Université Paris-Dauphine
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An introduction to evolution PDEs

CHAPTER 7 - A CRASH COURSE ABOUT KINETIC EQUATIONS

- STILL A DRAFT -

Contents

1. Introduction

Kinetic equations are a class of equations which aims to describe the evolution of gases in out of equilibrium statistical physics. In the classical theory the state of each particle of the gas is a couple (x, v) of the position of particle $x \in \Omega \subset \mathbb{R}^d$ and the the velocity of the particle $v \in \mathbb{R}^d$, and then the gas is described by the density $f = f(t, x, v) \geq 0$ of particles. The most famous of the kinetic equations are the Boltzmann equation which describe a gas which particules interact through binary collisions and the Vlasov equation which describe a gas of charged particules which move in a electromagnetic field environment. The Boltzmann equation writes

$$\partial_t f + v \cdot \nabla_x f = Q(f, f),$$

where Q is the Boltzmann quadratic collision operator which only acts on the velocity variable. On the other hand, the Vlasov equation writes

$$\partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = 0, \tag{1.1}$$

where E is a force field which can be a given exterior electric or magnetic field, a self-induced electric field (given through the Poisson equation) or a self-induced pair electromagnetic field (given through the Maxwell system).

The aim of the notes is to present three distinct aspects about the kinetic theory:

- remarkable properties of velocity averages of the solution to the simplest kinetic equation: the free transport equation;

- an existence of solution result for the space homogeneous Landau equation (this one being a kind of approximation of the Boltzmann equation);
- a result about hypocoercivity (quantitative and constructive rate of convergence to the equilibrium) for one of the simplest linear kinetic equation: the relaxation equation.

2. Dispersion, moments and averaging lemmas for the free transport equation

In this section we consider the free transport equation

$$\partial_t f + v \cdot \nabla_x f = 0, \quad f_{|t=0} = f_0, \tag{2.1}$$

and the denoted by $\mathcal{S}(t)$ the associated semigroup defined through the characteristics formula

$$[\mathcal{S}(t) f_0](x,v] := f(t,x,v) = f_0(x - vt, v). \tag{2.2}$$

The free transport equation is norm preserving, namely

$$\|f(t,\cdot)\|_{L^p_{xv}} = \|f_0\|_{L^p_{xv}}, \quad \forall\, t \ge 0, \tag{2.3}$$

for any $1 \le p \le \infty$, what is an immediate consequence of the characteristics formula (2.2): there is no gain of integrability nor regularity for the the solutions to the free transport equation. Despite of the above hyperbolic nature of free transport equation, we may establish some kind of regularization properties on velocity averages. More precisely, for a given weight function $\varphi : \mathbb{R}^d \to \mathbb{R}$, we define and consider the velocity average

$$\rho_\varphi = \mathcal{A} h = \mathcal{A}_\varphi h := \int_{\mathbb{R}^d} h(v)\, \varphi(v)\, dv, \quad \forall\, h, \tag{2.4}$$

when this quantity makes sense, and in particular when $\varphi = 1$, we write

$$\rho = \rho_h := \int_{\mathbb{R}^d} h(v)\, dv, \quad \forall\, h.$$

From (2.3) for any $\varphi \in C_c(\mathbb{R}^d)$, we already have that the averaging

$$\rho_\varphi(t,x) := \int_{\mathbb{R}^d} f(t,x,v)\, \varphi(v)\, dv = [\mathcal{A}_\varphi \mathcal{S}(t) f_0](x)$$

of the solution f to the free transport equation satisfies

$$\|\rho(t,\cdot)\|_{L^p_x} \lesssim \|f_0\|_{L^p_{xv}}, \quad \forall\, t \ge 0, \tag{2.5}$$

for any $1 \le p \le \infty$ and any $f_0 \in L^p_{xv}$. In the following sections, we establish several smoothing properties on $\rho(t)$, or equivalently, several regularity properties on the time indexed family of operators $\mathcal{A}\mathcal{S}(t)$.

2.1. Dispersion lemmas and decay properties.

We present two versions of the so-called *dispersion property of the free transport equation* which are related of the fact that particles move away from any bounded region or also to the fact that two particules with different velocities go far away one to the other when they start from the same point. The first result is very based on two nice conservations and a standard interpolation argument.

Lemma 2.1. *There exists $C = C(d)$ such that for any solution to the transport equation with nice initial datum, we have*

$$\|\rho(t,\cdot)\|_{L^{(d+2)/d}(\mathbb{R}^d)} \le C\, t^{-2d/(d+2)}\, \|f_0\|^{2/(d+2)}_{L^\infty} \||x|^2 f_0\|^{d/(d+2)}_{L^1}, \quad \forall\, t > 0.$$

Proof of Lemma 2.1. We consider the multiplicator function

$$m(x,v) := \frac{x}{|x|} \cdot \frac{v}{|v|}$$

For nice f_0, we have

$$\begin{aligned} \|f(t,\cdot)\|_{L^\infty} &= \|f_0\|_{L^\infty}, \\ \int_{\mathbb{R}^{2d}} f(t,x,v) |x - vt|^2\, dxdv &= \int_{\mathbb{R}^{2d}} f_0(x,v) |x|^2\, dxdv. \end{aligned}$$

For any $t > 0$, $x \in \mathbb{R}^d$, $\delta > 0$, we then compute

$$\begin{aligned} |\rho(t,x)| &= \int_{\mathbb{R}^d} f(t,x,v)\mathbf{1}_{|x-vt|\le\delta}\,dv + \int_{\mathbb{R}^d} f(t,x,v)\mathbf{1}_{|x-vt|\ge\delta}\,dv \\ &\le \|f(t,\cdot)\|_{L^\infty_v}\int_{\mathbb{R}^d}\mathbf{1}_{|x-vt|\le\delta}\,dv + \frac{1}{\delta^2}\int_{\mathbb{R}^d} f(t,x,v)|x-vt|^2\,dv \\ &\le \|f_0\|_{L^\infty_{xv}} C(d)(\delta/t)^d + \frac{1}{\delta^2}\int_{\mathbb{R}^{2d}} f_0(x,v)|x|^2\,dv \\ &\le C'(d)t^{-2d/(d+2)}\|f_0\|^{2/(d+2)}_{L^\infty_{xv}}\|f_0(x,\cdot)|x|^2\|^{d/(d+2)}_{L^1_v}, \end{aligned}$$

where in the last line we have optimized δ. We deduce

$$|\rho(t,x)|^{(d+2)/d} \le C''(d)t^{-2}\|f_0\|^{2/d}_{L^\infty_{xv}}\|f_0(x,\cdot)|x|^2\|_{L^1_v},$$

and we conclude by integrating in the position variable. □

Lemma 2.2. *For $f_0 \in L^q_x(L^p_v)$, $1 \le q \le p \le \infty$, there holds*

$$\|\mathcal{A}\mathcal{S}(t)f_0\|_{L^p_x} \lesssim \frac{1}{t^{d/(qp')}}\|f_0\|_{L^q_x(L^p_v)}, \quad \forall\, t > 0. \tag{2.6}$$

Proof of Theorem 2.2. When $f_0 \in L^q_x(L^q_v)$, we obviously have

$$\|\mathcal{A}\mathcal{S}(t)f_0\|_{L^q} \le \|\varphi\|_{L^\infty}\|\mathcal{S}(t)f_0\|_{L^q} = \|\varphi\|_{L^\infty}\|f_0\|_{L^q}, \quad \forall\, t \ge 0.$$

When $f_0 \in L^q_x(L^\infty_v)$, we use the representation formula

$$|f(t,x,v)| = |f_0(x-vt,v)| \le F_0(x-vt), \quad F_0(z) := \|f_0(z,\cdot)\|_{L^\infty},$$

to get

$$\begin{aligned} \|\mathcal{A}\mathcal{S}(t)f_0\|_{L^\infty} &\le \int_{\mathbb{R}^d} F_0(x-vt)\,\varphi(v)\,dv = \frac{1}{t^d}\int_{\mathbb{R}^d} F_0(w)\,\varphi\big(\frac{x-w}{t}\big)\,dw \\ &\le \frac{1}{t^{d/q}}\|\varphi\|_{L^{q'}}\|F_0\|_{L^1}, \end{aligned}$$

which is nothing but (2.6) with $p = \infty$. We deduce the general case $p \in (1,\infty)$ by interpolating the two previous bound. □

2.2. **Transfert of regularity.** As in the previous lemma in which some integrability in the v variable for the initial data has been transferred into a gain of integrability in the x for the for the averaging of the solution, we show in the following result how regularity on the v variable for the initial data into may be transferred into a gain of regularity in the x for the for the averaging of the solution.

Lemma 2.3. *For $f_0 \in L^p(\mathbb{R}^d \times \mathbb{R}^d)$ such that $\nabla_v f_0 \in L^p(\mathbb{R}^d \times \mathbb{R}^d)$, $1 \le p \le \infty$, there holds*

$$\|\nabla\mathcal{A}\mathcal{S}(t)f_0\|_{L^p_x} \lesssim \left(1 + \frac{1}{t}\right)(\|f_0\|_{L^p_{xv}} + \|\nabla_v f_0\|_{L^p_{xv}}), \quad \forall\, t > 0. \tag{2.7}$$

Proof of Lemma 2.3. We introduce the differential operator

$$D_t := t\nabla_x + \nabla_v, \tag{2.8}$$

and we observe that D_t commutes with the free transport operator $\partial_t + v\cdot\nabla_x$. The solution f to the free transport equation (2.1) then also satisfies

$$\partial_t(D_t f) + v\cdot\nabla_x(D_t f) = 0.$$

From the L^p-norm preservation (2.3) for the free transport flow on f_t and $D_t f_t$, we deduce

$$\forall\, t \ge 0, \quad \|f(t,\cdot)\|_{L^p} = \|f_0\|_{L^p}, \quad \|D_t f(t,\cdot)\|_{L^p} = \|D_0 f_0\|_{L^p} = \|\nabla_v f_0\|_{L^p}.$$

Finally, we calculate

$$\begin{aligned} \nabla_x\rho(t,x) &= \int_{\mathbb{R}^d}\left(\frac{D_t}{t} - \nabla_v\right) f(t,x,v)\,\varphi(v)dv \\ &= \frac{1}{t}\int_{\mathbb{R}^d}(D_t f)(t,x,v)\,\varphi(v)dv + \int_{\mathbb{R}^d} f(t,x,v)\,\nabla_v\varphi(v)dv, \end{aligned}$$

and we conclude the proof thanks to the previous estimates. □

2.3. **Higher moment estimate.** We start with an interesting moment identity.

Lemma 2.4. *For $\beta \le 1$ and for any solution to the transport equation with nice initial datum, we have*

$$\int_0^t \int f_s \frac{|v|}{|x|^\beta} \big(1-\beta(\hat x\cdot\hat v)^2\big)\, dxdvds$$
$$= \int f_0 \frac{x}{|x|^\beta}\cdot \hat v\, dxdv - \int f_t \frac{x}{|x|^\beta}\cdot \hat v\, dxdv,$$

with $\hat u := u/|u|$.

Proof of Lemma 2.4. We consider the multiplicator function

$$m := \frac{x}{|x|^\beta}\cdot\frac{v}{|v|}, \quad \beta\le 1,$$

and we observe that

$$v\cdot\nabla_x m = \frac{|v|}{|x|^\beta}\big(1-\beta(\hat x\cdot\hat v)^2\big).$$

Integrating the free transport equation, we have

$$\int f_t m\, dxdv + \int_0^t \int f_s v\cdot\nabla_x m\, dxdvds = \int f_0 m\, dxdv,$$

and we immediately conclude. □

We deduce some (small) gain of moment estimate in the v variable which can be seen again as a transfer of information from one to the other variable. This time the transfer is reverse: we transfer some moment estimate in the x variable as a a gain of moment in the v variable.

Corollary 2.5. *For $0 \le f_0 \in L^1$ such that $f_0(|x|^\alpha + |v|^\alpha) \in L^1$, $\alpha > 0$, we have*

$$\int_0^T \int_{B_R}\int_{\mathbb{R}^3} f(t,x,v)|v|^{\alpha+1}\, dxdvdt \le C.$$

Proof of Lemma 2.4. We only consider the case $\alpha\in(0,1]$. We compute

$$\begin{aligned}\int_{\mathbb{R}^{2d}} f(t,x,v)|v|^\alpha\, dxdv &= \int_{\mathbb{R}^{2d}} f_0(x,v)|v|^\alpha\, dxdv\\ \int_{\mathbb{R}^{2d}} f(t,x,v)|x-vt|^\alpha\, dxdv &= \int_{\mathbb{R}^{2d}} f_0(x,v)|x|^\alpha\, dxdv.\end{aligned}$$

so that

$$\begin{aligned}\int_{\mathbb{R}^{2d}} f(t,x,v)|x|^\alpha\, dxdv &\le C_\alpha \int_{\mathbb{R}^{2d}} f(t,x,v)(|x-vt|^\alpha + |v|^\alpha)\, dxdv\\ &\le C'_\alpha(1+t^\alpha).\end{aligned}$$

Using the previous lemma with $\beta := 1-\alpha < 1$, we get

$$\alpha\int_0^T\int_{\mathbb{R}^{2d}} f_t \frac{|v|}{|x|^{1-\alpha}}\, dvdxdt \le \int f_0|x|^\alpha + \int f_T|x|^\alpha,$$

and we conclude thanks to the previous estimate. □

2.4. **Regularity (averaging lemma).** We conclude this section with a version of probably the most famous of the averaging lemma: this one makes possible to truly gain some regularity in the x variable for the averaging (it is not a transfert of regularity from one variable to another).

Theorem 2.6. *For any $f_0\in L^2(\mathbb{R}^{2d})$ and $\varphi\in C_c(\mathbb{R}^d)$, we have*

$$\int_0^\infty \|\mathcal{A}\mathcal{S}(t)f_0\|_{\dot H^{1/2}_x}\, dt \lesssim \|f_0\|_{L^2_{x,v}},$$

where we use the notations introduced in (2.2) and (2.4).

During the proof, we will use the following classical trace result.

Lemma 2.7. *There exists a constant* $C_d \in (0,\infty)$ *such that for any* $\phi \in H^{d/2}(\mathbb{R}^d)$ *and any* $u \in \mathbb{R}^d$, $|u| = 1$, *the function* $\phi_u(s) := \phi(su)$ *satisfies*

$$\|\phi_u\|_{L^2(\mathbb{R})} \le C_d \|\phi\|_{H^{d/2}(\mathbb{R})} = C_d \left(\int_{\mathbb{R}^d} |\check{\mathcal{F}}\phi|^2(w)\, \langle w\rangle^d \, dw \right)^{1/2},$$

where $\check{\mathcal{F}}$ *stands for the (inverse) Fourier transform operator.*

Proof of Theorem 2.6. For a given function h which depends on the x variable or on the (x,v) variable, we denote by $\hat h$ its Fourier transform on the x variable and by $\mathcal{F}h$ its Fourier transform on both variables x and v.

We fix $f_0 \in L^2(\mathbb{R}^{2d})$ and $\varphi \in C_c(\mathbb{R}^d)$. We denote by f be the solution to the free transport equation (2.1) and by ρ the average function

$$\rho(t,x) := \int_{\mathbb{R}^d} f(t,x,v)\,\varphi(v)\,dv = [A_\varphi \mathcal{S}(t) f_0](x).$$

In Fourier variables, the free transport equation (2.1) writes

$$\partial_t \hat f + iv\cdot\xi \hat f = 0, \quad \hat f_{|t=0} = \hat f_0,$$

so that

$$\hat f(t,\xi,v) = e^{iv\cdot\xi\, t} \hat f_0(\xi,v)$$

and

$$\hat\rho(t,\xi) = \int_{\mathbb{R}^d} e^{iv\cdot\xi\, t} \hat f_0(\xi,v)\,\varphi(v)\,dv = \mathcal{F}(f_0\,\varphi)(\xi, t\xi).$$

We deduce

$$\int_0^\infty |\hat\rho(t,\xi)|^2\,dt \le \int_{\mathbb{R}} |\mathcal{F}(f_0\,\varphi)(\xi,t\xi)|^2\,dt.$$

Performing one change of variable, introducing the notation $\sigma_\xi = \xi/|\xi|$ and using Lemma 2.7, we deduce

$$\begin{aligned}\int_{\mathbb{R}} |\mathcal{F}(f_0\,\varphi)(\xi,t\xi)|^2\,dt &= \frac{1}{|\xi|}\int_{\mathbb{R}} |\mathcal{F}(f_0\,\varphi)(\xi,s\,\sigma_\xi)|^2\,ds \\ &\lesssim \frac{1}{|\xi|}\int_{\mathbb{R}^d} |(\hat f_0\,\varphi)(\xi,w)|^2 \langle w\rangle^d\,dw.\end{aligned}$$

Thanks to Plancherel identity, we then obtain

$$\int_0^\infty\!\!\int_{\mathbb{R}^d} |\xi|\,|\hat\rho(t,\xi)|^2\,d\xi dt \lesssim \int_{\mathbb{R}^d}\!\int_{\mathbb{R}^d} |(f_0\,\varphi)(x,w)|^2 \langle w\rangle^d\,dwdx = \|\varphi\|^2_{L^2_{d/2}}\,\|f_0\|^2_{L^2_{xv}},$$

which ends the proof. □

3. The space homogeneous Landau equation

We aim to establish the existence of solutions to the Landau equation

$$\partial_t f(t,v) = Q(f,f)(t,v), \quad f(0,v) = f_0(v), \tag{3.1}$$

on the density function $f = f(t,v) \ge 0$, $t \ge 0$, $v \in \mathbb{R}^d$, $d \ge 2$, where the Landau kernel is defined by the formula

$$Q(f,f)(v) := \frac{\partial}{\partial v_i}\Big\{ \int_{\mathbb{R}^d} a_{ij}(v-v_*)\Big(f(v_*)\frac{\partial f}{\partial v_j}(v) - f(v)\frac{\partial f}{\partial v_j}(v_*)\Big)\,dv_*\Big\}.$$

Here and the sequel we use Einstein's convention of sommation of repeated indices. The matrix $a = (a_{ij})$ is defined by

$$a(z) = |z|^2\Pi(z), \quad \Pi_{ij}(z) := \delta_{ij} - \hat z_i \hat z_j, \quad \hat z_k := \frac{z_k}{|z|},$$

so that Π is the is the orthogonal projection on the hyperplan $z^\perp := \{y \in \mathbb{R}^d;\ y\cdot z = 0\}$. We present the proof as an exercice (pick it up from Exam 2019-2020)

3.1. **Physical properties and a priori estimates.**

(1) Observe that $a(z)z = 0$ for any $z \in \mathbb{R}^d$ and $a(z)\xi\xi \geq 0$ for any $z, \xi \in \mathbb{R}^d$. Here and below, we use the bilinear form notation $auv = {}^t vau = v \cdot au$. In particular, the symmetric matrix a is positive but not strictly positive.

(2) For any nice functions $f, \varphi : \mathbb{R}^d \to \mathbb{R}$, $f \geq 0$, prove that

$$\int Q(f,f)\varphi \, dv = \frac{1}{2} \iint a(v - v_*)\big(f \nabla_* f_* - f_* \nabla f\big)\big(\nabla \varphi - \nabla_* \varphi_*\big) \, dv dv_*,$$

where $f_* = f(v_*)$, $\nabla_* \psi_* = (\nabla \psi)(v_*)$. Deduce that

$$\int Q(f,f)\varphi \, dv = 0, \quad \text{for } \varphi = 1, v_i, |v|^2,$$

and

$$-D(f) := \int Q(f,f) \log f \, dv \leq 0.$$

Establish then

$$\Big| \int Q(f,f)\varphi \, dv \Big| \leq D(f)^{1/2} \Big(\frac{1}{2} \iint f f_* a(v - v_*)(\nabla \varphi - \nabla_* \varphi_*)(\nabla \varphi - \nabla_* \varphi_*) \, dv dv_* \Big)^{1/2}.$$

(3) For $H_0 \in \mathbb{R}$, we define $\mathcal{E}_{H_0}$ the set of functions

$$\begin{aligned} \mathcal{E}_{H_0} \quad := \quad & \Big\{ f \in L^1_2(\mathbb{R}^d);\ f \geq 0,\ \int f \, dv = 1,\ \int f\, v \, dv = 0, \\ & \int f\, |v|^2 \, dv \leq d,\ H(f) := \int f \, \log f \, dv \leq H_0 \Big\}. \end{aligned}$$

Prove that there exists a constant C_0 such that

$$H_-(f) := \int f (\log f)_- dv \leq C_0, \quad \forall\, f \in \mathcal{E}_{H_0},$$

and define $D_0 := H_0 + C_0$. Deduce that for any nice positive solution f to the Landau equation such that $f_0 \in \mathcal{E}_{H_0}$, there holds

$$f \in \mathcal{F}_T := \Big\{ g \in C([0,T]; L^1_2);\ g(t) \in \mathcal{E}_{H_0},\ \forall\, t \in (0,T),\ \int_0^T D(g(t)) \, dt \leq D_0 \Big\}.$$

We say that $f \in C([0,T); L^1)$ is a weak solution to the Landau equation if $f \in \mathcal{F}_T$ and (3.1) holds in the distributional sense. Why the definition is meaningful?

(4) Prove that

$$Q(f,f) = \partial_i(\bar a_{ij} \partial_j f - \bar b_i f) = \partial^2_{ij}(\bar a_{ij} f) - 2\partial_i(\bar b_i f) = \bar a_{ij} \partial^2_{ij} f - \bar c f,$$

with

$$\bar a_{ij} = \bar a^f_{ij} := a_{ij} * f, \quad \bar b_i = \bar b^f_i := b_i * f, \quad \bar c = \bar c^f := c * f, \tag{3.2}$$

and

$$b_i := \sum_{j=1}^d \partial_j a_{ij} = -(d-1) z_i, \quad c := \sum_{i=1}^d \partial_i b_i = -(d-1)d.$$

Prove that there existe $C \in (0,\infty)$ such that

$$|\bar a_{ij}| \leq C(1 + |v|^2), \quad |\bar b_i| \leq C(1 + |v|),$$

3.2. **On the ellipticity of $\bar a$.**
We fix $H_0 \in \mathbb{R}$ and $f \in \mathcal{E}_{H_0}$.
(5a) Show that there exists a function $\eta \ge 0$ (only depending of D_0) such that

$$\forall\, A \subset \mathbb{R}^d, \quad \int_A f\, dv \le \eta(|A|)$$

and $\eta(r) \to 0$ when $r \to 0$. Here $|A|$ denotes the Lebesgue measure of A. Deduce that

$$\forall\, R, \varepsilon > 0, \quad \int f\, \mathbf{1}_{|v|\le R}\, \mathbf{1}_{|v_i|\le \varepsilon}\, dv \le \eta_R(\varepsilon)$$

and $\eta_R(r) \to 0$ when $r \to 0$.
(5b) Show that

$$\int f \mathbf{1}_{|v|\le R} \ \ge 1 - \frac{d}{R^2}.$$

(5c) Deduce from the two previous questions that

$$\forall\, i = 1, \dots, d, \quad T_i := \int f v_i^2 dv \ge \lambda,$$

for some constant $\lambda > 0$ which only depends on D_0. Generalize the last estimate into

$$\forall\, \xi \in \mathbb{R}^d, \quad T(\xi) := \int f |v \cdot \xi|^2 dv \ge \lambda |\xi|^2.$$

(6) Deduce that

$$\forall\, v, \xi \in \mathbb{R}^d, \quad \bar a(v)\xi\xi := \sum_{i,j=1}^d \bar a_{ij}(v)\xi_i\xi_j \ge (d-1)\lambda\, |\xi|^2.$$

Prove that any weak solution formally satisfies

$$\frac{d}{dt}\, H(f) = -\int \bar a_{ij}\, \frac{\partial_i f \partial_j f}{f} - \int \bar c f,$$

and thus the following bound on the Fisher information

$$I(f) := \int \frac{|\nabla f|^2}{f} \in L^1(0,T)$$

3.3. **The Aubin-Lions lemma.**
During the next section, we will have to prove a modified version of the following classical compactness lemma that we present here with full details.

Lemma 3.1 (Aubin-Lions)**.** *Consider a sequence (u_n) which satisfies*

$$\begin{aligned} &(i) \quad (u_n) \ \text{ is bounded in } \ L^2_{tx}, \\ &(ii) \quad (\partial_t u_n) \ \text{ is bounded in } \ L^2_t(H^{-s}_x), \ s \in \mathbb{R}_+, \\ &(iii) \quad (\nabla_x u_n) \ \text{ is bounded in } \ L^2_{tx}. \end{aligned}$$

Then, there exists $u \in L^2_{tx}$ and a subsequence $(u_{n'})$ such that $u_{n'} \to u$ strongly in $L^2((0,T)\times B_R)$ as $n \to \infty$ for any $R > 0$.

Idea of the proof. Step 1. We may write $\partial_t u_n = D^s g_n$ with (g_n) bounded in L^2_{tx}. We introduce a sequence of mollifiers (ρ_ε), that is $\rho_\varepsilon(x) := \varepsilon^{-d}\rho(\varepsilon^{-1}x)$ with $0 \le \rho \in \mathcal{D}(\mathbb{R}^d)$, $\langle \rho \rangle = 1$. We observe that

$$\frac{\partial}{\partial t} \int_{\mathbb{R}^d} u_n(t,y)\, \rho_\varepsilon(x-y)\, dx = \int_{\mathbb{R}^d} g_n(t,y)\, D^s \rho_\varepsilon(x-y)\, dy,$$

where the RHS term is bounded in $L^2((0,T)\times\mathbb{R}^d)$ uniformly in n for any fixed $\varepsilon > 0$. We also clearly have

$$\nabla_x \int_{\mathbb{R}^d} u_n(t,y)\, \rho_\varepsilon(x-y)\, dx = -\int_{\mathbb{R}^d} u_n \nabla_y \rho_\varepsilon(x-y)\, dy,$$

where again the RHS term is bounded in $L^2((0,T)\times\mathbb{R}^d)$ uniformly in n for any fixed $\varepsilon > 0$. In other words, $u_n * \rho_\varepsilon$ is bounded in $H^1((0,T)\times\mathbb{R}^d)$. Thanks to the Rellich-Kondrachov Theorem, we get that (up to the extraction of a subsequence) $(u_n * \rho_\varepsilon)_n$ is strongly convergent in $L^2((0,T)\times B_R)$, for any $R > 0$. On the other hand, from (i) and the Banach-Alaoglu weak compacteness theorem, we know that there exists $u \in L^2_{tx}$ and a subsequence $(u_{n'})$ such that $u_{n'} \rightharpoonup u$ weakly in L^2_{tx}. All together, for any fixed $\varepsilon > 0$, we then get

$$u_n * \rho_\varepsilon \to u * \rho_\varepsilon \text{ strongly in } L^2((0,T) \times B_R) \text{ as } n \to \infty.$$

Step 2. We now observe that

$$\begin{aligned} \int_{(0,T)\times\mathbb{R}^d} |w - w * \rho_\varepsilon|^2 \, dxdt &= \int_{(0,T)\times\mathbb{R}^d} \Big| \int_{\mathbb{R}^d} (w(t,x) - w(t,x-y))\rho_\varepsilon(y) \, dy \Big|^2 dxdt \\ &= \int_{(0,T)\times\mathbb{R}^d} \Big| \int_{\mathbb{R}^d} \int_0^1 \nabla_x w(t,z_s) \cdot y \rho_\varepsilon(y) \, dsdy \Big|^2 dxdt, \end{aligned}$$

with $z_s := x + sy$. From the Jensen (or Cauchy-Schwarz) inequality, we deduce

$$\begin{aligned} \int_{(0,T)\times\mathbb{R}^d} |w - w * \rho_\varepsilon|^2 \, dxdt &\le \varepsilon^2 \int_{(0,T)\times\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^1 |\nabla_x w(t,z_s)|^2 \frac{1}{\varepsilon^d} \frac{|y|^2}{\varepsilon^2} \rho(\frac{y}{\varepsilon}) \, dsdydxdt \\ &\le \varepsilon^2 \int_{(0,T)\times\mathbb{R}^d} |\nabla_x w(t,z)|^2 dtdz \int_{\mathbb{R}^d} |z| \rho(z) \, dy \\ &\le \varepsilon^2 C_\rho \|\nabla_x w\|^2_{L^2_{tx}}. \end{aligned}$$

We conclude that $u_n \to u$ in $L^2((0,T) \times B_R)$ by writing

$$u_n - u = (u_n - u_n * \rho_\varepsilon) + (u_n * \rho - u * \rho) + (u * \rho_\varepsilon - u)$$

and using the previous convergence and estimates. □

3.4. **Weak stability.**

We consider here a sequence of weak solutions (f_n) to the Landau equation such that $f_n \in \mathcal{F}_T$ for any $n \ge 1$.

(7) Prove that

$$\int_0^T \int |\nabla_v f_n| \, dvdt \le C_T$$

and that

$$\frac{d}{dt} \int f_n \varphi \, dv \text{ is bounded in } L^\infty(0,T), \quad \forall \varphi \in C^2_c(\mathbb{R}^d).$$

Deduce that (f_n) belongs to a compact set of $L^1((0,T)\times\mathbb{R}^d)$. Up to the extraction of a subsequence, we then have

$$f_n \to f \text{ strongly in } L^1((0,T) \times \mathbb{R}^d).$$

Deduce that

$$Q(f_n, f_n) \rightharpoonup Q(f,f) \text{ weakly in } \mathcal{D}((0,T) \times \mathbb{R}^d)$$

and that f is a weak solution to the Landau equation.

(8) (Difficult, here $d = 3$) Take $f \in \mathcal{E}_{H_0}$ with energy equals to d. Establish that $D(f) = 0$ if, and only if,

$$\frac{\nabla f}{f} - \frac{\nabla f_*}{f_*} = \lambda(v, v_*)(v - v_*), \quad \forall v, v_* \in \mathbb{R}^d,$$

for some scalar function $(v, v_*) \mapsto \lambda(v, v_*)$. Establish then that the last equation is equivalent to

$$\log f = \lambda_1 |v|^2/2 + \lambda_2 v + \lambda_3, \quad \forall v \in \mathbb{R}^d,$$

for some constants $\lambda_1 \in \mathbb{R}$, $\lambda_2 \in \mathbb{R}^d$, $\lambda_3 \in \mathbb{R}$. Conclude that

$$D(f) = 0 \text{ if, and only if, } f = M(v) := (2\pi)^{-3/2} \exp(-|v|^2/2).$$

(9) (very difficult, because needs many steps) Prove that for any global weak solution f associated to $f_0 \in L^1_3 \cap \mathcal{E}_{H_0}$ with energy equals d, there holds $f(t) \rightharpoonup M$ when $t \to \infty$. (Hint. Accept that the energy $M_2(f(t)) = d$ and prove that the third moment $M_3(f(t))$ is uniformly bounded).

3.5. **Existence.**
(10) We fix $k = d+4$. Show that $\mathcal{H} := L^2_k \subset L^1_3$ and that $H_0 := H(f_0) \in \mathbb{R}$ if $0 \le f_0 \in L^2_k$. In the sequel, we first assume that $f_0 \in \mathcal{E}_{H_0} \cap \mathcal{H}$.
(11) For $f \in C([0,T];\mathcal{E}_{H_0})$, we define $\bar a$, $\bar b$ and $\bar c$ thanks to (3.2) and then

$$\tilde a_{ij} := \bar a_{ij} + \varepsilon |v|^2 \delta_{ij}, \quad \tilde b_i := \bar b_i - \varepsilon \frac{d+2}{2} v_i, \quad \varepsilon \in (0,\lambda).$$

We define $\mathcal{V} := H^1_{k+2}$ and then

$$\forall\, g \in \mathcal{V}, \quad Lg := \partial_i \big(\tilde a_{ij} \partial_j g - \tilde b_i g \big) \in \mathcal{V}'.$$

Show that for some constant $C_i \in (0,\infty)$, there hold

$$(Lg,g)_{\mathcal{H}} \le -\varepsilon \|g\|^2_{\mathcal{V}} + C_1 \|g\|^2_{\mathcal{H}}, \quad |(Lg,h)_{\mathcal{H}}| \le C_2 \|g\|_{\mathcal{V}} \|h\|_{\mathcal{V}}, \quad \forall\, g,h \in \mathcal{V}.$$

Deduce that there exists a unique variational solution

$$g \in \mathcal{X}_T := C([0,T];\mathcal{H}) \cap L^2(0,T;\mathcal{V}) \cap H^1(0,T;\mathcal{V}')$$

to the parabolic equation

$$\partial_t g = Lg, \quad g(0) = f_0.$$

Prove furthermore that $g \in \mathcal{F}_T$.
(12) Prove that there exists a unique fonction

$$f_\varepsilon \in C([0,T]; L^2_k) \cap L^2(0,T; H^1_k) \cap \mathcal{F}_T$$

solution to the nonlinear parabolic equation

$$\partial_t f_\varepsilon = \partial_i (\tilde a^{f_\varepsilon}_{ij} \partial_j f_\varepsilon + \tilde b^{f_\varepsilon}_i f_\varepsilon), \quad f_\varepsilon(0) = f_0,$$

where $\tilde a^{f_\varepsilon}_{ij}$ denotes the
(13) For $f_0 \in \mathcal{E}_{H_0}$ and $T > 0$, prove that there exists at least one weak solution $f \in \mathcal{F}_T$ to the Landau equation.

4. Hypocoercivity

We present a L^2 hypocoercivity theory for proving spectral gap and thus exponential rate of convergence to the equilibrium for the relaxation equation

$$\partial_t f + v \cdot \nabla_x f = \rho M - f,$$

where

$$\rho = \rho_f(t,x) := \int_{\mathbb{R}^d} f\, dv, \quad M = M(v) := (2\pi)^{-d/2} e^{-|v|^2/2},$$

for a (variation of density) function $f = f(t,x,v)$, $t \ge 0$, $x \in \mathbb{T}^d$ (the torus of $\mathbb{R}^d$) and $v \in \mathbb{R}^d$. At a more general and abstract level, the approach is suitable for operators $\mathcal{L}$ which splits as

$$\mathcal{L} = \mathcal{T} + \mathcal{S}, \quad \mathcal{T} := -v \cdot \nabla_x,$$

and $\mathcal{S}$ is a collisional operator which satisfies the microscopic coercivity estimate

$$(-\mathcal{S} f, f)_{\mathcal{H}_v} \ge \kappa^\perp \|f\|^2_{\mathcal{H}_v}, \quad \forall\, f \in \mathcal{H}_{v0}, \tag{4.1}$$

where

$$\mathcal{H}_v := L^2(M^{-1}), \quad \mathcal{H}_{v0} := \big\{ f \in \mathcal{H}_v;\ \pi f = 0 \big\}, \quad \pi f := (f, M)_{\mathcal{H}_v} M, \tag{4.2}$$

and M is the unique positive and normalized steady state. In particular, we have $\mathcal{S} M = 0$ and $\operatorname{Ker} \pi = \operatorname{vect} M$. We observe that the relaxation operator writes

$$\mathcal{S} f := -\pi^\perp f, \quad \pi^\perp = I - \pi, \quad \pi f = M \rho_f.$$

Using that $\pi^* = \pi$ in $\mathcal{H}_v$, so that $\pi^* \pi^\perp = 0$, we deduce

$$(Sf, f)_{\mathcal{H}_v} = -(\pi^\perp f, \pi^\perp f + \pi f)_{\mathcal{H}} = -\|\pi^\perp f\|^2_{\mathcal{H}_v},$$

which is nothing but (4.1). For further references, it is worth emphasizing that the relaxation operator satisfy

$$|\langle \varphi v g \rangle| + |\langle \varphi \mathcal{S} g \rangle| \lesssim \|g\|_{\mathcal{H}_v}, \tag{4.3}$$

for any $g \in \mathcal{H}_v$ and $\varphi = 1$ or v_k. Another example which falls in this class of operators is the Fokker-Planck operator

$$\mathcal{S} f := \Delta_v f + \operatorname{div}(v f).$$

We work in the Hilbert space $\mathcal{H}_0 \subset \mathcal{H} := L^2(M^{-1} dx dv)$ which is the orthogonal to the line of equilibria and we introduce a one parameter twisted norm associated to the quadratic form

$$|||f|||^2 := \|f\|^2_{\mathcal{H}} - \alpha(\nabla\Delta^{-1}\rho, j)_{L^2(\mathbb{T}^d)}, \quad \forall\, f \in \mathcal{H}, \tag{4.4}$$

where $\rho = \rho_f$ and $j = j_f$ are the mass defined in (4.2) and the mean velocity defined by

$$j_f := j[f] = \langle f\, v \rangle = \int_{\mathbb{R}^d} f\, v\, dv$$

and where for $\xi : \mathbb{T}^d \to \mathbb{R}$ with zero mean, $u := \Delta^{-1}\xi$ denotes the solution to the elliptic problem

$$\Delta u = \xi, \quad u \in H^1(\mathbb{T}^d).$$

In that framework, it is worth recalling the classical estimates

$$\|\Delta^{-1}\eta_1\|_{H^2} \lesssim \|\eta_1\|_{L^2} \quad \text{and} \quad \|\Delta^{-1}\partial_x\eta_2\|_{H^1} \lesssim \|\eta_2\|_{L^2}, \tag{4.5}$$

for any $\eta_i \in L^2$, η_1 with zero mean.
The associated Dirichlet form to the operator $\mathcal{L}$ and twisted norm (4.4) writes

$$D[f] := (-\mathcal{L}f, f) + \alpha(\nabla\Delta^{-1}\rho, j[\mathcal{L}f]) + \alpha(\nabla\Delta^{-1}\rho[\mathcal{L}f], j).$$

Theorem 4.1. *For $\alpha > 0$ small enough the above Dirichlet form D is positive on $\mathcal{H}_0$, namely, there exist $\alpha, \lambda > 0$ such that*

$$D[f] := ((-\mathcal{L}f, f)) \geq \lambda |||f|||^2, \quad \forall\, f \in \mathcal{H}_0,$$

with

$$\mathcal{H}_0 := \{g \in \mathcal{H}; \langle\langle g \rangle\rangle = 0\}, \qquad \langle\langle g \rangle\rangle := \int_\Omega \langle g \rangle dx.$$

Proof of Theorem 4.1. We fix $f \in \mathcal{H}_0$. We compute separately each terms. The first term involved in the definition of $D[f]$ is bounded by below as always by

$$D_0[f] := (-\mathcal{L}f, f)_{\mathcal{H}} \geq \kappa^\perp \|f^\perp\|^2_{\mathcal{H}}, \tag{4.6}$$

thanks to the skew symmetry of $\mathcal{T}$ and the microscopic coercivity estimate (4.1).
One the one hand and before dealing with the second term, we observe that

$$j[\mathcal{L}f] = j[\mathcal{T}\pi f] + j[\mathcal{T} f^\perp] + j[\mathcal{S} f^\perp], \tag{4.7}$$

with

$$j_k[\mathcal{T}\pi f] \;\; = \;\; -\int v_k v_\ell \partial_{x_\ell}\rho_f M(v)\, dv = -\partial_{x_k}\rho_f.$$

We also observe that $\langle \rho_f \rangle = 0$ because $f \in \mathcal{H}_0$, so that $\Delta^{-1}\rho_f \in H^2(\mathbb{T}^d)$ is well defined. As a consequence, we have

$$\begin{aligned}
D_1[f] &:= \alpha(\nabla\Delta^{-1}\rho, j[\mathcal{L}f]) \\
&= \alpha(\nabla\Delta^{-1}\rho, -\nabla\rho - D_x\langle v \otimes v f^\perp\rangle + \langle v\mathcal{S} f^\perp\rangle) \\
&= \alpha\|\rho\|^2_{L^2} + \alpha(D_x\nabla\Delta^{-1}\rho, \langle v \otimes v f^\perp\rangle) + \alpha(\nabla\Delta^{-1}\rho, \langle v\mathcal{S} f^\perp\rangle) \\
&\geq \alpha\|\rho\|^2_{L^2} - \alpha\|\Delta^{-1}\rho\|_{H^2}\|\langle v \otimes v f^\perp\rangle\|_{L^2} - \alpha\|\Delta^{-1}\rho\|_{H^1}\|\langle v\mathcal{S} f^\perp\rangle\|_{L^2},
\end{aligned}$$

where in the third line we have performed one integration by part for each of the two first terms. We then deduce

$$D_1[f] \geq \alpha\|\rho\|^2_{L^2} - \alpha K\|\rho\|_{L^2}\|f^\perp\|_{\mathcal{H}} \tag{4.8}$$

for some constant $K \in (0, \infty)$, by using estimates (4.5) and (4.3).
On the other hand, we observe that $j = j[f^\perp]$ as well as $\rho[\mathcal{L}f] = \rho[\mathcal{L}f^\perp]$ with zero mean, and more precisely that

$$\rho[\mathcal{L}f] = \int [v \cdot \nabla_x\rho M + v \cdot \nabla_x f^\perp + \mathcal{S} f^\perp]\, dv = \nabla_x\langle v f^\perp\rangle.$$

We deduce

$$\begin{aligned} D_2[f] &:= \alpha(\nabla\Delta^{-1}\rho[\mathcal{L}f], j) \\ &= \alpha(\nabla\Delta^{-1}\nabla_x\langle vf^\perp\rangle, \langle vf^\perp\rangle) \geq -\alpha\|\Delta^{-1}\nabla_x\langle vf^\perp\rangle\|_{H^1}\|\langle vf^\perp\rangle\|_{L^2}, \end{aligned}$$

and then

$$D_2[f] \geq -\alpha K\|f^\perp\|^2_{\mathcal{H}}, \tag{4.9}$$

for some constant $K \in (0,\infty)$, by using estimates (4.5) and (4.3). Putting together the three contributions (4.6), (4.8) and (4.9) and using the Young inequality, we get

$$D[f] \geq \kappa^\perp\|f^\perp\|^2_{\mathcal{H}} + \alpha\|\rho\|^2_{L^2} - \frac{1}{2}\alpha^{3/2}K\|\rho\|^2_{L^2} - (\alpha + \frac{1}{2}\alpha^{1/2})K\|f^\perp\|^2_{\mathcal{H}}.$$

We kill the two last terms by taking $\alpha > 0$ small enough and we obtain

$$D[f] \geq \frac{1}{2}\min(\kappa^\perp, \alpha)\big\{\|f^\perp\|^2_{\mathcal{H}} + \|\rho\|^2_{L^2}\big\}.$$

We conclude by recalling that $\|f^\perp\|^2_{\mathcal{H}} + \|\rho\|^2_{L^2} = \|f\|^2_{\mathcal{H}}$. □

5. Notes

Most of the results (if not all) of section 2 may be found in the review paper of Perthame [13]. Lemma 2.1 is due to Perthame [12]. Lemma 2.2 is due to Bardos and Degond [1]. Lemma 2.3 is due to Gualdani, Mischler and Mouhot [8]. Lemma 2.4 and Corollary 2.5 are due to Perthame and Lions [10], improving a similar previous result by Perthame [11]. Theorem 2.6 is very similar to Bouchut-Desvillettes' version [2, Theorem 2.1] (see also [3] for a related discrete version) of the classical averaging Lemma initiated in the famous articles of Golse et al. [7].

The mathematical approach for the Landau equation presented in section 3 has mainly been developed by Villani [14], and by Villani and Desvillettes [4, 5]

In section 4, the hypocoercivity result established in Theorem 4.1 and its proof are probably a rephrasing of material developed by Hérau [9] and Dolbeault-Mouhot-Schmeiser [6]. We also refer to [15] for an introduction to this topic.

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