed the name of the variables in the second term

$$\iint b\, F\, \frac{f^2}{F^2} = \iint b_*\, F_*\, \frac{(f_*)^2}{(F_*)^2}$$

and we have used the fact that F is a stationary solution in the third term

$$\int b\, F\, dv_* = \int b_*\, F_*\, dv_*.$$

For a general law of conservation ϕ, the computation is almost the same

$$\begin{aligned}(\mathcal{L}f, \phi\, f/F) &= \iint b_*\, \phi\, F_* \frac{f_*}{F_*}\frac{f}{F} - \frac{1}{2}\iint b\, \phi_*\, F\, \frac{f^2}{F^2} - \frac{1}{2}\iint b\, \phi_*\, F\, \frac{f^2}{F^2} \\ &= \iint b_*\, \phi\, F_* \frac{f_*}{F_*}\frac{f}{F} - \frac{1}{2}\iint b_*\, \phi\, F_*\, \frac{(f_*)^2}{(F_*)^2} - \frac{1}{2}\iint b_*\, \phi\, F_*\, \frac{f^2}{F^2} \\ &= -\frac{1}{2}\iint b_*\, \phi\, F_* \left(\frac{f_*}{F_*} - \frac{f}{F}\right)^2.\end{aligned}$$

A theorem. We now consider the same situation as in example 1, and we assume furthermore that there exist some constants $0 < k_0 \le k_1 < \infty$ such that

$$\forall\, v, v_* \in \mathcal{V}, \qquad k_0 \le k(v, v_*) \le k_1.$$

We consider the scattering equation (4.1) in that case, that we complement with an initial condition

$$f(0, v) = f_0(v) \quad \forall\, v \in \mathcal{V}.$$

Theorem 4.1. *Assume $f_0 \in L^1(\mathcal{V})$, $\mathcal{V} = \mathbb{R}^d$.*
(1) There exists a unique global solution $f \in C([0,\infty); L^1(\mathcal{V}))$ to the scattering equation (4.1). That solution is mass conserving

$$\int_{\mathcal{V}} f(t, v)\, dv = \int_{\mathcal{V}} f_0(v)\, dv =: \langle f_0 \rangle$$

and satisfies the maximum principle

$$f_0 \ge 0 \quad \Rightarrow \quad f(t, .) \ge 0 \quad \forall\, t \ge 0.$$

(2) In the large time asymptotic, the solution converges to the unique stationary solution with same mass

$$\|f(t, .) - \langle f_0 \rangle\, F\|_E \le e^{-k_0 t/2}\, \|f_0 - \langle f_0 \rangle\, F\|_E,$$

where $\|\cdot\|_E$ is the Hilbert norm defined by

$$\|f\|_E^2 := \int_{\mathcal{V}} f^2\, F^{-1}\, dv.$$

For the proof of point (1) we refer to the precedent chapters where the needed arguments have been introduced. We are going to give now the (formal) proof of point (2).
Functional inequality and long time behaviour. The following functional inequality holds true: for any function $f \in E$, we have

$$D_2(f) \ge k_0 \|f - \langle f \rangle\, F\|_E^2. \tag{4.8}$$

It is worth observing that the Cauchy-Schwarz inequality implies

$$|\langle f \rangle| \le \int_{\mathcal{V}} (|f| F^{-1/2})\, F^{1/2} \le \Big(\int_{\mathcal{V}} f^2 F^{-1}\Big)^{1/2} \Big(\int_{\mathcal{V}} F\Big)^{1/2} = \|f\|_E,$$

so that the mass $\langle f \rangle$ is well defined if $f \in E$. Let us accept for a while the inequality (4.8) and let us prove then the convergence result (2) in Theorem 4.1. Thanks to (4.6), the fact that F is a stationary solution, the fact that f is mass conserving and (4.8), we have

$$\frac{d}{dt}\|f - \langle f \rangle F\|_E^2 = -D_2(f) \le -k_0 \|f - \langle f \rangle \, F\|_E^2,$$

and we conclude by applying the Gronwall lemma.

Let us prove now the functional inequality (4.8). From the lower bound assumption made on k, the following first inequality holds

$$D_2(f) := \iint b_* \, F_* \left(\frac{f_*}{F_*} - \frac{f}{F} \right)^2 \ge k_0 \iint F \, F_* \left(\frac{f_*}{F_*} - \frac{f}{F} \right)^2.$$

On the other hand, by integrating (in the v_* variable) the identity

$$f \, F_* - f_* \, F = \left(\frac{f}{F} - \frac{f_*}{F_*} \right) F \, F_*,$$

we get

$$g = F \int_{\mathcal{V}} \left(\frac{f}{F} - \frac{f_*}{F_*} \right) F_* \, dv_*$$

with $g = f - \langle f \rangle F$. Thanks to the Cauchy-Schwarz inequality, we deduce

$$g^2 \le \int_{\mathcal{V}} \left(\frac{f}{F} - \frac{f_*}{F_*} \right)^2 F \, F_* \, dv_* \times \int_{\mathcal{V}} F \, F_* \, dv_*,$$

so that we get the second inequality

$$\int_{\mathcal{V}} \frac{g^2}{F} \, dv \quad \le \quad \int_{\mathcal{V}} \int_{\mathcal{V}} \left(\frac{f}{F} - \frac{f_*}{F_*} \right)^2 F \, F_* \, dv_* \, dv.$$

We conclude by gathering these two estimates. □

Exercise 4.2. *Consider the mass conservative scattering equation* (4.1) *and assume that*

$$K_1 \le B(v) \le K_2, \quad 0 \le b(v, v_*) \le K_3 \qquad \forall \, v, v_* \in \mathcal{V} := \mathbb{R}^d,$$

as well as

$$\int_{\mathbb{R}^d} (|v_*|^2 - |v|^2) \, b(v, v_*) \le K_4 \quad \forall \, v \in \mathbb{R}^d,$$

for some constants $K_i \in (0, \infty)$. Show that there exists a positive and unit mass steady state $f_1 \in L^2 \cap L^1_2$ (Hint. Use Theorem 1.2) and that any unit mass solution converges to that steady state (Hint. Use the GRE method).

5. Markov and stochastic semigroup

From now on, we will be interested in Stochastic semigroups which is a class of semigroups which enjoy both a positivity and a "conservativity" property. The importance of Stochastic semigroups comes from its deep relation with Markov processes in stochastic theory as well as from the fact that a quite satisfactory description of the longtime behaviour of such a semigroups can be performed.

We start with the notion of positivity. It can be formulated in the abstract framework of Banach lattices $(X, \|\cdot\|, \ge)$ which are Banach spaces endowed with compatible order relation or equivalently with an appropriate positive cone X_+. To be more concrete, we just observe that the following three examples are Banach lattices when endowed with their usual order relation:

• $X := C_0(E)$, the space of continuous functions which tend to 0 at infinity (when E is not a compact set) endowed with the uniform norm $\| \cdot \|$;

• $X := L^p(E) = L^p(E, \mathcal{E}, \mu)$, the Lebesgue space of functions associated to the Borel σ-algebra $\mathcal{E}$, a positive σ-finite measure μ and an exponent $p \in [1, \infty]$;

• $X := M^1(E) = (C_0(E))'$, the space of Radon measures defined as the dual space of $C_0(E)$.

Here E denotes a σ-locally compact metric space (typically $E \subset \mathbb{R}^d$) and in the last example the positivity can be defined by duality: $\mu \ge 0$ if $\langle \mu, \varphi \rangle \ge 0$ for any $0 \le \varphi \in C_0(E)$.

Lemma 5.1. *Consider X a Banach lattice (one of the above examples), a bounded linear operator A on X and its dual operator A^* on X'. The following equivalence holds:*
(1) A is positive, namely $Af \geq 0$ for any $f \in X$, $f \geq 0$;
(2) A^* is positive, namely $A^*\varphi \geq 0$ for any $\varphi \in X'$, $\varphi \geq 0$.

The (elementary) proof is left as an exercise. We emphasize that $\langle f, \varphi \rangle \geq 0$ for any $\varphi \in X'_+$ (resp. for any $f \in X_+$) implies $f \in X_+$ (resp. $\varphi \in X'_+$).

There are two "equivalent" (or "dual") ways to formulate the notion of Stochastic and Markov semigroup.

Definition 5.2. *On a Banach lattice $Y \supset C_0(E)$ we say that (P_t) ia a Markov semigroup if*
(1) (P_t) is a continuous semigroup in Y;
(2) (P_t) is positive, namely $P_t \geq 0$ for any $t \geq 0$;
(3) (P_t) is conservative, namely $\mathbf{1} \in Y$ and $P_t\mathbf{1} = \mathbf{1}$ for any $t \geq 0$.

Definition 5.3. *On a Banach lattice $X \subset M^1(E)$ we say that (S_t) ia a stochastic semigroup if*
(1) (S_t) is a (strongly or weakly $*$ continuous) continuous semigroup in X;
(2) (S_t) is positive, namely $S_t \geq 0$ for any $t \geq 0$;
(3) (S_t) is conservative, namely $\langle S_t f \rangle = \langle f \rangle$, $\forall\, t \geq 0$, $\forall\, f \in X$, where $\langle g \rangle := \langle g, \mathbf{1} \rangle$.

The two notions are dual. In particular, if (P_t) is a Markov semigroup on $Y \supset C_0(E)$, the dual semigroup (S_t) defined by $S_t := P_t^*$ on $X := Y'$ is a stochastic semigroup. In the sequel we will only consider stochastic semigroups defined on $X \subset L^1(E)$.

Stochastic semigroup and semigroup of contractions for the L^1 are closely linked.

Proposition 5.4. *A Stochastic semigroup is a semigroup of contractions for the L^1 norm. In the other way round, a mass conservative semigroup of contractions for the L^1 norm is positive, and thus it is a Stochastic semigroup.*

Proof of Proposition 5.4. We fix $f \in X$ and $t \geq 0$. We write

$$\begin{aligned} |S_t f| &= |S_t f_+ - S_t f_-| \\ &\leq |S_t f_+| + |S_t f_-| \\ &= S_t f_+ + S_t f_- \\ &= S_t |f|, \end{aligned}$$

where we have used the positivity property in the third line. We deduce

$$\int |S_t f| \leq \int S_t |f| = \int |f|,$$

because of the mass conservation. The reciprocal part is left to the reader. □

Consider $f \geq 0$. From both the contraction property and the mass conservation, we have

$$\|S_t f\|_1 \leq \|f\|_1 = \int f = \int S_t f.$$

As a consequence,

$$\|(S_t f)_-\|_{L^1} = \frac{1}{2} \int (|S_t f| - S_t f) \leq 0$$

so that $(S_t f)_- = 0$ and thus $S_t f \geq 0$. That proves the positivity property.

We may also characterize a Stochastic semigroup in terms of its generator.

Theorem 5.5. *Let $S = S_{\mathcal{L}}$ be a strongly continuous semigroup on a Banach space $X \subset L^1$. There is equivalence between*
(a) $S_{\mathcal{L}}$ is a Stochastic semigroup;
(b) $\mathcal{L}^* 1 = 0$ and $\mathcal{L}$ satisfies Kato's inequality

$$(\text{sign}\, f)\mathcal{L} f \leq \mathcal{L}|f|, \quad \forall\, f \in D(\mathcal{L}).$$

Partial proof of Theorem 5.5. Step 1. We prove $(a) \Rightarrow (b)$. On the one hand, for any $f \in D(\mathcal{L})$ and any $0 \le \psi \in D(\mathcal{L}^*)$, we have

$$\begin{aligned}
\langle \psi, (\mathrm{sign} f)\mathcal{L} f\rangle &= \lim_{t\to 0} \frac{1}{t}\langle \psi, (\mathrm{sign} f)(S(t)f - f)\rangle \\
&\le \lim_{t\to 0} \frac{1}{t}\langle \psi, |S(t)f| - |f|\rangle \\
&\le \lim_{t\to 0} \frac{1}{t}\langle \psi, S(t)|f| - |f|\rangle \\
&= \lim_{t\to 0} \frac{1}{t}\langle S^*(t)\psi - \psi, |f|\rangle \\
&= \langle \mathcal{L}^*\psi, |f|\rangle,
\end{aligned}$$

where we have used the inequality $(\mathrm{sign} f)g \le |g|$ in the second line and the positivity assumption in the third line. That inequality is the weak formulation of Kato's inequality. On the other hand and similarly, for any $f \in D(\mathcal{L})$, we have

$$\begin{aligned}
\langle \mathcal{L}^*\mathbf{1}, f\rangle &= \langle \mathbf{1}, \mathcal{L} f\rangle \\
&= \lim_{t\to 0} \frac{1}{t}\langle \mathbf{1}, S(t)f - f\rangle = 0,
\end{aligned}$$

by just using the mass conservation property. The reciprocal part is left to the reader. □

Step 2. We prove $(b) \Rightarrow (a)$. On the one hand, for any $f \in D(\mathcal{L})$ and $t \ge 0$, we denote $f_t := S_t f$ and we write

$$\langle S_t f - f\rangle = \Big\langle \int_0^t \mathcal{L} f_s\, ds, \mathbf{1}\Big\rangle = \int_0^t \langle f_s, \mathcal{L}^*\mathbf{1}\rangle\, ds = 0.$$

On the other hand, in order to conclude it is enough to prove that (S_t) is a semigroup of contractions. We consider $f \in D(\mathcal{L}^2)$, $t \ge 0$, $n \in \mathbb{N}^*$, we introduce the notation $f_t := S_t f$, $t_k := kt/n$, and we write

$$\begin{aligned}
|S_t f| - |f| &= \sum_{k=0}^{n-1} (|f_{t_{k+1}}| - |f_{t_k}|) \\
&\le \sum_{k=0}^{n-1} \mathrm{sign} f_{t_{k+1}} (f_{t_{k+1}} - f_{t_k}) \\
&= \sum_{k=0}^{n-1} \mathrm{sign} f_{t_{k+1}} \int_{t_k}^{t_{k+1}} \mathcal{L} f_s\, ds \\
&= \sum_{k=0}^{n-1} \mathrm{sign} f_{t_{k+1}} \Big\{ \frac{1}{n}\mathcal{L} f_{t_{k+1}} + \int_{t_k}^{t_{k+1}} \mathcal{L}(f_s - f_{t_{k+1}})\, ds \Big\} \\
&\le \sum_{k=0}^{n-1} \Big\{ \frac{1}{n}\mathcal{L}|f_{t_{k+1}}| + \mathrm{sign} f_{t_{k+1}} \int_{t_k}^{t_{k+1}} \int_{t_{k+1}}^{s} (S_u \mathcal{L}^2 f)\, du ds \Big\},
\end{aligned}$$

where we have used the inequality $(\mathrm{sign} f)g \le |g|$ in the second line and Kato's inequality in the last line. Taking the mean and using the mass conservation, we have

$$\begin{aligned}
\|S_t f\| - \|f\| &\le \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \int_s^{t_{k+1}} \|S_u \mathcal{L}^2 f\|\, du ds \\
&\le \frac{1}{n} \int_0^t \|S_u \mathcal{L}^2 f\|\, du \to 0,
\end{aligned}$$

as $n \to \infty$. □

Exercise 5.6. *Consider $S_{\mathcal{L}^*}$ a (constant preserving) Markov semigroup and $\Phi : \mathbb{R} \to \mathbb{R}$ a concave function. Prove that $\mathcal{L}^*\Phi(m) \le \Phi'(m)\mathcal{L}^* m$. (Hint. Use that $\Phi(a) = \inf\{\ell(a);\ \ell$ affine such that $\ell \ge \Phi\}$ in order to prove $S^*_t(\Phi(m)) \le \Phi(S^*_t m)$ and $\Phi(b) - \Phi(a) \ge \Phi'(a)(b - a)$).*

6. Asymptotic of Stochastic semigroups

6.1. **Strong positivity condition and Doeblin Theorem.** We consider the case of a strong positivity condition.

Theorem 6.1 (Doeblin). *Consider a Stochastic semigroup S_t such that*

$$S_T f \ge \alpha\, \nu\, \langle f \rangle, \quad \forall\, f \in X_+,$$

for some constants $T > 0$ and $\alpha \in (0,1)$ and some probability measure ν. There holds

$$\|S_t f\|_{L^1} \le C\, e^{at} \|f\|_{L^1}, \quad \forall\, t \ge 0,\ \forall\, f \in X,\ \langle f \rangle = 0,$$

for some constants $C \ge 1$ and $a < 0$.

Proof of Theorem 6.1. We fix $f \in X$ such that $\langle f \rangle = 0$ and we define $\eta := \alpha\nu\langle f_+ \rangle = \alpha\nu\langle f_- \rangle$. We write

$$\begin{aligned} |S_T f| &= |S_T f_+ - \eta - S_T f_- + \eta| \\ &\le |S_T f_+ - \eta| + |S_T f_- - \eta| \\ &= S_T f_+ - \eta + S_T f_- - \eta, \end{aligned}$$

where in the last equality we have used the Doeblin condition. Integrating, we deduce

$$\begin{aligned} \int |S_T f| &\le \int S_T f_+ - \alpha\langle \nu \rangle \langle f_+ \rangle + \int S_T f_- - \alpha \langle \nu \rangle \langle f_- \rangle \\ &\le \int f_+ - \alpha\, \langle f_+ \rangle + \int f_- - \alpha\, \langle f_- \rangle \\ &\le (1-\alpha) \int |f|. \end{aligned}$$

By induction, we obtain $a := [\log(1-\alpha)]/T$ and $C := \exp[|a|T]$. □

6.2. **Geometric stability under Harris and Lyapunov conditions.** We consider now a semigroup S with generator $\mathcal{L}$ and we assume that

(H1) there exists some weight function $m : \mathbb{R}^d \to [1,\infty)$ satisfying $m(x) \to \infty$ as $x \to \infty$ and there exist some constants $\alpha > 0, b > 0$ such that

$$\mathcal{L}^* m \le -\alpha\, m + b;$$

(H2) for any $R > 0$, there exists a constant $T \ge T_0 > 0$ and a positive and not zero measure $\nu = \nu_R$ such that

$$S_T f \ge \nu \int_{B_R} f, \quad \forall\, f \in X_+.$$

Theorem 6.2 (Doeblin). *Consider a Stochastic semigroup S on $X := L^1(m)$ which satisfies (H1) and (H2). There holds*

$$\|S_t f\|_{L^1(m)} \le C\, e^{at} \|f\|_{L^1(m)}, \quad \forall\, t \ge 0,\ \forall\, f \in X,\ \langle f \rangle = 0,$$

for some constants $C \ge 1$ and $a < 0$.

We start with a variant of the key argument in the above Doeblin's Theorem.

Lemma 6.3 (Doeblin's variant). *Under assumption (H2), if $f \in L^1(m)$, with $m(x) \to \infty$ as $|x| \to \infty$, satisfies*

$$\|f\|_{L^1} \ge \frac{4}{m(R)} \|f\|_{L^1(m)} \quad \text{and} \quad \langle f \rangle = 0, \tag{6.1}$$

we then have

$$\|S_T f\|_{L^1} \le \big(1 - \frac{\langle \nu \rangle}{2}\big) \|f\|_{L^1}.$$

Proof of Lemma 6.3. From the hypothesis (6.1), we have

$$\begin{aligned}\int_{B_R} f_\pm &= \int f_\pm - \int_{B_R^c} f_\pm \\ &\geq \frac{1}{2}\int |f| - \frac{1}{m(R)}\int |f| m \geq \frac{1}{4}\int |f|.\end{aligned}$$

Together with (H2), we get

$$S_T f_\pm \geq \frac{\nu}{4}\int |f| =: \eta.$$

We deduce

$$|S_T f| \leq |S_T f_+ - \eta| + |S_T f_- - \eta| = S_T f_+ - \eta + S_T f_- - \eta = S_T |f| - 2\eta,$$

and next

$$\int |S_T f| \leq \int S_T |f| - 2\int \eta = \int |f| - \langle \nu \rangle \frac{1}{2}\int |f|,$$

which is nothing but the announced estimate. □

Proof of Theorem 6.2. We split the proof in several steps. We fix $f_0 \in L^1(m)$, $\langle f_0 \rangle = 0$ and we denote $f_t := S_t f_0$.

Step 1. From (H1), we have

$$\frac{d}{dt}\|f_t\|_{L^1(m)} \leq -\alpha \|f_t\|_{L^1(m)} + b\|f_t\|_{L^1},$$

from what we deduce

$$\|f_t\|_{L^1(m)} \leq e^{-\alpha t}\|f_0\|_{L^1(m)} + \big(1 - e^{-\alpha t}\big)\frac{b}{\alpha}\|f_0\|_{L^1} \quad \forall t \geq 0.$$

In other words, for any $T \geq T_0 > 0$, we have

$$\|S_T f_0\|_{L^1(m)} \leq \gamma_L \|f_0\|_{L^1(m)} + K\|f_0\|_{L^1}, \tag{6.2}$$

with $\gamma_L \in (0,1)$ and $K > 0$, both constants depending only of T_0. We fix $R > 0$ large enough such that $K/A \leq (1-\gamma_L)/2$ with $A := m(R)/4$.

Step 2. On the one hand, we recall that

$$\|S_T f_0\|_{L^1} \leq \|f_0\|_{L^1}, \quad \forall T \geq 0. \tag{6.3}$$

On the other hand, because of Lemma 6.3, there exists $\gamma_H \in (0,1)$ and $T \geq T_0$ only depending on R defined above such that

$$\|S_T f_0\|_{L^1} \leq \gamma_H \|f_0\|_{L^1} \quad \text{when} \quad \|f_0\|_{L^1} \geq A\|f_0\|_{L^1(m)}. \tag{6.4}$$

The two estimates (6.3) and (6.4) together give

$$\|S f_0\|_{L^1} \leq \gamma_H \|f_0\|_{L^1} + \frac{1-\gamma_H}{A}\|f_0\|_{L^1(m)}. \tag{6.5}$$

Step 3. The two previous steps together, we deduce that

$$U^{n+1} = M U^n$$

with

$$U^n := \begin{pmatrix} \|S_T^n f_0\|_{L^1(m)} \\ \|S_T^n f_0\|_{L^1} \end{pmatrix} \quad \text{and} \quad M := \begin{pmatrix} \gamma_L & K \\ \frac{1-\gamma_H}{A} & \gamma_H \end{pmatrix}.$$

The eigenvalues of M are

$$\mu_\pm := \frac{1}{2}\big(T \pm \sqrt{T^2 - 4D}\big),$$

with

$$T := \mathrm{tr} M = \gamma_L + \gamma_H, \quad D := \det M = \gamma_L\gamma_H - (1-\gamma_H)\frac{K}{A}.$$

We observe that

$$\gamma_L \gamma_H > D > \gamma_L\gamma_H - (1-\gamma_H)(1-\gamma_L) = T - 1,$$

so that

$$(\gamma_H - \gamma_L)^2 = T^2 - 4\gamma_L\gamma_H < T^2 - 4D < T^2 - 4(T-1) = (T-2)^2$$

and finally

$$\theta := \max(|\mu_+|, |\mu_-|) < \max(\gamma_H, \gamma_L, |T-1|, 1) = 1.$$

We have established that $\|M^n\| \le C\,\theta^n \to 0$ for some constant $C \ge 1$, and we then conclude as in the proof of Theorem 6.1. □

7. An example: the renewal equation

We will discuss now the renewal equation for which we apply some of the results of the preceding sections in order to get some insight about its qualitative behavior in the large time asymptotic. We are thus interesting by the renewal equation

$$\left\{ \begin{array}{l} \partial_t f + \partial_x f + a f = 0 \\ f(t,0) = \rho_{f(t)}, \quad f(0,x) = f_0(x), \end{array} \right. \tag{7.1}$$

where $f = f(t,x)$, $t \ge 0$, $x \ge 0$, and

$$\rho_g := \int_0^\infty g(y)\, a(y)\, dy.$$

Here f typically represents a population of cells (particles) which are aging (getting holder), die (disappear) with rate $a \ge 0$, born again (reappear) with age $x = 0$ and has distribution f_0 at initial time. At least at a formal level, any solution of (7.1) satisfies

$$\frac{d}{dt}\int_0^\infty f\,dx = \int_0^\infty (-\partial_x f - af)\,dx = \big[-f\big]_0^\infty - \int_0^\infty af\,dx = 0,$$

so that the mass is conserved. Similarly, we have

$$\frac{d}{dt}\int_0^\infty |f|\,dx = \int_0^\infty (-\partial_x |f| - a|f|)\,dx = \big[-|f|\big]_0^\infty - \int_0^\infty a|f|\,dx \le 0,$$

so that the sign of the solution is preserved by observing that $g_- = (|g| + g)/2$ and using the above two informations. That seems to indicate that if (7.1) defines a semigroup, this one is a L^1 Stochastic semigroup.

Preliminarily, we consider the (simpler) transport equation with boundary condition

$$\left\{ \begin{array}{l} \partial_t f + \partial_x f + a f = 0 \\ f(t,0) = \rho(t), \quad f(0,x) = f_0(x), \end{array} \right. \tag{7.2}$$

with f_0 and ρ are given data. We observe that when f is smooth (C^1) and satisfies (7.2), we have

$$\frac{d}{ds}[f(t+s,x+s)e^{A(x+s)}] = 0, \quad A(x) := \int_0^x a(y)\,dy,$$

from what we deduce

$$f(t,x)e^{A(x)} = f(t-s,x-s)e^{A(x-s)},$$

when both terms are well defined. Choosing either $s = t$ or $s = x$, we get

$$f(t,x) = f_0(x-t)\,e^{A(x-t)-A(x)}\,\mathbf{1}_{x>t} + \rho(t-x)\,e^{-A(x)}\,\mathbf{1}_{x<t}. \tag{7.3}$$

In the other way round, we may check that for any smooth functions a, f_0, ρ, the above formula gives a classical solution to (7.2) at least in the region $\{(t,x) \in \mathbb{R}^2_+,\ x \ne t\}$, and thus a weak solution to (7.2) in the sense

$$\int_0^\infty\!\!\int_0^\infty f\,(-\partial_t\varphi - \partial_x\varphi + a\varphi)\,dxdt - \int_0^\infty f_0(x)\varphi(0,x)\,dx - \int_0^\infty \rho(t)\varphi(t,0)\,dt = 0, \tag{7.4}$$

for any $\varphi \in C^1_c(\mathbb{R}^2_+)$. It is worth noticing that this last equation is also the weak formulation of the evolution equation with source term

$$\partial_t f + \partial_x f + af = \rho(t)\delta_0, \quad f(0,x) = f_0(x),$$

defined on the all line (that is for any $x \in \mathbb{R}$).

At least at a formal level, for any solution f to (7.2), we may compute

$$\frac{d}{dt}\int_0^\infty |f|\,dx = \big[-|f|\big]_0^\infty - \int_0^\infty a|f|\,dx \le |\rho(t)|,$$

so that

$$\sup_{[0,T]} \|f(t)\|_{L^1} \le \|f_0\|_{L^1} + \int_0^T |\rho(t)|\,dt. \tag{7.5}$$

Lemma 7.1. *Assume $a \in L^\infty$. For any $f_0 \in L^1(\mathbb{R}_+)$ and $\alpha \in L^1(0,T)$ there exists a unique weak solution $f \in C([0,T]; L^1(\mathbb{R}_+))$ associated to equation (7.2).*

Proof Lemma 7.1. Step 1. Existence. When $a \in C_b(\mathbb{R}_+)$ and $f_0, \rho \in C^1_c(\mathbb{R}_+)$ the solution is explicitly given thanks to the characteristics formula (7.3). In the general case, we consider three sequences (a_ε), $(f_{0,\varepsilon})$ and (ρ_ε) of $C_b(\mathbb{R}_+)$ and $C^1_c(\mathbb{R}_+)$ which converge appropriately, namely $a_\varepsilon \to a$ a.e. and (a_ε) bounded in L^∞, $f_{0,\varepsilon} \to f_0$ in $L^1(\mathbb{R}_+)$ and $\rho_\varepsilon \to \rho$ in $L^1(0,T)$, and we see immediately from (7.5) that the functions (f_ε) and f defined thanks to the characteristics formula (7.3) satisfy $f_\varepsilon \to f$ in $C([0,T]; L^1)$. As a consequence, we may pass to the limit in (7.2) and we deduce that f is a weak solution to equation (7.2).

Step 2. Uniqueness. Consider two weak solutions f_1 and f_2 to equation (7.2). The difference $f := f_2 - f_1$ satisfies

$$\int_0^\infty \int_0^\infty f\,(-\partial_t \varphi - \partial_x \varphi + a\varphi)\,dxdt = 0, \tag{7.6}$$

for any $\varphi \in C^1_c(\mathbb{R}^2_+)$ and thus also for any $\varphi \in C_c(\mathbb{R}^2_+) \cap W^{1,\infty}(\mathbb{R}^2_+)$. Introducing the semigroup

$$(S_t g)(x) := g(x-t)\, e^{A(x-t)-A(x)}\, \mathbf{1}_{x>t},$$

associated to equation (7.2) with no boundary term, its dual is

$$(S^*_t \psi)(x) := \psi(x+t)\, e^{A(x)-A(x+t)}, \quad \forall\, \psi \in L^\infty(\mathbb{R}_+),$$

and (S^*_t) is well-defined as a semigroup in $C_c \cap W^{1,\infty}(\mathbb{R}_+)$. Now, for $\psi \in C^1_c(\mathbb{R}^2_+)$, we define

$$\begin{aligned} \varphi(t,x) &:= \int_t^T (S^*_{s-t}\psi(s,\cdot))(x)\,ds \\ &= \int_t^T \psi(s, x+s-t)\, e^{A(x)-A(x+s-t)}\,ds \in C_c(\mathbb{R}^2_+) \cap W^{1,\infty}(\mathbb{R}^2_+), \end{aligned}$$

and we compute

$$\partial_x \varphi(t,x) = \int_t^T [\partial_x \psi(s,x+s-t) + \psi(s,x+s-t)(a(x) - a(x+s-t))]\, e^{A(x)-A(x+s-t)}\,ds,$$

from what we deduce

$$\begin{aligned} \partial_t \varphi(t,x) &= -\psi(t,x) + \int_t^T [-\partial_x \psi(s,x+s-t) + \psi(s,x+s-t) a(x+s-t)]\, e^{A(x)-A(x+s-t)}\,ds \\ &= -\psi(t,x) - \partial_x \varphi(t,x) + a(x)\varphi(t,x). \end{aligned}$$

Using then this test function φ in (7.6), we get

$$\int_0^\infty \int_0^\infty f\,\psi\,dxdt = 0, \quad \forall\, \psi \in C^1_c(\mathbb{R}^2_+),$$

and finally $f_1 = f_2$. □

We are now in position to come back to the renewal equation (7.1).

Lemma 7.2. *Assume $a \in L^\infty$. For any $f_0 \in L^1(\mathbb{R}_+)$, there exists a unique global weak solution $f \in C(\mathbb{R}_+; L^1(\mathbb{R}_+))$ associated to equation (7.1). We may then associate to the renewal evolution a Stochastic semigroup.*

Proof Lemma 7.2. We define $\mathcal{E}_T := C([0,T]; L^1(\mathbb{R}_+))$ and for any $g \in \mathcal{E}_T$, we define $f := \Phi(g) \in \mathcal{E}_T$ the unique solution to equation (7.2) associated to f_0 and $\rho(t) := \rho_{g(t)} \in C([0,T])$. For two given functions $g_1, g_2 \in \mathcal{E}_T$ and the two associated images $f_i := \Phi(g_i)$, we observe that $f := f_2 - f_1$ is a weak solution to equation (7.2) associated to $f(0) = 0$ and $\rho(t) := \rho_{g_2(t)-g_1(t)}$. The estimate (7.5) reads here

$$\begin{aligned}\sup_{[0,T]} \|(f_2 - f_1)(t)\|_{L^1} &\le \int_0^T |\rho_{g_2(t)-g_1(t)})|\,dt \le \int_0^T \int_0^\infty a(y)|(g_2 - g_1)(t,y)\,dydt \\ &\le T\,\|a\|_{L^\infty} \sup_{[0,T]} \|(g_2 - g_1)(t)\|_{L^1}.\end{aligned}$$

Taking first T small enough such that $T\,\|a\|_{L^\infty} < 1$, we get the existence and uniqueness of a fixed point $f = \Phi(f) \in \mathcal{E}_T$, which is nothing but a weak solution to the renewal equation (7.1). Iterating the argument, we get the desired global weak solution $f \in C(\mathbb{R}_+; L^1(\mathbb{R}_+))$.

We may apply the results of the first section in the semigroup chapter 3 in order to get the existence of a semigroup S_t associated to the evolution problem (7.1). This semigroup is clearly positive. That can be seen by construction for instance. Indeed, if $g \in \mathcal{E}_{T,+} := \{g \in \mathcal{E}_T,\ g \ge 0\}$, then $f = \Phi(g) \in \mathcal{E}_{T,+}$ from the representation formula (7.3), and the fixed point argument can be made in that set. Next, from (7.4), we classical deduce (see chapter 2) that

$$\int_0^\infty f\,\varphi_R\,dx = \int_0^\infty f_0\,\varphi_R\,dx + \int_0^t \int_0^\infty (\partial_x \varphi_R + a\varphi_R)\,dxds + \int_0^t \rho(s)\,ds$$

for $\varphi_R(x) := \varphi(x/R)$, $\varphi \in C^1_c(\mathbb{R}_+)$, $\mathbf{1}_{[0,1]} \le \varphi \le \mathbf{1}_{[0,2]}$. We get the mass conservation by passing to the limit as $R \to \infty$. □

Lemma 7.3. *Assume furthermore* $\liminf a \ge a_0 > 0$. *There exists a unique stationary solution* $F \in W^{1,\infty}(\mathbb{R}_+)$ *to the stationary problem*

$$\partial_x F + aF = 0, \quad F(0) = \rho_F, \quad F \ge 0, \quad \langle F \rangle = 1.$$

Proof Lemma 7.3. From the first equation we have $F(x) = Ce^{-A(x)}$, so that the boundary condition is immediately fulfilled and the normalized condition is fulfilled by choosing $C := \langle e^{-A(x)} \rangle^{-1}$. It is worth noticing that the additional assumption implies $\langle e^{-A(x)} \rangle < \infty$ so that $C > 0$ and the same is true for F. □

Lemma 7.4. *We still assume* $a \in L^\infty$ *and* $\liminf a \ge a_0 > 0$. *There exist* $C \ge 1$ *and* $\alpha < 0$ *such that for any* $f_0 \in L^1(\mathbb{R}_+)$ *the associated global solutionf to the renewal equation* (7.1) *satisfies*

$$\|f(t) - \langle f_0 \rangle F\|_{L^1} \le C\,e^{\alpha t}\,\|f_0 - \langle f_0 \rangle F\|_{L^1}, \quad \forall\, t \ge 0.$$

Proof Lemma 7.4. We check Harris condition. We observe that $a \ge a_0/2\,\mathbf{1}_{x \ge x_0}$ for some $x_0 > 0$. We then set $T := 2x_0 > 0$ and we take $0 \le f_0 \in L^1(\mathbb{R}_+)$. From (7.3), we have

$$f(T,x) \ge \rho_{f(T-x,\cdot)}\,e^{-A(x)}\,\mathbf{1}_{x<T/2}. \tag{7.7}$$

with

$$\begin{aligned}\rho_{f(T-x,\cdot)} &= \int_0^\infty a(y) f(T-x,y)\,dy \\ &\ge \frac{a_0}{2} \int_{x_0}^\infty f(T-x,y)\,dy,\end{aligned}$$

Using the representation formula (7.3) again, we have

$$\begin{aligned}f(T-x,y) &\ge f_0(y+x-T)\,e^{-(A(y)-A(y-(x-T)))}\,\mathbf{1}_{y>T-x} \\ &\ge f_0(y+x-T)\,e^{-(x-T)\|a\|_\infty}\,\mathbf{1}_{y>T-x},\end{aligned}$$

so that

$$\begin{aligned}\rho_{f(T-x,\cdot)} &\ge \frac{a_0}{2} \int_{x_0}^\infty f_0(y+x-T)\,\mathbf{1}_{y>T-x}\,dy\,e^{-(x-T)\|a\|_\infty} \\ &\ge \frac{a_0}{2} \int_0^\infty f_0(z)\,\mathbf{1}_{z>x_0+x-T}\,dz\,e^{-(x-T)\|a\|_\infty}.\end{aligned}$$

Together with (7.7), we obtain

$$\begin{aligned} f(T,x) &\geq \frac{a_0}{2}\int_0^\infty f_0(z)\,\mathbf{1}_{z>x_0+x-T}\,dz\,e^{-(x-T)\|a\|_\infty}\,e^{-A(x)}\,\mathbf{1}_{x<T/2} \\ &= \nu(x)\int_0^\infty f_0(z)\,dz, \quad \nu(x) := \frac{a_0}{2}\,e^{-(x-T)\|a\|_\infty}\,e^{-A(x)}\,\mathbf{1}_{x<T/2}, \end{aligned}$$

which is precisely a Harris type lower bound. We conclude thanks to Theorem 6.1. □

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8. Appendix

Theorem 8.1 (Brouwer-Schauder-Tychonoff). *Consider a locally convex topological vector space X and $\mathcal{Z} \subset X$ a convex set which is metrizable and compact for the induced topology. Then, any continuous function $\varphi : \mathcal{Z} \to \mathcal{Z}$ has a least one fixed point.*

Remark 8.2. *The examples we have in mind are the following:*

1. A Banach space X endowed with its norm $\|\cdot\|_X$ and a convex and bounded set $\mathcal{Z} \subset X$ which is furthermore compact for the strong topology. Typically $X = L^p$ and $\mathcal{Z} := \{f \in W^{1,p} \cap L^p_1;\ \|f\|_{W^{1,p}\cap L^p_1} \le 1\}$.

2. A separable and reflexive Banach space X endowed with the weak topology $\sigma(X,X')$ and a bounded, closed and convex set $\mathcal{Z} \subset X$. Because X' is separable, the topology $\sigma(X,X')$ on the bounded set $\mathcal{Z}$ is metrizable, and the set $\mathcal{Z}$ is both topologically and sequentially compact.

3. $X = L^1(\Omega)$, $\Omega \subset \mathbb{R}^d$ open set, endowed with the weak topology $\sigma(L^1, L^\infty)$ and a bounded, closed and convex set $\mathcal{Z} \subset X$ such that $\mathcal{Z}$ is uniformly equi-integrable both locally and at the infinity. For instance there exist $\omega : \Omega \to [1,\infty)$, $\omega(x) \to \infty$ when $|x| \to \infty$, $\Phi : \mathbb{R} \to \mathbb{R}_+$, $\Phi(s)/s \to \infty$ when $|s| \to \infty$ and $C \in \mathbb{R}_+$, such that

$$\mathcal{Z} \subset \Big\{ f \in L^1(\Omega),\ \int_\Omega (\Phi(f) + |f|\,\omega)\,dx \le C \Big\}.$$

As a consequence, $\mathcal{Z}$ is both topologically and sequentially compact for the weak topology $\sigma(L^1, L^\infty)$.

4. A Banach space X such that $X = Y'$ for a separable Banach space Y endowed with the weak $*$ topology $\sigma(X,Y)$ and a convex and bounded set $\mathcal{Z} \subset X$ which is furthermore closed for the weak $*$ topology $\sigma(X,Y)$. Because $\mathcal{Z} \subset \{f \in X,\ \|f\|_X \le C\}$, for some $C \in \mathbb{R}_+$, and that last set is topologically and sequentially compact for the weak $*$ topology $\sigma(X,Y)$, the same is true for $\mathcal{Z}$.

Proof of Theorem 8.1. By assumption $\mathcal{Z}$ is endowed with a metrizable topology associated to a family of seminorms $(p_i)_{i\in I}$ with $I = \{0\}$ or $I = \mathbb{N}$. We assume that we are in the second case, the first case being simpler, and we also assume without restriction that (p_i) is increasing. We split the proof into two steps.

Step 1. By compactness of $\mathcal{Z}$, for any $\varepsilon > 0$ and $n \in I$, there exists a finite set J and some vectors $e_j \in \mathcal{Z}$, $j \in J$, such that

$$\mathcal{Z} \subset \bigcup_{j\in J} \{p_n(x - e_j) < \varepsilon/2\}. \tag{8.1}$$

We then define φ_ε by

$$\varphi_\varepsilon(x) := \sum_i \theta_i(x)\,e_i, \quad \theta_i(x) = \frac{q_i(x)}{\sum_{j\in J} q_j(x)}, \quad q_i(x) := \max(\varepsilon - p_n(\varphi(x) - e_i), 0).$$

For any $i \in J$, the mapping $x \mapsto q_i(x)$ is continuous and moreover, for any $x \in \mathcal{Z}$, there exists at least one $i_x \in J$ such that $q_{i_x}(x) \ge \varepsilon/2$. As a consequence, $x \mapsto \sum q_j(x)$ is continuous and larger than $\varepsilon/2$, which in turn imply that φ_ε is a continuous mapping. Because

$$0 \le \theta_i(x) \quad \text{and} \quad \sum_{i\in I} \theta_i(x) = 1,$$

we get that $\varphi_\varepsilon : \mathcal{Z}_\varepsilon \subset \mathcal{Z} \to \mathcal{Z}_\varepsilon$, where $\mathcal{Z}_\varepsilon$ is the convex hull of the points $(e_j)_{j\in J}$. In particular, $\mathcal{Z}_\varepsilon$ is a convex and compact subset of the finite dimensional space $\mathrm{Vect}(e_j;\ j \in J)$ endowed with the topology induced by the family of seminorms $(p_i)_{i\in I}$, which therefore is a normable topology (it is associated to a seminorm p_m, m large, which is a norm on that finite dimensional space). We may apply the Brouwer theorem and we get the existence of at least a fixed point. Namely, there exists $x_\varepsilon \in \mathcal{Z}_\varepsilon$ such that $\varphi_\varepsilon(x_\varepsilon) = x_\varepsilon$. We next observe that for any $x \in \mathcal{Z}$, there holds

$$\begin{aligned} p_n(\varphi(x) - \varphi_\varepsilon(x)) &= p_n\big(\sum_{j\in I} \theta_j(x)(\varphi(x) - e_j)\big) \\ &\le \sum_{j\in I} \theta_j(x) p_n(\varphi(x) - e_j) \le \varepsilon \end{aligned}$$

because $p_n(\varphi(x) - e_j) \le \varepsilon$ when $\theta_j(x) \neq 0$.

Step 2. For any $n \in \mathbb{N}^*$, we take $\varepsilon_n = 1/n$ in the previous construction, and we write φ_n instead of φ_{ε_n} as well instead x_n instead of x_{ε_n}. With this notation, we have for any $n \ge m \ge 1$

$$\varphi_n(x_n) = x_n \quad \text{and} \quad p_m(\varphi(x) - \varphi_n(x)) \le \frac{1}{n}, \tag{8.2}$$

because (p_n) is an increasing sequence. By compactness of $\mathcal{Z}$ there exist a subsequence, still denoted as (x_n), and $\bar x \in \mathcal{Z}$ such that $x_n \to \bar x$. By continuity of φ and thanks to (8.2), we deduce

$$p_m(\varphi(\bar x) - \bar x) \le p_m(\varphi(\bar x) - \varphi(x_n)) + p_m(\varphi(x_n) - \varphi_n(x_n)) + p_m(\varphi_n(x_n) - x_n) + p_m(x_n - \bar x) \to 0$$

as $n \to \infty$, for any $m \ge 1$. Because (p_n) separates points, we conclude with $\varphi(\bar x) = \bar x$. □

9. Bibliographic discussion

Theorem 1.2 in section 1.1 is an abstract version and generalization of a technical lemma classically used in the proof of the Poincaré-Bendixson Theorem about the qualitative behaviour of solutions to a 2d system of ode. I learned the material of sections 1.2 and 1.4 in Haraux's book [2]. The result in section 1.4 belongs to folklore (it has been used several times in order to prove the convergence of the solution of the Boltzmann equation to the corresponding Maxwellian equilibrium).

The computations presented in section 2 and leading to the General Relative Entropy are taken from [3]. The case $\phi = 1$ corresponds to the usual probability framework and then can be found in many earlier papers of the probability community.

The material of section 3 on the Fokker-Planck equation is a simplified presentation of more or less recent results on this very active line of research. The dissipativity estimate established in Proposition 3.3 is taken from [1] (see also [4]). The most general case when no structure assumption is made on E is inspired from a Master degree work by M. Ndao [5].

The material of section 4 on the scattering belongs to folklore. I learned it from A. Mellet.

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