CHAPTER 6 - MORE ABOUT LONGTIME ASYMPTOTIC
remarkable solutions, entropy and positivity techniques

- STILL A DRAFT -

I write in blue color additional material with respect to what has been taught during the classes. I write in brown color the significative changes with respect to the previous version (of January 11).

Contents

In this chapter we introduce several techniques which are useful for analyzing the longtime asymptotic of quite general evolution PDEs. On the one hand, it is an introduction to entropy (or Lyapunov) methods for general (possibly nonlinear) dynamical system and an illustration on some exemples of evolution PDEs (linear, positivity preserving) of parabolic type. On the other hand it is an introduction of the analysis of stochastic semigroup following Harris-Meyn-Tweedie type approach. The aim is thus to develop some quite general tools which make possible to get a better understanding of the longtime asymptotic issue.

1. Existence of steady states

In this section we present a general a dynamic system argument for proving the existence of a steady state for a time autonomous evolution PDE. We illustrate the technique on the Fokker-Planck equation

$$\partial_t f = \mathcal{L} f := \Delta f + \operatorname{div}(E f), \tag{1.1}$$

on the density $f = f(t,x)$, $t \ge 0$, $x \in \mathbb{R}^d$, where the force field $E \in \mathbb{R}^d$ is

- either a given fixed (exterior) vectors field, see the next section and adapt the proofs;
- or a function of the density, the McKean-Vlasov model.

1.1. **Dynamic system and steady states.** We recall a result already presented during the *Review course on differential calculus for ODEs and PDEs* and which is yet useful for proving steady state for ODEs.

Definition 1.1. *We say that $(S_t)_{t\ge 0}$ is a dynamical system (or a continuous (possibly nonlinear) semigroup) on a metric space $(\mathcal{Z}, d)$ if*

(S1) $\forall t \ge 0$, $S_t \in C(\mathcal{Z}, \mathcal{Z})$ (continuously defined on $\mathcal{Z}$);

(S2) $\forall x \in \mathcal{Z}$, $t \mapsto S_t\, x \in C([0,\infty), \mathcal{Z})$ (trajectories are continuous);

(S3) $S_0 = I$; $\forall\, s, t \ge 0$, $S_{t+s} = S_t\, S_s$ (semigroup property).

We say that $\bar z \in \mathcal{Z}$ is invariant (or is a steady state, a stationary point) if $S_t \bar z = \bar z$ for any $t \ge 0$. We denote by $\mathcal{E}$ the set of all steady states,

$$\mathcal{E} := \{y \in \mathcal{Z};\ S_t y = y\ \forall\, t \ge 0\}.$$

We remark that $\mathcal{E}$ is closed by definition ($\mathcal{E} = \cap_{t\ge 0}(S_t - I)^{-1}(\{0\})$).

Theorem 1.2. (Dynamic system and steady state). *Consider a bounded and convex subset $\mathcal{Z}$ of a Banach space X which is sequentially compact when it is endowed with the metric associated to the norm $\|\cdot\|_X$ (strong topology), to the weak topology $\sigma(X, X')$ or to the weak-$\star$ topology $\sigma(X, Y)$, $Y' = X$. Then any dynamical system $(S_t)_{t\ge 0}$ on $\mathcal{Z}$ admits at least one steady state, that is $\mathcal{E} \ne \emptyset$.*

Proof of Theorem 1.2. For any $t > 0$, there exists $z_t \in \mathcal{Z}$ such that $S_t z_t = z_t$ thanks to the Schauder or the Tychonov point fixed Theorem. On the one hand, from the semigroup property *(S3)*

$$S_{i\, 2^{-m}} z_{2^{-n}} = z_{2^{-n}} \quad \text{for any} \quad i, n, m \in \mathbb{N}, \quad m \le n. \tag{1.2}$$

On the other hand, by compactness of $\mathcal{Z}$, we may extract a subsequence $(z_{2^{-n_k}})_k$ which converges weakly to a limit $\bar z \in \mathcal{Z}$. By the continuity assumption *(S1)* on S_t, we may pass to the limit $n_k \to \infty$ in (1.2) and we obtain $S_t\, \bar z = \bar z$ for any dyadic time $t \ge 0$. We conclude that $\bar z$ is a stationary point by the trajectorial continuity assumption *(S2)* on S_t and the density of the dyadic real numbers in the real line. ☐

1.2. **Nonlinear McKean-Vlasov equation.** We consider the evolution PDE

$$\partial_t f = \mathcal{L} f := \Delta f + \operatorname{div}((a * f + x) f), \quad f(0) = f_0, \tag{1.3}$$

with $a \in L^\infty$. The analysis follows the classical steps.

1.2.1. *A priori estimates.* We compute (formally) successively

$$\frac{d}{dt}\int f = 0,$$
$$\frac{d}{dt}\int f_\pm \le 0,$$

and (more details are given in section 1.2.4)

$$\frac{d}{dt}\int f^2 m^2 = -\int |\nabla f|^2 m^2 + \int f^2 \psi m^2, \tag{1.4}$$

where ψ is the usual function (see the next Lemma and section 1.2.4), as well as thus

$$\frac{d}{dt}\int f^2 m^2 \le -\frac{1}{2}\int |\nabla f|^2 m^2 + C\int f - \int f^2 m^2, \tag{1.5}$$

if $L^2_m \subset L^1$. At least formally, the set

$$\mathcal{Z} := \left\{f \in L^2_k;\ f \ge 0,\ \int f = 1,\ \|f\|_{L^2_k} \le \mathscr{R}\right\} \tag{1.6}$$

is invariant if $\mathscr{R} > 0$ is large enough: $f(t) \in \mathcal{Z}$ for any $t \ge 0$ if $f_0 \in \mathcal{Z}$.

More precisely, the L^2_m estimate is a consequence of the following result based on integration by parts.

Lemma 1.3. *For any $f \in \mathcal{D}(\mathbb{R}^d)$ and any weight function $m : \mathbb{R}^d \to \mathbb{R}_+$, we have*

$$\int (\mathcal{L}f) f^{p-1} m^p = -(p-1) \int |\nabla f|^2 f^{p-2} m^p + \int f^p m^p \psi$$

with

$$\psi := (p-1) \frac{|\nabla m|^2}{m^2} + \frac{\Delta m}{m} + \left(1 - \frac{1}{p}\right) \mathrm{div} E - E \cdot \frac{\nabla m}{m}.$$

Proof of Lemma 1.3. It is a good exercise! Just perform two integrations by part: one on the term which involves the Laplacian, another on the term which involves the $E \cdot \nabla f$ function. □

Observe that (at least formally):

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^d} |f|^p m^p &= \frac{p}{2} \int_{\mathbb{R}^d} (|f|^2)^{p/2-1} \partial_t (f \, \bar f) m^p \\ &= \frac{p}{2} \int_{\mathbb{R}^d} |f|^{p-2} (\mathcal{L} f \, \bar f + f \, \bar{\mathcal{L}} f) m^p, \end{aligned}$$

so that defining $f^* := \|f\|^{2-p}_{L^p(m)} \, \bar f \, |f|^{p-2}$, we get

$$\begin{aligned} \frac{d}{dt} \|f\|^2_{L^p(m)} &= \frac{2}{p} (\|f\|^p_{L^p(m)})^{2/p-1} \frac{d}{dt} \|f\|^p_{L^p(m)} = \int_{\mathbb{R}^d} (\mathcal{L} f \, f^* + \bar f^* \, \mathcal{L} f) m^p \\ &= 2 \, \Re e \langle \mathcal{L} f, f^* \rangle. \end{aligned} \tag{1.7}$$

As a consequence, (1.7) together with Lemma 1.3 lead to some differential inequality on the L^p-norm which provides an a priori estimate on a solution of (1.1) when the function ψ in Lemma 1.3 is uniformly bounded above.

As a consequence of the previous identity we obtain several existence results. In the sequel we assume that

$$E := E_1 + E_2$$

with

$$E_1 = E_1(t,x) \in L^\infty((0,T) \times \mathbb{R}^d), \tag{1.8}$$

and $E_2 = E_2(x) \in W^{1,\infty}_{loc}$ and, for some $\gamma \ge 2$, (we have taken $\gamma = 2$ during the classes)

$$|E_2(x)| \le K_1 \, \langle x \rangle^{\gamma-1}, \quad |\mathrm{div} E_2(x)| \le K_2 \, \langle x \rangle^{\gamma-2}, \quad E_2(x) \cdot x \ge |x|^\gamma \quad \forall x \in \mathbb{R}^d. \tag{1.9}$$

This framework contains the particular case

$$E := x + a * f, \quad a \in L^\infty, \quad f \in L^\infty_t L^1_x.$$

We define

$$H := L^2(m), \quad V := H^1(m) \cap L^2(m_1), \tag{1.10}$$

with $m = m_1 = \langle x \rangle^k$, $k \ge 0$, or with $m := e^{\kappa \langle x \rangle^\gamma}$, $m_1 := \langle x \rangle^{\gamma-1} e^{\kappa \langle x \rangle^\gamma}$, $\kappa := \gamma/4$. We next define

$$X_T := C([0,T]; H) \cap L^2(0,T; V).$$

1.2.2. *Existence.* If $f_0 \in L^2_m$, $m := e^{\alpha \langle x \rangle^2}$, $\alpha > 0$ small, we may apply J.-L. Lions theory and we obtain the existence and uniqueness of a solution $f \in X_T$ to the linear problem

$$\partial_t f = \mathcal{L}_g f := \Delta f + \mathrm{div}((a * g + x) f), \quad f(0) = f_0, \tag{1.11}$$

for any $g \in L^\infty(0,T; L^1)$. It is worth emphasizing that we use here the Poincaré inequality in $\mathbb{R}^d$ in order to prove that the associated Dirichlet form is coercive. We refer to the tutorial where this problem has been tackled without the term $\mathrm{div}(xf)$ so that the same technique applies in a L^2 space with polynomial weight of degree $k > d/2$ (see Section 1.2.3 below).

Next, for any $f_0 \in L^2_k$, $k > d/2$, we claim that there exists a solution $f \in X_T$, to equation (1.11) in the sense that

$$\frac{d}{dt} \int f \varphi = - \int \nabla f \cdot \nabla \varphi - \int f (a * g + x) \cdot \nabla \varphi,$$

for any $\varphi \in C_c(\mathbb{R}^d)$. There is indeed no difficulty for passing to the limit in the above formulation for a sequence of strongly decaying solutions associated to strongly decaying initial data and given by the previous existence result. It is worth emphasizing that such a sequence (f_n) of solutions is a Cauchy sequence in X_T if the sequence (f_{0n}) of initial data is a Cauchy sequence in L^2_k, what we see by writing (1.4) on the difference $f_n - f_m$. The issue is rather about the uniqueness of such a solution (see Section 1.2.4).

1.2.3. *Well-posedness when* $m := e^{\kappa \langle x\rangle^\gamma}$.

We just present the argument in the case $E = a * f$ and $m := \langle x\rangle^k$ as it has been considered in a tutorial. We claim that there is no difficulty to adapt the proof to the case $E = x + a * f$ when $m := e^{\kappa \langle x\rangle^\gamma}$. We emphasize again on the fact that in that case we may work with variational solutions by taking advantage of the gain of moment provided by the strong Poincaré inequality.

We first consider the Fokker-Planck equation

$$\partial_t f = \mathcal{L} f = \Delta f + \mathrm{div}(E f), \quad E \in L^\infty((0,T)\times\mathbb{R}^d). \tag{1.12}$$

Proposition 1.4. *For any $f_0 \in H$, there exists a unique variational solution $f \in X_T$ to the Fokker-Planck equation* (1.12)*. Moreover, if $f_0 \geq 0$ then $f(t) \geq 0$ for any $t \geq 0$; if $f_0 \in L^1$ then $f(t) \in L^1$ and $\langle f(t)\rangle = \langle f_0\rangle$ for any $t \geq 0$.*

Proof of Proposition 1.4. We observe that the (possibly time dependent) bilinear form

$$\begin{aligned} a(t,f,g) &:= -\int \mathcal{L}(t) f\, g\, m^2 \\ &= \int \{m^2\, \nabla f \cdot \nabla g - g\, \nabla f \nabla m^2 - f m^2\, E\cdot \nabla g - f g\, E \cdot \nabla m^2\}\, dx \end{aligned}$$

is continuous in V. Moreover, using twice the Young inequality, we see that it satisfies the following coercivity-dissipativity estimate

$$\begin{aligned} a(t,f,f) &= \int \{m^2\, |\nabla f|^2 - f\, \nabla f \nabla m^2 - f m^2\, E\cdot\nabla f - f^2\, E\cdot\nabla m^2\}\, dx \\ &\leq -\frac{1}{2}\int |\nabla f|^2\, m^2 + \int \{C_1^2 + \|E\|^2_{L^\infty} + \|E\|_{L^\infty} C_1\} f^2 m^2\, dx, \end{aligned}$$

with $C_1 := \|m^{-2}\nabla m^2\|_{L^\infty}$. We conclude to the existence and the uniqueness of a variational solution $f \in X_T$ by applying Lions' Theorem presented in Chapter 1. □

As announced, we next consider the McKean-Vlasov equation

$$\partial_t f = \mathcal{L} f = \Delta f + \mathrm{div}((a * f) f), \quad a \in L^\infty(\mathbb{R}^d). \tag{1.13}$$

Proposition 1.5. *Assume $a \in L^\infty$. For any $f_0 \in L^2_q$, $q > d/2$, there exists a unique solution*

$$f \in X_T := C([0,T]; L^2_q) \cap L^2(0,T; H^1_q), \quad \forall T > 0,$$

to the McKean-Vlasov equation (1.13)*.*

Proof of Proposition 1.5. Step 1. A priori bounds. On the one hand, we clearly have

$$\int_{\mathbb{R}^d} |f|\, dx \leq \int_{\mathbb{R}^d} |f_0|\, dx,$$

and, using Proposition 1.4 with $E = a * f$, we then deduce

$$\frac{1}{2}\frac{d}{dt}\int f^2 m^2 \leq -\frac{1}{2}\int |\nabla f|^2 m^2 + C_3 \int f^2 m^2,$$

with

$$C_3 := \frac{3}{2} C_1^2 + \frac{3}{2}\|a\|^2_{L^\infty}\|f_0\|^2_{L^1}$$

Step 2. Existence. To prove the existence we consider the mapping $g \mapsto f$ defined for $g \in C([0,T]; L^2_k)$, $k > d/2$, so that $L^2_k \subset L^1$, by solving the linear evolution PDE

$$\partial_t f = \Delta f + \mathrm{div}((a * g) f).$$

For the linear (and g dependent) problem, by repeating the same computations as in step 1 and using the Gronwall lemma, we also have

$$\sup_{[0,T]} \|f\|_{L^1} \leq \|f_0\|_{L^1}, \quad \sup_{[0,T]} \|f\|_{L^2_k} \leq \mathcal{A}_T,$$

where $\mathcal{A}_T$ only depends on $\|f_0\|_{L^2_k}$, k, a and T. We then define

$$\mathcal{C}_T := \{f \in C([0,T]; L^2_k),\ \|f(t)\|_{L^1} \le \|f_0\|_{L^1},\ \|f(t)\|_{L^2_k} \le \mathcal{A}_T\}$$

and we have $\Phi : \mathcal{C}_T \to \mathcal{C}_T$. We consider two solutions

$$\partial_t f_i = \Delta f_i + \mathrm{div}((a * g_i) f_i)$$

so that the differences $f = f_2 - f_1$ and $g := g_2 - g_1$ satisfy

$$\partial_t f = \Delta f + \mathrm{div}((a * g_1) f) + \mathrm{div}((a * g) f_2).$$

As a consequence, adapting the proof of Proposition 1.5, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|f\|^2_{L^2_m} &= -\int [\nabla f + (a * g_1) f + (a * g) f_2] \cdot \nabla(f m^2) \\ &\le \int \{2C_1^2 + \|a * g_1\|^2_{L^\infty} + \|a * g_1\|_{L^\infty} C_1\} f^2 m^2\, dx + \int (a * g)^2 f_2^2 m^2\, dx. \end{aligned}$$

Because

$$\|a * g\|_{L^\infty} \le \|a\|_{L^\infty} \|g\|_{L^1} \le C_1' \|a\|_{L^\infty} \|g\|_{L^2_m},$$

we deduce

$$\frac{1}{2}\frac{d}{dt}\|f\|^2_{L^2_m} \le \{\frac{5}{2} C_1^2 + \frac{3}{2}\|a\|^2_{L^\infty}\|f_0\|^2_{L^1}\}\|f\|^2_{L^2_m} + C_1' \|f_2\|^2_{L^2_m}\|a\|^2_{L^\infty}\|g\|^2_{L^2_m}.$$

Using he Gronwall lemma, we then obtain

$$\sup_{[0,T]} \|f\|^2_{L^2} \le \varepsilon_T \sup_{[0,T]} \|g\|_{L^2},$$

with $\varepsilon_T \to 0$ as $T \to 0$. We conclude to the existence and uniqueness of a solution in X_T by the usual argument (Banach fixed point theorem on a small interval and iteration process). □

1.2.4. *Well-posedness when $m := \langle x \rangle^k$.*
We come back to the case $m := \langle x \rangle^k$ for the McKean-Vlasov equation (1.3).

Lemma 1.6. *Consider $f \in Y_T$ a solution to the Fokker-Planck equation*

$$\partial_t f = \Delta f + \mathrm{div}(E f) + \mathrm{div}(F), \quad f(0) = f_0,$$

with $E \in \mathrm{Lip}(\mathbb{R}^d; \mathbb{R}^d)$ and $F \in L^2_{tx}$. For any $\beta \in C^2(\mathbb{R})$ such that $\beta'' \in L^\infty$, there holds

$$\begin{aligned} \partial_t \beta(f) &= \Delta \beta(f) - \beta''(f)|\nabla f|^2 + (\mathrm{div} E) f \beta'(f) \\ &\quad + E \cdot \nabla \beta(f) + \mathrm{div}(F \beta'(f)) - F \beta''(f) \cdot \nabla f. \end{aligned}$$

As a consequence, we have

$$\begin{aligned} \frac{d}{dt}\int \beta(f)\varphi &= \int \{-\nabla \beta(f) \cdot \nabla \varphi - \beta''(f)|\nabla f|^2 \varphi + (\mathrm{div} E) f \beta'(f) \varphi \\ &\quad + E \cdot \nabla \beta(f) \varphi - F \beta'(f) \cdot \nabla \varphi - F \beta''(f) \cdot \nabla f \varphi\}. \end{aligned}$$

Proof of Lemma 1.6. For a smooth and rapidly decaying function f the two formulas come from the chain rule and integration. For $f \in Y_T$, we consider $f * \rho_\varepsilon$, we write de formulas and we pass to the limit $\varepsilon \to 0$. □

Let us first consider the linear Fokker-Planck equation

$$\partial_t f = \mathcal{L} f = \Delta f + \mathrm{div}((A + B) f), \quad A \in L^\infty,\ B = x. \tag{1.14}$$

• For $m^* := e^{\alpha \langle x \rangle^2}$, $\alpha > 0$, a previous argument tells us that for any $f_0 \in L^2_{m^*}$, we may associate a unique variational solution belonging to the usual corresponding space.
• For $m := \langle x \rangle^k$, $k > d/2$, using Lemma 1.6 and Proposition 1.4 with $E = B$ and $F = Af$, we may write

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int f^2 m^2 &= -\int |\nabla f|^2 m^2 + \int f^2 m^2 \psi_B - \int A f \cdot \nabla(f m^2) \\ &\le -\frac{1}{2}\int |\nabla f|^2 m^2 + C_1' \int f^2 m^2, \end{aligned}$$

for a constant $C_1' := C_1'(k, \|\mathrm{div}B - kB\cdot x\langle x\rangle^{-2}\|_{L^\infty}, \|A\|_{L^\infty})$. Arguing as mentioned at the end of Section 1.2.2, we deduce that for any $f_0 \in L^2_k$, there exist at least a solution $f \in X_T$ to the Fokker-Planck equation (1.14) in the distributional sense. Now, for two distributional solutions belonging to X_T, we may apply Lemma 1.6 to the difference f and we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int f^2\varphi &= \int \frac{1}{2}f^2\Delta\varphi - |\nabla f|^2\varphi + f^2[(\mathrm{div}B)\varphi - \frac{1}{2}\mathrm{div}(B\varphi)] \\ &\quad - \int [f^2 A\cdot\nabla\varphi + Af\cdot\nabla f\varphi],\end{aligned}$$

for any $\varphi \in \mathcal{D}(\mathbb{R}^d)$, as well as $f(0) = 0$. Because $f \in X_T$, we may choose $\varphi = m^2\chi_R \in \mathcal{D}(\mathbb{R}^d)$ for a usual truncation function χ_R and we may pass to the limit $R \to \infty$, getting the same equation with $\varphi := m^2$. We recover as an a posteriori estimate the classical estimate

$$\frac{d}{dt}\int f^2\varphi \le -\int |\nabla f|^2 m^2 + C_2\int f^2 m^2, \tag{1.15}$$

with $C_2 = C_2(m, B, \|A\|_{L^\infty})$. Because $f \in C([0,T]; L^2_m)$ and $f(0) = 0$, we obtain that $f \equiv 0$ thanks to the Gronwall lemma, and thus the uniqueness.

Exercise 1.7. *Prove that we have the same uniqueness result for solutions which belong to the space*

$$X_T^w := L^\infty(0,T; L^2_q) \cap C([0,T]; L^2_{weak}) \cap L^2(0,T : H^1_q), \quad q > d/2$$

We will first establish that any such a solution f satisfies

$$t \mapsto f_t^2 \text{ is continuous in } \mathcal{D}'(\mathbb{R}^d)$$

and thus

$$f \in C([0,T]; L^2_\ell) \cap L^2(0,T : H^1_\ell), \quad \forall \ell < q.$$

Proposition 1.8. *Fix $q > d/2$ and $T > 0$. For any $f_0 \in L^2_q$, there exists a unique solution $f \in X_T^w$ to the McKean-Vlasov equation* (1.3).

Proof of Proposition 1.8. One just has to repeat the proof of Proposition 1.5 with minor modifications using the last a posteriori bound (with $A := a * f$) and the last uniqueness result. □

1.3. Existence of steady state for the McKean-Vlasov equation.

Theorem 1.9. *There exists at least one stationary state $G \in \mathcal{Z}$ to the McKean-Vlasov equation* (1.3).

We aim to apply Theorem 1.2 to the McKean-Vlasov equation (1.3). We check that the assumptions of the Theorem 1.2 are satisfied. We set $H := L^2_m \subset L^1$.

The nonlinear flow $S_t : H \to H$ is well defined thanks to the well-posedness results established in the previous section. Furthermore the set $\mathcal{Z}$ defined in (1.6) is invariant. Clearly, the set $\mathcal{Z}$ is compact for the weak topology L^2. We choose to provide $\mathcal{Z}$ with this weak topology (the alternative to choose the strong topology is also possible and leads to similar arguments). We then have (S2) and (S3) in the Definition 1.1 of a dynamical system because of the strong continuity of the trajectories and the uniqueness of the solutions yet established. In order to establish (S1) we start with the classical Aubin-Lions Lemma (that is reverse with respect to what has been taught during the classes, but simplifies a bit the presentation).

Lemma 1.10 (Aubin-Lions). *Consider a sequence (f_n) such that*
(1) (f_n) is bounded in $L^2(0,T;H^1)$;
(2) f_n satisfies

$$\partial_t f_n = \Delta f_n + \mathrm{div} g_n$$

with (g_n) bounded in $L^2((0,T)\times B_R)$, for any $R > 0$.

Then (f_n) is strongly sequentially compact in $L^2((0,T)\times B_R)$ for any $R > 0$.

Proof of Lemma 1.10. Up to the extraction of a subsequence we may assume that there exists $f \in L^2(0,T;H^1)$ such that

$$f_n \rightharpoonup f, \ \nabla_x f_n \rightharpoonup \nabla_x f, \quad \text{weakly in } L^2((0,T)\times\mathbb{R}^d). \tag{1.16}$$

Step 1. We introduce a sequence of mollifiers (ρ_ε) and more precisely we set $\rho_\varepsilon(x) := \varepsilon^{-d}\rho(\varepsilon^{-1}x)$ with $0 \le \rho \in \mathcal{D}(\mathbb{R}^d)$, $\langle \rho \rangle = 1$. We observe that

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^d} f_n(t,y)\,\rho_\varepsilon(x-y)\,dx = \int_{\mathbb{R}^d} (f_n\,\Delta\rho_\varepsilon - g_n\cdot\nabla\rho_\varepsilon)\,dy,$$

where the RHS term is bounded in $L^2((0,T)\times B_R)$ uniformly in n for any fixed $\varepsilon > 0$. We also clearly have

$$\nabla_x \int_{\mathbb{R}^2} f_n(t,y)\,\rho_\varepsilon(x-y)\,dx = -\int_{\mathbb{R}^2} f_n(t,y)\nabla_y\rho_\varepsilon(x-y)\,dy,$$

where again the RHS term is bounded in $L^2((0,T)\times B_R)$ uniformly in n for any fixed $\varepsilon > 0$. In other words, $f_n*\rho_\varepsilon$ is bounded in $H^1((0,T)\times B_R)$. Thanks to the Rellich-Kondrachov Theorem, we get that (up to the extraction of a subsequence) $(f_n*\rho_\varepsilon)_n$ is strongly convergent in $L^2((0,T)\times B_R)$. Thanks to (1.16) and for any fixed $\varepsilon > 0$, we then get

$$f_n * \rho_\varepsilon \to f * \rho_\varepsilon \text{ strongly in } L^2((0,T)\times B_R) \text{ as } n\to\infty.$$

Step 2. Now, we observe that

$$\|g - g*\rho_\varepsilon\|^2_{L^2_{tx}} = \int_0^T \|g - g*\rho_\varepsilon\|^2_{L^2_x}\,dt \le \varepsilon^2\int_0^T \|g\|^2_{H^1_x}\,dt,$$

where the inequality has been proved in a previous chapter. We conclude by writing

$$f_n - f = (f_n - f_n*\rho_\varepsilon) + (f_n*\rho - f*\rho) + (f*\rho_\varepsilon - f)$$

and using the convergence established in step 1 and the above estimate. □

We deduce the following result which in particular implies (S1).

Lemma 1.11. *Let (f_{0n}) be a sequence of initial data which elements belong to $\mathcal{Z}$ and such that $f_{0n} \rightharpoonup f_0$ weakly in L^2. Denoting by f_n (resp. f) the solutions to the McKean-Vlasov equation (1.3) associated to f_{0n} (resp. f_0), we have $f_n \to f$ strongly in $L^2((0,T)\times\mathbb{R}^d)$ for any $T>0$ and $f_n(t) \rightharpoonup f(t)$ weakly in L^2_k for any $t\ge 0$, what is nothing but (S1).*

Proof of Lemma 1.11. Step 1. We fix $T>0$. Because (f_n) is bounded in $L^\infty(0,T;L^2_k)$, there exists a function $f\in L^\infty(0,T;L^2_k)$ and a subsequcne (not relabeled) such that

$$f_n \rightharpoonup f \text{ weakly in } L^\infty(0,T;L^2_k).$$

The solution f_n satisfies

$$\partial_t f_n = \Delta f_n + \mathrm{div} g_n, \quad g_n := (a*f_n)f_n, \tag{1.17}$$

with (g_n) bounded in $L^2((0,T)\times\mathbb{R}^d)$ because (f_n) is bounded in $L^\infty(0,T;L^2_k)\subset L^\infty(0,T;L^1)$ and $a\in L^\infty$, so that $(a*f_n)$ is bounded in $L^\infty((0,T)\times\mathbb{R}^d)$. We may apply Lemma 1.10 and we get that $f_n\to f$ strongly in $L^2((0,T)\times B_M)$ for any $M>0$. Together with the estimate

$$\|g\mathbf{1}_{B_M}\|_{L^2(0,T;L^2_\ell)} \le \frac{C_T}{\langle M\rangle^{k-\ell}}\|g\|_{L^\infty(0,T;L^2_k)},$$

for $\ell\in(d/2,k)$, we deduce that $f_n\to f$ strongly in $L^2(0,T;L^2_\ell)\subset L^1(0,T;L^1)$.

Step 2. As a consequence of Step 1, for any $R>0$, we have

$$\|a*(f_n-f)\|_{L^1((0,T)\times B_R)} \le \|a\|_{L^\infty}\|f_n-f\|_{L^1((0,T)\times\mathbb{R}^d)} \to 0$$

and $a*f_n\to a*f$ in $L^1((0,T)\times B_R)$ and a.e. (for another subsequence). Using also that

$$\|a*\zeta\|_{L^\infty} \le \|a\|_{L^\infty}\|\zeta\|_{L^\infty(0,T;L^2_k)},$$

we classically deduce (exercise) that

$$(a*f_n)f_n \to (a*f)f \quad \text{strongly in } L^1((0,T)\times B_R).$$

We may pass to the limit in (1.17) and we get that $f \in Y^w_T$ is a solution to the McKean-Vlasov equation (1.3) associated to the initial datum f_0. By uniqueness of the solution, that is the whole sequence (f_n) which converges to f.

Step 3. For any fixed $\varphi \in \mathcal{D}(\mathbb{R}^d)$ and any $t \in [0,T]$, we have

$$u^\varphi_n(t) := \int_{\mathbb{R}^d} f_n \varphi \, dx \ \text{ bounded in } \ C([0,T])$$

and

$$\frac{d}{dt} u^\varphi_n(t) = \int_{\mathbb{R}^d} (f_n \, \Delta\varphi - g_n \cdot \nabla\varphi) \, dy \ \text{ bounded in } \ L^2(0,T).$$

By Morrey estimate $H^1(0,T) \subset C^{1/2}([0,T])$ and the Ascoli theorem, we have $u^\varphi_n \to u^\varphi$ in $C([0,T])$ for a subsequence. Thanks to a Cantor diagonal process, we may assume that the same holds for any φ in a coutable subset $\mathscr{F}$ of $\mathcal{D}(\mathbb{R}^d)$ which is dense $L^2(\mathbb{R}^d)$, in particular

$$\int_0^T \psi(t) \int_{\mathbb{R}^d} f_n \varphi(x) \, dx dt \to \int_0^T \psi(t) u^\varphi(t) dt, \quad \forall \, \psi \in \mathcal{D}((0,T)).$$

Thanks to the previous step, we have

$$\int_0^T \psi(t) \int_{\mathbb{R}^d} f \varphi(x) \, dx dt = \int_0^T \psi(t) u^\varphi(t) dt.$$

In other words, we identify the limit as

$$u^\varphi(t) := \int_{\mathbb{R}^d} f(t,x) \varphi(x) \, dx, \quad \forall \, t \in [0,T],$$

and it is the whole sequence $(u_n(t))$ which converges to $u^\varphi(t)$ for any $t \in [0,T]$. Finally, because $(f_n(t))$ is bounded in L^2 for any fixed $t \in [0,T]$, there exists a subsequence $(f_{n'}(t))$ which converges weakly L^2 to a limit g_t. We have both

$$u^\varphi_{n'}(t) \to v^\varphi(t) := \int \varphi g_t \quad \text{and} \quad u^\varphi_{n'}(t) \to u^\varphi(t)$$

for any $\varphi \in \mathscr{F}$, so that $v^\varphi(t) = u^\varphi(t)$. On other words, it is the whole sequence $(f_n(t))$ which converges weakly L^2 to the limit $f(t)$. □

2. The first eigenvalue problem

2.1. **An abstract Krein-Rutman theorem (existence part).** In this section, we consider a real Banach lattice X, that is a real Banach space $(X, \|\cdot\|)$ endowed with a partial order denoted by $\geq$ (or $\leq$) such that the following holds:
(1) The set $X_+ := \{f \in X; \ f \geq 0\}$ is a nonempty convex cone (compatibility of the order with the vector space structure).
(2) For any $f \in X$, there exist some unique positive part $f_+ \in X_+$ and negative part $f_- \in X_+$ such that $f = f_+ - f_-$ which are minimal: $f = g - h$, $g, h \geq 0$ imply $g \geq f_+$ and $h \geq f_-$ (generation and properness of the positive cone). We set $|f| := f_+ + f_- \in X_+$ the absolute value of $f \in X$.
(3) For any $f, g \in X$, $|f| \leq |g|$ implies $\|f\| \leq \|g\|$ (compatibility of norm and order structures).

In the examples, we will only deal with the weighted Lebesgue space $X = L^p_m$ case endowed with its usual partial order: $f \geq 0$ in $L^p_k(E, \mathscr{E}, \mu)$ iff $f(x) \geq 0$ for μ-a.e. $x \in E$.

Theorem 2.12. *Consider a positive semigroup $S = S_\mathcal{L}$ on a Banach lattice X in duality with a Banach lattice Y (we take $X = Y'$ and Y separable). We assume*

- **(1)** *there exist $\kappa_0 \in \mathbb{R}$ and $\psi_0 \in Y_+ \backslash \{0\}$ such that $[S_t f]_0 \geq e^{\kappa_0 t} [f]_0$ for any $f \in X_+$, where X_0 denotes the vector space X endowed with the (semi) norm $\|f\|_{X_0} = [f]_0 := \langle |f|, \psi_0 \rangle$;*
- **(2)** *there exist two familly of operators v, w such that*

$$S = v + w * S, \tag{2.18}$$

and $C_v, C_w \geq 0$, $\kappa \in \mathbb{R}$ such that

$$\|v(t)\|_{\mathscr{B}(X)} \leq C_v e^{\kappa t}, \quad \|w(t)\|_{\mathscr{B}(X_0, X)} \leq C_w e^{\kappa t}; \tag{2.19}$$

- **(3)** *there holds $\kappa < \kappa_0$ and $X_+ \cap B_X$ is compact in X_0.*

Then there exists a pair $(\lambda_1, f_1) \in \mathbb{R}_+ \times X_+ \backslash \{0\}$ *such that* $\mathcal{L} f_1 = \lambda_1 f_1$.

When there exists a splitting $\mathcal{L} = \mathcal{A} + \mathcal{B}$ and $\kappa_{\mathcal{B}} \in \mathbb{R}$ such that $\mathcal{A}$ is bounded, the operator $\mathcal{B}$ generates a semigroup $S_{\mathcal{B}}$ and

$$\|(S_{\mathcal{B}}\mathcal{A})^{(*\ell)} * S_{\mathcal{B}}(t)\|_{\mathscr{B}(X)} = \mathcal{O}(e^{\alpha t}), \quad \|(S_{\mathcal{B}}\mathcal{A})^{(*\ell)}(t)\|_{\mathscr{B}(X_0,X)} = \mathcal{O}(e^{\alpha t}),$$

for any $t \ge 0$, $\ell \ge 0$ and $\alpha > \kappa_{\mathcal{B}}$, then (2.18) holds with any $\kappa \in (\kappa_{\mathcal{B}}, \kappa_0)$ and

$$v := \sum_{\ell=0}^{N-1} S_{\mathcal{B}} * (\mathcal{A} S_{\mathcal{B}})^{(*\ell)}, \quad w := (S_{\mathcal{B}}\mathcal{A})^{(*N)}. \tag{2.20}$$

Proof of Theorem 2.12. Step 1. We define the set

$$\mathcal{C} := \{f \in X_+, \ [f]_0 = 1, \ \|f\| \le R\},$$

for a convenient constant $R > 0$ to be fixed later. We next define the nonlinear mapping

$$\Phi_t : \mathcal{C} \to X, \quad f \mapsto \frac{S_t f}{[S_t f]_0}.$$

Thanks to assumption **(1)**, we may observe that it is well defined because

$$[S_t f]_0 \ge e^{\kappa_0 t}[f]_0 = e^{\kappa_0 t} > 0. \tag{2.21}$$

For any $f \in \mathcal{C}$, we thus immediately have $\Phi_t f \ge 0$ and $[\Phi_t f]_0 = 1$. On the other hand, from assumption **(1)** again and the semigroup property, we have

$$[S(t)f]_0 \ge e^{\kappa_0 (t-s)}[S(s)f]_0. \tag{2.22}$$

For $f \in \mathcal{C}$ and $t \ge 0$, we next compute

$$\begin{aligned}
\|\Phi_t f\| &\le C_v e^{\kappa t} \frac{\|f\|}{[S(t)f]_0} + \int_0^t C_w e^{\kappa(t-s)} \frac{[S(s)f]_0}{[S(t)f]_0}\, ds \\
&\le C_v e^{(\kappa-\kappa_0)t}\|f\| + C_w \int_0^t e^{(\kappa-\kappa_0)(t-s)}\, ds \\
&\le C_v e^{(\kappa-\kappa_0)t} R + \frac{C_w}{\kappa_0 - \kappa},
\end{aligned}$$

where we have used the estimate (2.19) in the first line, the lower bounds (2.21) and (2.22) in the second line and then just the fact that $\kappa - \kappa_0 < 0$ in the last line. Fixing T_0 such that $C_v e^{(\kappa-\kappa_0)T_0} \le 1/2$ and next $R \ge 2C_w/(\kappa_0 - \kappa)$, we have thus $\Phi_{T_0} : \mathcal{C} \to \mathcal{C}$. We also notice that Φ_{T_0} is continuous for the weak $*\sigma(X, Y)$ topology. Thanks to the Tykonov fixed point Theorem, there exists $f_{T_0} \in \mathcal{C}$ such that $\Phi_{T_0} f_{T_0} = f_{T_0}$.

Step 2. In other words, we have established the existence of $f_{T_0} \in X$ such that

$$f_{T_0} \ge 0, \quad [f_{T_0}]_0 = 1, \quad S_{T_0} f_{T_0} = e^{\lambda_1 T_0} f_{T_0}, \tag{2.23}$$

with $\lambda_1 := (1/T_0) \log[S_{T_0} f_{T_0}]_0 \in \mathbb{R}$. We then write

$$0 = e^{-\lambda_1 T_0} S_{T_0} f_{T_0} - f_{T_0} = (\mathcal{L} - \lambda_1) \int_0^{T_0} e^{-\lambda_1 t} S_t f_{T_0} dt,$$

and we define

$$f_1 := \int_0^{T_0} e^{-\lambda_1 t} S_t f_{T_0} dt \in D(\mathcal{L}) \cap X_+ \setminus \{0\},$$

which satisfies $\mathcal{L} f_1 = \lambda_1 f_1$. □

2.2. **Existence of first eigenvalue for a Fokker-Planck type equation.** We consider the Fokker-Planck type equation

$$\partial_t f = \mathcal{L} f = \Delta f + \mathrm{div}(E f) + c f, \tag{2.24}$$

on the density $f = f(t,x)$, $t \geq 0$, $x \in \mathbb{R}^d$, where the force field $E \in \mathbb{R}^d$ is a given fixed (exterior) vector field and c is a potential function.

Theorem 2.13. *Assume $E = x$ and $0 \leq c \in L^\infty(\mathbb{R}^d)$, $c \not\equiv 0$. There exists a solution (λ_1, f_1) to the first eigenvalue problem with $\lambda_1 \geq 0$ and $0 \leq f_1 \in H^2_k$, $\forall\, k$.*

Proof of Theorem 2.13. We apply Theorem 2.12 with $X = L^2_k$, $k > d/2$.
(1) We observe that

$$\mathcal{L}^* 1 = c \geq 0,$$

so that **(1)** holds with $\psi_0 = 1$ and $\kappa_0 = 0$.
(2) We introduce the spliting

$$\mathcal{A} f := M \chi_R f, \; \mathbf{1}_{B_R} \leq \chi_R \in C_c(\mathbb{R}^d), \quad \mathcal{B} := \mathcal{L} - \mathcal{A}.$$

On the one hand, for any $\kappa_\mathcal{B} < 0$, we may find M and R large enough such that

$$S_\mathcal{B} : L^p_k \to L^p_k, \quad \mathcal{O}(e^{\kappa_\mathcal{B} t}),$$

for some $k > 0$ when $p = 1$ and for some $k > d/2$ when $p = 2$. Using Nash argument, we also have

$$S_\mathcal{B} : L^1_k \to L^2_k, \quad \mathcal{O}\Big(\frac{e^{\kappa_\mathcal{B} t}}{t^{d/4}}\Big).$$

Choosing $d \in \{1,2,3\}$ for simplicity, we may take $N = 2$ in (2.20) and we deduce that

$$\begin{aligned} \|S_\mathcal{B} \mathcal{A} * S_\mathcal{B}(t) \mathcal{A} f\|_{L^2_k} &\lesssim \int_0^t e^{\kappa_\mathcal{B}(t-s)} \frac{e^{\kappa_\mathcal{B} s}}{s^{d/4}} ds \|\mathcal{A} f\|_{L^1_k} \\ &\lesssim t^{1-d/4} e^{\kappa_\mathcal{B} t} \|f\|_{L^1} \lesssim e^{\kappa t} \|f\|_{L^1} \end{aligned}$$

for any $t \geq 0$ and any $\kappa \in (\kappa_\mathcal{B}, 0)$. We have established **(2)**.
(3) We obviously have $L^2_k \subset L^1$, $B_{L^2_k}$ is weakly compact and $f \mapsto \|f\|_{L^1}$ is continuous from X_+ into $\mathbb{R}$ for the weak L^2_k topology. ☐

2.3. **The Krein-Rutman theorem for the Fokker-Planck equation.**

Proposition 2.14. *The operator $\mathcal{L}$ satisfies "Kato's inequality" and the "strong maximum principle" in $H = L^2_k$, $k > d/2$.*

Proof of Proposition 2.14. Step 1. Kato's inequality. For a convex function $\beta : \mathbb{R} \to \mathbb{R}$ such that $\beta(s) = s\beta'(s)$, we clearly have

$$\mathcal{L}\beta(f) = \beta''(f)|\nabla f|^2 + \beta'(f)\mathcal{L} f \geq \beta'(f) \mathcal{L} f.$$

Step 2. Strong maximum principle. Consider $f \in H \backslash \{0\}$ such that $\mathcal{L} f = 0$. By a bootstrap regularization argument, we classically have $f \in W^{2,d}_{loc}(\mathbb{R}^d) \subset C(\mathbb{R}^d)$. By assumption there exist then $x_0 \in \mathbb{R}^d$, $c, r > 0$, such that $|f(x)| \geq c$ on $B(x_0, r)$. From Lemma 1.3, we also have that $\mathcal{L} - a$ is -1-dissipative for $a \geq 0$ large enough, in the sense that

$$\forall\, h \in D(\mathcal{L}) \quad ((\mathcal{L} - a)h, h)_H \leq -\|h\|^2_H. \tag{2.25}$$

We next observe that for $\sigma > 0$ large enough, the function $g(x) := c \exp(\sigma r^\gamma - \sigma |x - x_0|^\gamma)$ satisfies $g = c$ on $\partial B(x_0, r)$ and

$$\begin{aligned} (-\mathcal{L} + a) g &= \big[-\sigma^2 \gamma^2 |x - x_0|^{2(\gamma-1)} + \sigma\gamma(d + \gamma - 2)|x - x_0|^{\gamma - 2} \\ &\qquad - \mathrm{div} E + E \cdot (x - x_0) \gamma\sigma |x - x_0|^{\gamma-2} - a \big]\, g \leq 0 \quad \text{on} \quad B(x_0, r)^c. \end{aligned}$$

We define $h := (g - |f|)_+$ and $\Omega := \mathbb{R}^d \backslash B(x_0, r)$. We have $h \in H^1_0(\Omega, m dx)$ and

$$\begin{aligned} (\mathcal{L} - a) h &\geq \theta'(g - |f|)\, \mathcal{L}(g - |f|) - a\, h \\ &= \theta'(g - |f|)\, [(\mathcal{L} - a) g + a|f|] \geq 0, \end{aligned}$$

where we have used the notation $\theta(s) = s_+$. Thanks to a straightforward generalization of (2.25) to $H^1_0(\Omega, m)$, we deduce

$$0 \le ((\mathcal{L} - a)h, h)_{L^2(\Omega,m)} \le -\|h\|^2_{L^2(\Omega,m)},$$

and then $h = 0$. That implies $|f| \ge g$ on Ω, next $|f| > 0$ on $\mathbb{R}^d$ and then $f > 0$ or $f < 0$ because $f \in C(\mathbb{R}^d)$. □

Theorem 2.15. *Under the same conditions on E and c, there exists a solution $0 \le \phi_1 \in H^1_{-k}$, $\forall\, k > d/2$, to the dual eigenvalue problem $\mathcal{L}^*\phi_1 = \lambda_1\phi_1$. Furthermore, $f_1 > 0$, $\phi_1 > 0$ and they are the unique (up to normalization) solutions to the eigenvalue problem with positive eigenvectors.*

Proof of Theorem 2.15. Step 1. Positivity of f_1. Since $(\mathcal{L} - \lambda_1)f_1 = 0$, $f_1 \ge 0$ and $f_1 \not\equiv 0$, the strong maximum principle implies that $f_1 > 0$ on $\mathbb{R}^d$.

Step 2. Dual problem. We observe that

$$\mathcal{L}^*\phi = \Delta\phi - E\cdot\nabla\phi + c\phi,$$

so that for $m = \langle x\rangle^{-k}$, we have

$$\begin{aligned}\int (\mathcal{L}^*\phi) m\,dx &= \int \phi(\Delta m + \mathrm{div}(Em) + cm)dx \\ &\le \int \phi[\mathcal{O}(\langle x\rangle^{-k-2}) - k\langle x\rangle^{-k}]dx\end{aligned}$$

and similarly

$$\begin{aligned}\int (\mathcal{L}^*\phi)\phi m^2dx &= -\int |\nabla\phi|^2 m^2 dx + \int \phi^2(\frac{1}{2}\Delta m^2 + \frac{1}{2}\mathrm{div}(Em^2) + cm^2)dx \\ &\le \int \phi^2[\mathcal{O}(\langle x\rangle^{-2k-2}) + (d/2 - k)\langle x\rangle^{-2k}]dx.\end{aligned}$$

We may also write

$$S_\mathcal{L} = S_\mathcal{B} + ... + S_\mathcal{L} * (\mathcal{A}S_\mathcal{B})^{(*N)},$$

so that

$$S_{\mathcal{L}^*} = S_{\mathcal{B}^*} + ... + (S_{\mathcal{B}^*}\mathcal{A})^{(*N)} * S_{\mathcal{L}^*}$$

and by duality we have

$$\begin{aligned}&S_{\mathcal{B}^*} : L^p_{-k} \to L^p_{-k},\ \mathcal{O}(e^{\kappa_\mathcal{B} t}),\ \forall\, k > d/p';\\ &S_{\mathcal{B}^*} : L^1_{-k} \to L^2_{-k},\ \mathcal{O}(t^{-d/4}e^{\kappa_\mathcal{B} t}),\ \forall\, k > d/2.\end{aligned}$$

On the other hand, we have

$$\frac{d}{dt}\int \phi_t f_1 = \int (\mathcal{L}^* - \lambda_1)\phi_t f_1 = \int \phi_t(\mathcal{L}^* - \lambda_1)f_1 = 0$$

for any solution ϕ_t to the dual evolution equation $\partial_t\phi_t = (\mathcal{L}^* - \lambda_1)\phi_t$, and thus

$$\int \phi_t f_1 = \int \phi_0 f_1, \quad \forall\, t \ge 0.$$

It is worth observing that

$$\int |\phi f_1| \le \|\phi\|_{L^2_{-k}}\|f_1\|_{L^2_k},$$

so that the previous integral is well defined. The two pieces of information together imply

$$\begin{aligned}\|\bar S^*(t)\phi\|_{L^2_{-k}} &\le C_v e^{\kappa t}\|\phi\|_{L^2_{-k}} + \int_0^t C_w e^{\kappa(t-s)}[\bar S^*(s)\phi]_{f_1}ds \\ &\le C_v e^{\kappa t}R + \frac{C_w}{\kappa}\|f_1\|_{L^2_{-k}}[\phi]_{f_1} \le R,\end{aligned}$$

for $t \ge T_0$ large enough and $\phi \in \mathcal{Z}$, with

$$\mathcal{Z} := \{\psi \in L^2_{-k};\ \psi \ge 0,\ [\psi]_{f_1} = 1,\ \|\psi\|_{L^2_{-k}} \le R\},$$

with $R > 0$ fixed and large enough. We may apply Theorem 1.2 or argue in the following way. We define

$$U_T := \frac{1}{T}\int_0^T \bar S^*(t)\phi dt.$$

Because of the previous analysis, we have $U_T \in \mathcal{Z}$ for any $T \ge T_1 > 0$ if we choose $\phi \in \mathcal{Z}$. We deduce that there exists (T_k) such that $T_k \to \infty$ and $\phi_1 \in \mathcal{Z}$ such that $U_{T_k} \rightharpoonup \phi_1$ weakly in L^2_{-k}. We compute

$$\begin{aligned} \bar S^*(s)\phi_1 - \phi_1 &= \lim_{k\to\infty}\Big\{\frac{1}{T_k}\int_s^{T_k+s} \bar S^*(t)\phi dt - \frac{1}{T_k}\int_0^{T_k} \bar S^*(t)\phi dt\Big\} \\ &= \lim_{k\to\infty}\Big\{\frac{1}{T_k}\int_{T_k}^{T_k+s} \bar S^*(t)\phi dt - \frac{1}{T_k}\int_0^{s} \bar S^*(t)\phi dt\Big\} = 0, \end{aligned}$$

for any $s \ge 0$. That implies that $(\mathcal{L}^* - \lambda_1)\phi_1 = 0$.
Step 3. Positivity of ϕ_1. For the same reason as in step 1, we have $\phi_1 > 0$ on $\mathbb{R}^d$. □

3. Relative entropy for linear and positive PDE and longtime behavior

3.1. **General relative entropy.** We briefly discuss the long-time asymptotic for the linear and nonlinear Fokker-Planck equations (1.1).
We consider the general evolution PDE

$$\partial_t f = \Delta f - a\cdot\nabla f + cf + \int b\, f_*, \quad \int b\, f_* := \int b(x,x_*)f(x_*)\,dx_*, \quad b \ge 0. \tag{3.1}$$

If $g > 0$ is a solution

$$\partial_t g = \Delta g - a\cdot\nabla g + cg + \int b\, g_*$$

and if $\phi \ge 0$ is a solution to the dual evolution problem

$$-\partial_t\phi = \Delta\phi + \mathrm{div}(a\,\phi) + c\,\phi + \int b_*\,\phi_*, \quad \int b_*\,\phi_* := \int b(x_*,x)\,\phi(x_*)\,dx_*,$$

we can exhibit a family of entropies associated to the evolution PDE (3.1). More precisely, we establish the following result (and in fact a bit more accurate formulation of it).

Theorem 3.1. *For any real values convex function H, the generalized entropy functional*

$$f \mapsto \mathcal{H}(f) := \int_{\mathbb{R}^d} H(f/g)\,g\,\phi, \tag{3.2}$$

is an Lyapunov function for the evolution PDE (3.1) *(meaning that is is decaying function of time along the flows of the evolution PDE).*

Step 1. First order PDE. We assume that

$$\begin{aligned} \partial_t f &= -a\cdot\nabla f + cf \\ \partial_t g &= -a\cdot\nabla g + cg \\ -\partial_t\phi &= \mathrm{div}(a\,\phi) + c\,\phi, \end{aligned}$$

and we show that

$$\partial_t(H(X)g\phi) + \mathrm{div}(aH(X)g\phi) = 0, \quad X = f/g.$$

We compute

$$\begin{aligned} &\partial_t(H(X)g\phi) + \mathrm{div}(aH(X)g\phi) \\ &= H'(X)g\phi\,[\partial_t X + a\nabla X] + H(x)\,[\partial_t(g\phi) + \mathrm{div}(ag\phi)] \end{aligned}$$

The first term vanishes because

$$\partial_t X + a\nabla X = \frac{1}{g}\,(\partial_t f + a\nabla f) - \frac{f}{g^2}\,(\partial_t g + a\nabla g) = \frac{1}{g}\,(cf) - \frac{f}{g^2}\,(cg) = 0.$$

The second term also vanishes because

$$\partial_t(g\phi) + \mathrm{div}(ag\phi) = \phi\,[\partial_t g + a\nabla g] + g\,[\partial_t\phi + \mathrm{div}(a\phi)] = \phi\,[-cg] + g\,[+c\phi] = 0.$$

Step 2. Second order PDE. We assume that

$$\begin{aligned} \partial_t f &= \Delta f + cf \\ \partial_t g &= \Delta g + cg \\ -\partial_t \phi &= \Delta \phi + c\,\phi, \end{aligned}$$

and we show

$$\partial_t(H(X)g\phi) - \mathrm{div}(\phi\nabla(H(X)g)) + \mathrm{div}(gH(X)\nabla\phi) = -H''(X)g\phi|\nabla X|^2.$$

We first observe that

$$\begin{aligned} \Delta X &= \mathrm{div}\Big(\frac{\nabla f}{g} - f\,\frac{1}{g^2}\,\nabla g\Big) \\ &= \frac{\Delta f}{g} - 2\nabla f\,\frac{\nabla g}{g^2} + 2\,f\,\frac{|\nabla g|^2}{g^3} - \frac{f}{g^2}\,\Delta g \\ &= \frac{\Delta f}{g} - \frac{f\,\Delta g}{g^2} - 2\,\frac{\nabla g}{g}\cdot\nabla X, \end{aligned}$$

which in turn implies

$$\partial_t X - \Delta X = 2\,\frac{\nabla g}{g}\cdot\nabla X.$$

We then compute

$$\begin{aligned} &\partial_t(H(X)g\phi) - \mathrm{div}(\phi\nabla(H(X)g)) + \mathrm{div}(gH(X)\nabla\phi) = \\ &= (\partial_t H(X))\,g\phi + H(X)\,\partial_t(g\phi) - \phi\,\mathrm{div}[gH'(X)\nabla X + H(X)\nabla g] + gH(X)\Delta\phi \\ &= H'(X)g\phi\,\big\{\partial_t X - \Delta X - 2\frac{\nabla g}{g}\cdot\nabla X\big\} - g\phi\,H''(X)\,|\nabla X|^2 + H(X)\,[\partial_t(g\phi) - \phi\Delta g + g\Delta\phi] \\ &= -g\phi\,H''(X)\,|\nabla X|^2, \end{aligned}$$

since the first term and the last term independently vanish.

Step 3. Integral equation. We assume that

$$\begin{aligned} \partial_t f &= cf + \int bf_* \\ \partial_t g &= cg + \int bg_* \\ -\partial_t \phi &= c\,\phi + \int b_*\phi_*, \end{aligned}$$

with the notations

$$\int b\psi_* := \int b(x,x_*)\,\psi(x_*)\,dx_*, \qquad \int b_*\psi_* := \int b(x_*,x)\,\psi(x_*)\,dx_*,$$

and we show

$$\partial_t(H(X)g\phi) + \int H(X)gb_*\phi_* - \int bH(X_*)g_*\phi = -\int bg_*\phi\Big\{H(X_*) - H(X) - H'(X)(X_* - X)\Big\}$$

We compute indeed

$$\begin{aligned} \partial_t(g\phi H(X)) &= H(X)g\partial_t\phi + H(X)\phi\partial_t g + H'(X)\phi(\partial_t f - X\partial_t g) \\ &= -\int H(X)gb_*\phi_* + \int bH(X_*)g_*\phi \\ &\quad + \int bg_*\phi\Big\{-H(X_*) + H(X) + H'(X)X_* - H'(X)X\Big\} \end{aligned}$$

Step 4. Conclusion. For any solutions (f, g, ϕ) to the system of (full) equations, we have summing up the three computations

$$\begin{aligned}&\partial_t(g\phi H(X)) + \\ &+\mathrm{div}(aH(X)g\phi) - \mathrm{div}(\phi\nabla(H(X)g)) + \mathrm{div}(gH(X)\nabla\phi) + \int bH(X_*)g_*\phi - \int H(X)gb_*\phi_* \\ &= -g\phi\, H''(X)\,|\nabla X|^2 - \int bg_*\phi\Big\{H(X_*) - H(X) - H'(X)(X_* - X)\Big\}.\end{aligned}$$

Since when we integrate in the x variable the term on the second line vanishes, we find out

$$\frac{d}{dt}\mathcal{H}(f) = -D_{\mathcal{H}}(f),$$

with

$$D_{\mathcal{H}}(f) := \int g\phi\, H''(X)\,|\nabla X|^2 + \int\!\!\int bg_*\phi\Big\{H(X_*) - H(X) - H'(X)(X_* - X)\Big\} \ge 0,$$

so that (3.2) is proved. □

Exercise 3.2. *We consider a semigroup $S_t = e^{tL}$ of linear and bounded operators on L^1 and we assume that*
(i) $S_t \ge 0$;
(ii) $\exists\, g > 0$ *such that* $Lg = 0$, *or equivalently* $S_t g = g$ *for any* $t \ge 0$;
(iii) $\exists\, \phi \ge 0$ *such that* $L^*\phi = 0$, *or equivalently* $\langle S_t h, \phi\rangle = \langle h, \phi\rangle$ *for any* $h \in L^1$ *and* $t \ge 0$.
Our aim is to generalize to that a bit more general (and abstract) framework the general relative entropy principle we have presented for the evolution PDE (3.1).
(a) Prove that for any real affine function ℓ, *there holds* $\ell[(S_t f)/g]g = S_t[\ell(f/g)g]$.
(b) Prove that for any convex function H *and any* f, *there holds* $H[(S_t f)/g]g \le S_t[H(f/g)g]$. *(Hint. Use the fact that* $H = \sup_{\ell \le H} \ell$*).*
(c) Deduce that

$$\int H[(S_t f)/g]g\phi \le \int H[f/g]g\phi, \quad \forall\, t \ge 0.$$

3.2. **Dissipation of entropy method.** We briefly present some general result about the longtime asymptotic of dynamical system that we will take over in the next section on a concrete example of application.

Consider a dynamical system $(S_t)_{t\ge0}$ on a metric space $(\mathcal{Z}, d)$. We say that a functional $\mathcal{H} : \mathcal{Z} \to \mathbb{R}$ is an entropy if there exists a *dissipation of entropy* functional $\mathcal{D} : \mathcal{Z} \to \mathbb{R}_+$ such that for any $z \in \mathcal{Z}$ there holds

$$\frac{d}{dt}\mathcal{H}(S_t z) = -\mathcal{D}(S_t z) \le 0 \quad \forall\, t > 0,$$

or equivalently

$$\mathcal{H}(S_t z) + \int_0^t \mathcal{D}(S_s z)\, ds = \mathcal{H}(z). \tag{3.3}$$

As a consequence $t \mapsto \mathcal{H}(S_t z)$ is a decreasing function, and more importantly here, under the additional lower bound assumption

$$\mathcal{H}_z > -\infty, \quad \mathcal{H}_z := \inf_{y\in\omega_0(z)} \mathcal{H}(y), \tag{3.4}$$

there holds

$$\int_0^\infty \mathcal{D}(S_s z)\, ds \le \mathcal{H}(z) - \mathcal{H}_z < \infty. \tag{3.5}$$

We define

$$\omega_{\mathcal{D}}(z) := \{y \in \omega_0(z);\ \mathcal{D}(S_t y) = 0\ \forall\, t \ge 0\},$$

and we observe that $\mathcal{E}_z \subset \omega_{\mathcal{D}}(z)$ at least when (3.3) holds. (not clear ?)

Theorem 3.3. (Dissipation of entropy method - weak version). *Consider a dynamical system $(S_t)_{t\ge0}$ on a metric space $(\mathcal{Z},d)$ and $z\in\mathcal{Z}$. We assume*

(S4′) $(S_t z)_{t\ge0}$ is "locally uniformly compact" in the sense that $(\mathcal{S}^{z,T}_t)_{t\ge0}$ is relatively compact in $C([0,T];\mathcal{Z})$ for any fixed time $T\in\mathbb{R}_+$, where we have defined $s\mapsto\mathcal{S}^{z,T}_t(s):=S_{t+s}z$;

(H1) there exists a lsc dissipation of entropy functional $\mathcal{D}$ on $\mathcal{Z}$ such that $t\mapsto\mathcal{D}(S_t z)\in L^1$.

Then, we have $\omega(z)\subset\omega_{\mathcal{D}}(z)$, and therefore $d(S_t z,\omega_{\mathcal{D}}(z))\to0$ as $t\to\infty$.

Proof of Theorem 3.3. We define $z^t:=\mathcal{S}^{z,T}_t\in C([0,T];\mathcal{Z})$, $T>0$, and we observe that

$$\int_0^T\mathcal{D}(z^t(s))\,ds=\int_t^{t+T}\mathcal{D}(S_s z)\,ds\le\int_t^\infty\mathcal{D}(S_s z)\,ds.$$

Consider $y\in\omega(z)$ and a sequence $t_n\to\infty$ such that $S_{t_n}z\to y$ as $n\to\infty$. From the compactness assumption *(S4′)* and a diagonal Cantor procedure, there exist a subsequence $(t_{n'})$ and a function $z^*\in C([0,\infty);\mathcal{Z})$ such that $z^{t_{n'}}\to z^*$ in $C([0,T];\mathcal{Z})$ for any $T>0$ and obviously $z^*(s)=S_s y$ for any $s\ge0$. From the assumptions *(H1)* made on the dissipation of entropy and the above inequality, we then deduce

$$\int_0^T\mathcal{D}(z^*(s))\,ds\le\liminf_{n'\to\infty}\int_{t_{n'}}^\infty\mathcal{D}(S_s z)\,ds=0.$$

As a consequence $\mathcal{D}(z^*(s))=0$ for any $s\ge0$ and then $y\in\omega_{\mathcal{D}}(z)$. We conclude thanks to the general result Theorem 6.6.-(iii) about the ω-limit set which have been presented during the *Review course on differential calculus for ODEs and PDEs*. □

Theorem 3.4. (Dissipation of entropy method - strong version). *We assume furthermore that*

$$\omega_{\mathcal{D}}(z)\quad\text{is discrete.}\tag{3.6}$$

Then, $\omega(z)$ is a singleton and $\omega(z)\subset\mathcal{E}_z$. More explicitly, we have $\omega(z)=\{z^*\}\subset\mathcal{E}_z\cap\omega_{\mathcal{D}}(z)$ for some $z^*\in\mathcal{Z}$ or equivalently $S_t z\to z^*$ as $t\to\infty$.

Proof of Theorem 3.4. From Theorem 3.3 we have $\omega(z)\subset\omega_{\mathcal{D}}(z)$ which is assumed to be discrete. We conclude thanks to (iv) in Theorem 6.6 (in *Review course on differential calculus for ODEs and PDEs*). We conclude thanks to the general result Theorem 6.6.-(iv) about the ω-limit set which have been presented during the *Review course on differential calculus for ODEs and PDEs*. □

3.3. **Long-time behaviour.** We consider the Fokker-Planck equation

$$\partial_t f=\Delta f+\operatorname{div}(xf)+cf=:\mathcal{L}f,$$

$c\in L^\infty$, for which we have yet established the existence of a solution (λ_1,f_1,ϕ_1) to the eigentriplet problem

$$\lambda_1\in\mathbb{R},\quad\mathcal{L}f_1=\lambda_1 f_1,\ f_1>0,\quad\mathcal{L}^*\phi_1=\lambda_1\phi_1,\ \phi_1>0.$$

We also have established that any solution f to the rescaled Fokker-Planck equation

$$\partial_t f=\bar{\mathcal{L}}f:=\mathcal{L}f-\lambda_1 f,\tag{3.7}$$

satisfies

$$\frac{d}{dt}\mathcal{H}(f)=-\mathcal{D}(f)\le0$$

with

$$\mathcal{H}(f):=\int_{\mathbb{R}^d}H(f/f_1)f_1\phi_1dx,\quad H:\mathbb{R}\to\mathbb{R}_+\text{ convex.}$$

In particular and more precisely, we have

$$\frac{d}{dt}\mathcal{H}_1(f)\le0,\quad\text{with}\quad\mathcal{H}_1(f):=\int|f-cf_1|\phi_1$$

for the choice $H_1(s):=|s-c|$, $c\in\mathbb{R}$, and

$$\frac{d}{dt}\mathcal{H}_2(f)=-\mathcal{D}_2(f),\tag{3.8}$$

with

$$\mathcal{H}_2(f) := \int |f - cf_1|^2 f_1^{-1}\phi_1, \quad \mathcal{D}_2(f) := \int f_1\phi_1 |\nabla(f/f_1)|^2,$$

for the choice $H_2(s) := (s-c)^2$, $c \in \mathbb{R}$.

Theorem 3.5. *For any* $f_0 \in L^1_{\phi_1}(\cap L^1_q)$*, the associated solution to the rescaled Fokker-Planck equation* (3.7) *satisfies*

$$f(t) \to f_1 \langle f_0, \phi_1\rangle \quad \textit{as} \quad t \to \infty.$$

Proof of Theorem 3.5. Step 1. If $f_0^2 f_1^{-1}\phi_1 \in L^1$ then $\mathcal{H}_2(f_0) < \infty$ (whatever is $c \in \mathbb{R}$) and integrating (3.8), we get

$$\sup_{t\ge 0} \mathcal{H}_2(f(t))t \le \mathcal{H}_2(f_0), \qquad \int_0^\infty \mathcal{D}_2(f(t))\, dt \le \mathcal{H}_2(f_0).$$

We fix $T > 0$ and we define $f_n(t) := f(t+n)$. We deduce that

$$\sup_{[0,T]} \mathcal{H}_2(f_n) \le \mathcal{H}_2(f_0),$$

so that (f_n) is bounded in $L^\infty(0,T;L^2(B_R))$, $\forall\, R > 0$, and

$$\int_0^T \mathcal{D}_2(f_n(t))\, dt \le \int_n^\infty \mathcal{D}_2(f_n(t))\, dt \to 0,$$

so that $(\nabla(f_n(t)/f_1))$ is bounded in $L^2((0,T)\times B_R$, $\forall\, R>0$. By compactness of the L^2 ball (and a Cantor diagonal process), there exist (f_{n_k}) and $\bar f \in L^2_{loc}$ such that

$$f_{n_k} \rightharpoonup \bar f \ \text{ and } \ \nabla(f_{n_k}(t)/f_1) \rightharpoonup \nabla(\bar f/f_1) \quad \text{weakly} \quad L^2((0,T)\times B_R).$$

By convexity / lsc of the norm, we get

$$\int_0^T\!\!\int_{B_R} \big|\nabla \frac{\bar f}{f_1}\big|^2 f_1\phi_1\, dxdt \le \liminf \int_0^T\!\!\int_{B_R} \big|\nabla \frac{f_{n_k}}{f_1}\big|^2 f_1\phi_1\, dxdt \le \liminf \int_0^T \mathcal{D}_2(f_n)dt = 0.$$

We may next pass to the limit as $R \to \infty$ (by monotonous convergence) and we deduce that $\mathcal{D}_2(\bar f/f_1) = 0$, so that $\nabla(\bar f/f_1) = 0$ and finally $\bar f = cf_1$ for some constant $c \in \mathbb{R}$.
On the other hand, by Nash argument (...), we have $\phi_1 \in L^\infty_{-\ell}$, for any $\ell > d$, as well as for some $q > \ell > d$,

$$\frac{d}{dt}\int |f|\langle x\rangle^q \le \dots \le C\int |f|\phi_1 - \int |f|\langle x\rangle^q,$$

so that

$$\sup_{t\ge 0} \int |f|\langle x\rangle^q \le C \max\Big(\int |f_0|\phi_1, \int |f_0|\langle x\rangle^q\Big).$$

We deduce that

$$\int f_0\phi_1 = \int f_{n_k}\phi_1 = \int_{B_R} f_{n_k}\phi_1 + \int_{B_R^c} f_{n_k}\phi_1 \to \int_{B_R} \bar f\phi_1 + \mathcal{O}(R^{\ell-q}) = \int \bar f\phi_1.$$

Together with $\bar f = cf_1$ and the normalization $\langle f_1, \phi_1\rangle = 1$, we identify $c = c_0 := \langle f_0, \phi_1\rangle$.
Step 2. From the very definition of f_n, the Aubin-Lions Lemma 1.10 and the L^1_q estimate, we have

$$f_{n_k} \to c_0 f_1 \quad \text{in} \quad L^2((0,T)\times B_R),$$

next

$$f(t_{n_k},\cdot) \to c_0 f_1 \quad \text{in} \quad L^1_{\phi_1}, \quad t_{n_k} \in (n_k, n_{n+1}).$$

But since

$$\mathcal{H}_1(f) := \int |f - c_0 f_1|\phi_1 \searrow,$$

we deduce that

$$\sup_{t\ge n_{k+1}} \int |f(t) - c_0 f_1|\phi_1 \le \int |f(t_{n_k}) - c_0 f_1|\phi_1 \to 0.$$

When $f_0 \in L^1_q$, we use an approximation argument $f_{0n} \to f_0$ in L^1_q and $\mathcal{H}_2(f_{0n}) < \infty$ in order to get the same conclusion. □

4. Asymptotic of Stochastic semigroups

4.1. **Generalities.** From now on, we will be interested in Stochastic semigroups which is a class of semigroups which enjoy both a positivity and a "conservativity" property. The importance of Stochastic semigroups comes from its deep relation with Markov processes in stochastic theory as well as from the fact that a quite satisfactory description of the longtime behaviour of such a semigroups can be performed.

We start with the notion of positivity. It can be formulated in the abstract framework of Banach lattices $(X, \|\cdot\|, \geq)$ which are Banach spaces endowed with compatible order relation or equivalently with an appropriate positive cone X_+. To be more concrete, we just observe that the following three examples are Banach lattices when endowed with their usual order relation:

• $X := C_0(E)$, the space of continuous functions which tend to 0 at infinity (when E is not a compact set) endowed with the uniform norm $\|\cdot\|$;

• $X := L^p(E) = L^p(E, \mathcal{E}, \mu)$, the Lebesgue space of functions associated to the Borel σ-algebra $\mathcal{E}$, a positive σ-finite measure μ and an exponent $p \in [1, \infty]$;

• $X := M^1(E) = (C_0(E))'$, the space of Radon measures defined as the dual space of $C_0(E)$.

Here E denotes a σ-locally compact metric space (typically $E \subset \mathbb{R}^d$) and in the last example the positivity can be defined by duality: $\mu \geq 0$ if $\langle \mu, \varphi \rangle \geq 0$ for any $0 \leq \varphi \in C_0(E)$.

Lemma 4.1. *Consider X a Banach lattice (one of the above examples), a bounded linear operator A on X and its dual operator A^* on X'. The following equivalence holds:*
(1) A is positive, namely $Af \geq 0$ for any $f \in X$, $f \geq 0$;
(2) A^* is positive, namely $A^*\varphi \geq 0$ for any $\varphi \in X'$, $\varphi \geq 0$.

The (elementary) proof is left as an exercise. We emphasize that $\langle f, \varphi \rangle \geq 0$ for any $\varphi \in X'_+$ (resp. for any $f \in X_+$) implies $f \in X_+$ (resp. $\varphi \in X'_+$).

There are two "equivalent" (or "dual") ways to formulate the notion of Stochastic and Markov semigroup.

Definition 4.2. *On a Banach lattice $Y \supset C_0(E)$ we say that (P_t) is a Markov semigroup if*
(1) (P_t) is a continuous semigroup in Y;
(2) (P_t) is positive, namely $P_t \geq 0$ for any $t \geq 0$;
(3) (P_t) is conservative, namely $\mathbf{1} \in Y$ and $P_t\mathbf{1} = \mathbf{1}$ for any $t \geq 0$.

Definition 4.3. *On a Banach lattice $X \subset M^1(E)$ we say that (S_t) is a stochastic semigroup if*
(1) (S_t) is a (strongly or weakly $*$ continuous) continuous semigroup in X;
(2) (S_t) is positive, namely $S_t \geq 0$ for any $t \geq 0$;
(3) (S_t) is conservative, namely $\langle S_t f \rangle = \langle f \rangle$, $\forall\, t \geq 0$, $\forall\, f \in X$, where $\langle g \rangle := \langle g, \mathbf{1} \rangle$.

The two notions are dual. In particular, if (P_t) is a Markov semigroup on $Y \supset C_0(E)$, the dual semigroup (S_t) defined by $S_t := P_t^*$ on $X := Y'$ is a stochastic semigroup. In the sequel we will only consider stochastic semigroups defined on $X \subset L^1(E)$.

Stochastic semigroup and semigroup of contractions for the L^1 are closely linked.

Proposition 4.4. *A Stochastic semigroup is a semigroup of contractions for the L^1 norm. In the other way round, a mass conservative semigroup of contractions for the L^1 norm is positive, and thus it is a Stochastic semigroup.*

Proof of Proposition 4.4. We fix $f \in X$ and $t \geq 0$. We write

$$\begin{aligned} |S_t f| &= |S_t f_+ - S_t f_-| \\ &\leq |S_t f_+| + |S_t f_-| \\ &= S_t f_+ + S_t f_- \\ &= S_t |f|, \end{aligned}$$

where we have used the positivity property in the third line. We deduce

$$\int |S_t f| \leq \int S_t |f| = \int |f|,$$

because of the mass conservation. For the reciprocal part, we consider $f \ge 0$. From both the contraction property and the mass conservation, we have

$$\|S_t f\|_1 \le \|f\|_1 = \int f = \int S_t f.$$

As a consequence,

$$\|(S_t f)_-\|_{L^1} = \frac{1}{2} \int (|S_t f| - S_t f) \le 0$$

so that $(S_t f)_- = 0$ and thus $S_t f \ge 0$. That proves the positivity property.

We may also characterize a Stochastic semigroup in terms of its generator.

Theorem 4.5. *Let $S = S_{\mathcal{L}}$ be a strongly continuous semigroup on a Banach space $X \subset L^1$. There is equivalence between*
(a) $S_{\mathcal{L}}$ is a Stochastic semigroup;
(b) $\mathcal{L}^* 1 = 0$ and $\mathcal{L}$ satisfies Kato's inequality

$$(\mathrm{sign}\, f)\mathcal{L} f \le \mathcal{L}|f|, \quad \forall\, f \in D(\mathcal{L}).$$

Partial proof of Theorem 4.5. *Step 1. We prove* $(a) \Rightarrow (b)$. On the one hand, for any $f \in D(\mathcal{L})$ and any $0 \le \psi \in D(\mathcal{L}^*)$, we have

$$\begin{aligned}
\langle \psi, (\mathrm{sign} f)\mathcal{L} f\rangle &= \lim_{t\to 0} \frac{1}{t} \langle \psi, (\mathrm{sign} f)(S(t)f - f)\rangle \\
&\le \lim_{t\to 0} \frac{1}{t} \langle \psi, |S(t) f| - |f|\rangle \\
&\le \lim_{t\to 0} \frac{1}{t} \langle \psi, S(t)|f| - |f|\rangle \\
&= \lim_{t\to 0} \frac{1}{t} \langle S^*(t)\psi - \psi, |f|\rangle \\
&= \langle \mathcal{L}^*\psi, |f|\rangle,
\end{aligned}$$

where we have used the inequality $(\mathrm{sign} f) g \le |g|$ in the second line and the positivity assumption in the third line. That inequality is the weak formulation of Kato's inequality. On the other hand and similarly, for any $f \in D(\mathcal{L})$, we have

$$\begin{aligned}
\langle \mathcal{L}^* \mathbf{1}, f\rangle &= \langle \mathbf{1}, \mathcal{L} f\rangle \\
&= \lim_{t\to 0} \frac{1}{t} \langle \mathbf{1}, S(t) f - f\rangle = 0,
\end{aligned}$$

by just using the mass conservation property.

Step 2. We prove $(b) \Rightarrow (a)$. On the one hand, for any $f \in D(\mathcal{L})$ and $t \ge 0$, we denote $f_t := S_t f$ and we write

$$\langle S_t f - f\rangle = \Big\langle \int_0^t \mathcal{L} f_s \, ds, \mathbf{1} \Big\rangle = \int_0^t \langle f_s, \mathcal{L}^* \mathbf{1}\rangle \, ds = 0.$$

On the other hand, in order to conclude it is enough to prove that (S_t) is a semigroup of contractions. We consider $f \in D(\mathcal{L}^2)$, $t \ge 0$, $n \in \mathbb{N}^*$, we introduce the notation $f_t := S_t f$, $t_k := kt/n$, and we write

$$\begin{aligned} |S_t f| - |f| &= \sum_{k=0}^{n-1} (|f_{t_{k+1}}| - |f_{t_k}|) \\ &\le \sum_{k=0}^{n-1} \mathrm{sign} f_{t_{k+1}} (f_{t_{k+1}} - f_{t_k}) \\ &= \sum_{k=0}^{n-1} \mathrm{sign} f_{t_{k+1}} \int_{t_k}^{t_{k+1}} \mathcal{L} f_s \, ds \\ &= \sum_{k=0}^{n-1} \mathrm{sign} f_{t_{k+1}} \Big\{ \frac{1}{n} \mathcal{L} f_{t_{k+1}} + \int_{t_k}^{t_{k+1}} \mathcal{L}(f_s - f_{t_{k+1}}) \, ds \Big\} \\ &\le \sum_{k=0}^{n-1} \Big\{ \frac{1}{n} \mathcal{L} |f_{t_{k+1}}| + \mathrm{sign} f_{t_{k+1}} \int_{t_k}^{t_{k+1}} \int_{t_{k+1}}^{s} (S_u \mathcal{L}^2 f) \, du ds \Big\}, \end{aligned}$$

where we have used the inequality $(\mathrm{sign} f) g \le |g|$ in the second line and Kato's inequality in the last line. Taking the mean and using the mass conservation, we have

$$\begin{aligned} \|S_t f\| - \|f\| &\le \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \int_{s}^{t_{k+1}} \|S_u \mathcal{L}^2 f\| \, du ds \\ &\le \frac{1}{n} \int_0^t \|S_u \mathcal{L}^2 f\| \, du \to 0, \end{aligned}$$

as $n \to \infty$. □

Exercise 4.6. *Consider $S_{\mathcal{L}^*}$ a (constant preserving) Markov semigroup and $\Phi : \mathbb{R} \to \mathbb{R}$ a concave function. Prove that $\mathcal{L}^* \Phi(m) \le \Phi'(m) \mathcal{L}^* m$. (Hint. Use that $\Phi(a) = \inf\{\ell(a);\ \ell$ affine such that $\ell \ge \Phi\}$ in order to prove $S_t^*(\Phi(m)) \le \Phi(S_t^* m)$ and $\Phi(b) - \Phi(a) \ge \Phi'(a)(b-a)$).*

4.2. **Strong positivity condition and Doblin Theorem.** We consider the case of a strong positivity condition.

Theorem 4.7 (Doeblin)**.** *Consider a Stochastic semigroup S_t such that*

$$S_T f \ge \alpha \, \nu \, \langle f \rangle, \quad \forall \, f \in X_+,$$

for some constants $T > 0$ and $\alpha \in (0,1)$ and some probability measure ν. There holds

$$\|S_t f\|_{L^1} \le C \, e^{at} \|f\|_{L^1}, \quad \forall \, t \ge 0, \ \forall \, f \in X, \ \langle f \rangle = 0,$$

for some constants $C \ge 1$ and $a < 0$.

Proof of Theorem 4.7. We fix $f \in X$ such that $\langle f \rangle = 0$ and we define $\eta := \alpha \nu \langle f_+ \rangle = \alpha \nu \langle f_- \rangle$. We write

$$\begin{aligned} |S_T f| &= |S_T f_+ - \eta - S_T f_- + \eta| \\ &\le |S_T f_+ - \eta| + |S_T f_- - \eta| \\ &= S_T f_+ - \eta + S_T f_- - \eta, \end{aligned}$$

where in the last equality we have used the Doeblin condition. Integrating, we deduce

$$\begin{aligned} \int |S_T f| &\le \int S_T f_+ - \alpha \langle \nu \rangle \langle f_+ \rangle + \int S_T f_- - \alpha \langle \nu \rangle \langle f_- \rangle \\ &\le \int f_+ - \alpha \, \langle f_+ \rangle + \int f_- - \alpha \, \langle f_- \rangle \\ &\le (1 - \alpha) \int |f|. \end{aligned}$$

By induction, we obtain $a := [\log(1-\alpha)]/T$ and $C := \exp[|a|T]$. □

4.3. **Geometric stability under Harris and Lyapunov conditions.** We consider now a semigroup S with generator $\mathcal{L}$ and we assume that
(H1) there exists some weight function $m : \mathbb{R}^d \to [1,\infty)$ satisfying $m(x) \to \infty$ as $x \to \infty$ and there exist some constants $\alpha > 0, b > 0$ such that

$$\mathcal{L}^* m \le -\alpha\, m + b;$$

(H2) for any $R > 0$, there exists a constant $T \ge T_0 > 0$ and a positive and not zero measure $\nu = \nu_R$ such that

$$S_T f \ge \nu \int_{B_R} f, \quad \forall\, f \in X_+.$$

Theorem 4.8 (Doeblin)**.** *Consider a Stochastic semigroup S on $X := L^1(m)$ which satisfies (H1) and (H2). There holds*

$$\|S_t f\|_{L^1(m)} \le C\, e^{at} \|f\|_{L^1(m)}, \quad \forall\, t \ge 0,\ \forall\, f \in X,\ \langle f \rangle = 0,$$

for some constants $C \ge 1$ and $a < 0$.

We start with a variant of the key argument in the above Doeblin's Theorem.

Lemma 4.9 (Doeblin's variant)**.** *Under assumption (H2), if $f \in L^1(m)$, with $m(x) \to \infty$ as $|x| \to \infty$, satisfies*

$$\|f\|_{L^1} \ge \frac{4}{m(R)} \|f\|_{L^1(m)} \quad \text{and} \quad \langle f \rangle = 0, \tag{4.1}$$

we then have

$$\|S_T f\|_{L^1} \le \big(1 - \frac{\langle \nu \rangle}{2}\big) \|f\|_{L^1}.$$

Proof of Lemma 4.9. From the hypothesis (4.1), we have

$$\begin{aligned} \int_{B_R} f_\pm &= \int f_\pm - \int_{B_R^c} f_\pm \\ &\ge \frac{1}{2} \int |f| - \frac{1}{m(R)} \int |f| m \ge \frac{1}{4} \int |f|. \end{aligned}$$

Together with (H2), we get

$$S_T f_\pm \ge \frac{\nu}{4} \int |f| =: \eta.$$

We deduce

$$|S_T f| \le |S_T f_+ - \eta| + |S_T f_- - \eta| = S_T f_+ - \eta + S_T f_- - \eta = S_T |f| - 2\eta,$$

and next

$$\int |S_T f| \le \int S_T |f| - 2 \int \eta = \int |f| - \langle \nu \rangle \frac{1}{2} \int |f|,$$

which is nothing but the announced estimate. □

Proof of Theorem 4.8. We split the proof in several steps. We fix $f_0 \in L^1(m)$, $\langle f_0 \rangle = 0$ and we denote $f_t := S_t f_0$.
Step 1. From (H1), we have

$$\frac{d}{dt} \|f_t\|_{L^1(m)} \le -\alpha \|f_t\|_{L^1(m)} + b \|f_t\|_{L^1},$$

from what we deduce

$$\|f_t\|_{L^1(m)} \le e^{-\alpha t} \|f_0\|_{L^1(m)} + \big(1 - e^{-\alpha t}\big) \frac{b}{\alpha} \|f_0\|_{L^1} \quad \forall\, t \ge 0.$$

In other words, for any $T \ge T_0 > 0$, we have

$$\|S_T f_0\|_{L^1(m)} \le \gamma_L \|f_0\|_{L^1(m)} + K \|f_0\|_{L^1}, \tag{4.2}$$

with $\gamma_L \in (0,1)$ and $K > 0$, both constants depending only of T_0. We fix $R > 0$ large enough such that $K/A < 1 - \gamma_L$ with $A := m(R)/4$.
On the other hand, we recall that

$$\|S_T f_0\|_{L^1} \le \|f_0\|_{L^1}, \quad \forall\, T \ge 0, \tag{4.3}$$

and because of Lemma 4.9, there exists $\gamma_H \in (0,1)$ and $T \ge T_0$ only depending on R defined above such that

$$\|S_T f_0\|_{L^1} \le \gamma_H \|f_0\|_{L^1} \quad \text{when} \quad A\,\|f_0\|_{L^1} \ge \|f_0\|_{L^1(m)}. \tag{4.4}$$

Step 2. We introduce the modified norm

$$|||g||| := \|g\|_{L^1} + \beta \|g\|_{L^1(m)}$$

and we observe that we have the alternative

$$A\,\|f_0\|_{L^1} \ge \|f_0\|_{L^1(m)} \quad \text{or} \quad A\,\|f_0\|_{L^1} < \|f_0\|_{L^1(m)}.$$

In the first case of the alternative, using the Lyapunov estimate (4.2) and the coupling estimate (4.4), we have

$$\begin{aligned} |||S_T f_0||| &= \|S_T f_0\|_{L^1} + \beta \|S_T f_0\|_{L^1(m)} \\ &\le (\gamma_H + \beta K)\|f_0\|_{L^1} + \beta \gamma_L \|f_0\|_{L^1(m)} \\ &\le \gamma_1 |||f_0|||, \end{aligned}$$

with $\gamma_1 := \max(\gamma_H + \beta K, \gamma_L) < 1$, by fixing from now on $\beta > 0$ small enough. In the second case of the alternative, using the Lyapunov estimate (4.2) and the non expansion estimate (4.4), we have

$$\begin{aligned} |||S_T f_0||| &= \|S_T f_0\|_{L^1} + \beta \|S_T f_0\|_{L^1(m)} \\ &\le (1 + \beta K - \beta\delta)\|f_0\|_{L^1} + \beta(\gamma_L + \delta/A) \|S_T f_0\|_{L^1(m)} \\ &\le \gamma_2 |||f_0|||, \end{aligned}$$

with $\gamma_2 := \max(1 + \beta K - \beta\delta, \gamma_L + \delta/A)$ for any $0 < \beta\delta < 1 + \beta K$. We take $\delta := K + \varepsilon$, $\varepsilon > 0$, so that we get

$$\gamma_2 = \max(1 - \beta\varepsilon, (\gamma_L + K/A) + \varepsilon/A) < 1,$$

by choosing $\varepsilon > 0$ small enough and by recalling from the very definition of A that $\gamma_L + K/A < 1$. In any cases, we have thus established that

$$|||S_T f_0||| \le \gamma |||f_0|||, \quad \text{with} \quad \gamma := \max(\gamma_1, \gamma_2) < 1.$$

We then conclude as in the proof of Theorem 4.7.

Step 3 (Alternative argument). Alternatively, the two estimates (4.3) and (4.4) together give

$$\|S f_0\|_{L^1} \le \gamma_H \|f_0\|_{L^1} + \frac{1-\gamma_H}{A} \|f_0\|_{L^1(m)}. \tag{4.5}$$

Together with step 1, we deduce that

$$U^{n+1} = M U^n$$

with

$$U^n := \begin{pmatrix} \|S_T^n f_0\|_{L^1(m)} \\ \|S_T^n f_0\|_{L^1} \end{pmatrix} \quad \text{and} \quad M := \begin{pmatrix} \gamma_L & K \\ \frac{1-\gamma_H}{A} & \gamma_H \end{pmatrix}.$$

The eigenvalues of M are

$$\mu_\pm := \frac{1}{2}(T \pm \sqrt{T^2 - 4D}),$$

with

$$T := \text{tr} M = \gamma_L + \gamma_H, \quad D := \det M = \gamma_L \gamma_H - (1 - \gamma_H)\frac{K}{A}.$$

We observe that

$$\gamma_L \gamma_H > D > \gamma_L \gamma_H - (1 - \gamma_H)(1 - \gamma_L) = T - 1,$$

so that

$$(\gamma_H - \gamma_L)^2 = T^2 - 4\gamma_L\gamma_H < T^2 - 4D < T^2 - 4(T-1) = (T-2)^2$$

and finally

$$\theta := \max(|\mu_+|, |\mu_-|) < \max(\gamma_H, \gamma_L, |T-1|, 1) = 1.$$

We have established that $\|M^n\| \le C\,\theta^n \to 0$ for some constant $C \ge 1$, and we then conclude as in the proof of Theorem 4.7. □