CHAPTER 0 - ON THE GRONWALL LEMMA

I write in blue color what has been taught during the classes.

There are many variants of the Gronwall lemma which simplest formulation tells us that any given function $u : [0,T) \to \mathbb{R}$, $T \in (0,\infty]$, of class C^1 satisfying the differential inequality

$$u' \le au \quad \text{on} \quad (0,T), \tag{0.1}$$

for $a \in \mathbb{R}$, also satisfies the pointwise estimate

$$u(t) \le e^{at}u(0) \quad \text{on} \quad [0,T). \tag{0.2}$$

We indeed establish (0.2) by a mere time integration of the differential inequality $(u\,e^{-at})' \le 0$ that we deduce from (0.1).

The aim of these notes is to give several generalizations and variants in differential, integral and discrete form of that first version which make possible to establish

- local in time estimates;
- large time (decay) estimates;
- uniform in time estimates (trap trick);
- uniqueness type result.

1. Local in time estimates (from differential inequality)

We give in this section some locally in time estimates for solutions to differential inequalities and we start with a first version.

Lemma 1.1 (classical differential version of Gronwall lemma)**.** *We assume that $u \in C([0,T);\mathbb{R})$, $T \in (0,\infty)$, satisfies the differential inequality*

$$u' \le a(t)u + b(t) \quad \text{on} \quad (0,T), \tag{1.1}$$

for some $a, b \in L^1(0,T)$. Then, u satisfies the pointwise estimate

$$u(t) \le e^{A(t)}u(0) + \int_0^t b(s)e^{A(t)-A(s)}\,ds, \quad \forall\, t \in [0,T), \tag{1.2}$$

where we have defined the primitive function

$$A(t) := \int_0^t a(s)\,ds.$$

Some examples and important special cases of the Gronwall lemma are

$$u' \le a(t)u \implies u(t) \le u(0)e^{A(t)}, \tag{1.3}$$

$$u' \le au + b \implies u(t) \le u(0)e^{at} + \frac{b}{a}(e^{at} - 1), \tag{1.4}$$

$$u' \le au + b(t) \implies u(t) \le u(0)e^{at} + \int_0^t e^{a(t-s)}\, b(s)\, ds, \tag{1.5}$$

$$u' + b(t) \le a(t)u,\ a,\, b \ge 0 \implies u(t) + \int_0^t b(s)\, ds \le u(0)e^{A(t)}. \tag{1.6}$$

Proof of Lemma 1.1. The differential inequality (1.1) means

$$-\langle u, \varphi' \rangle \le \langle au + b, \varphi \rangle$$

for any $0 \le \varphi \in \mathcal{D}(0,T)$. We set

$$v(t) = u(t)\, e^{-A(t)} - \int_0^t b(s) e^{-A(s)}\, ds,$$

and we observe that

$$v' \le 0 \quad \mathcal{D}'(0,T), \qquad v \in C([0,T]).$$

• When furthermore $v \in C^1$ (or even $v \in W^{1,1}$), we immediately conclude to

$$v(t) = v(0) + \int_0^t v'(s)\, ds \le v(0) = u(0),$$

from what (1.2) follows.

• In the general case when $v \in C([0,T])$, we proceed as follows. We fix $\varepsilon > 0$ and $\varrho \in C^1_c(0,\varepsilon)$ such that $\varrho \ge 0$, $\int \varrho = 1$. For any function $0 \le w \in C^1_c(\varepsilon,T)$, the function $\psi := -w + (\int_0^T w)\, \varrho$ belongs to $C_c(0,T)$ and $\int_0^T \psi = 0$. As a consequence ψ has a primitive φ such that $\varphi(0) = \varphi(T) = 0$. The function φ thus enjoys the following properties $\varphi \in C^1_c(0,T)$, $\varphi \ge 0$ and $\varphi' = \psi$. We deduce

$$\begin{aligned} 0 \ge \langle v', \varphi \rangle &= \int_0^T v \left\{ w - \left(\int_0^T w \right) \varrho \right\} dt \\ &= \int_0^T w \left\{ v - \int_0^T v\, \varrho \right\} dt. \end{aligned}$$

Because the above inequality is true for any $w \in C^1_c(\varepsilon,T)$, $w \ge 0$, it comes

$$v \le \int_0^T v\, \varrho \quad \text{on} \quad (\varepsilon, T).$$

Taking $\varrho = \varrho_\alpha$ for a mollifier sequence (ϱ_α) (i.e. $\varrho_\alpha \rightharpoonup \delta_0$ as $\alpha \to 0$) and letting $\alpha \to 0$, we deduce again $v \le v(0)$ on $(0,T)$. □

2. Local in time estimates (from integral inequality)

In many situations, it is not easy to deal with differential inequalities and it is much more natural to start from the associated integral inequality. The conclusion can be however the same.

Lemma 2.1 (integral version of Gronwall lemma)**.** *We assume that* $u \in C([0,T);\mathbb{R})$, $T \in (0,\infty)$, *satisfies the integral inequality*

$$u(t) \le u_0 + \int_0^t a(s)u(s)\,ds + \int_0^t b(s)\,ds \quad \text{on} \quad [0,T), \tag{2.1}$$

for some $0 \le a \in L^1(0,T)$ *and* $b \in L^1(0,T)$*. Then,* u *satisfies the same pointwise estimate*

$$u(t) \le u_0\, e^{A(t)} + \int_0^t b(s) e^{A(t)-A(s)}\,ds, \quad \forall\, t \in (0,T).$$

Remark 2.2. *Lemma 2.1 can be seen as an integral (and thus weak) version of Lemma 1.1, but we emphasize that we additionally need to assume* $a \ge 0$ *here.*

Some examples and important special cases of the Gronwall lemma are

$$u(t) \le \int_0^t a(s)u(s)\,ds \implies u(t) \equiv 0, \tag{2.2}$$

$$u(t) \le u_0 + \int_0^t a(s)u(s)\,ds \implies u(t) \le u_0\, e^{A(t)}, \tag{2.3}$$

$$u(t) + \int_0^t |b(s)| ds \le u_0 + \int_0^t a(s)u(s)\,ds \implies u(t) + \int_0^t |b(s)| ds \le u(0) e^{A(t)}. \tag{2.4}$$

Proof of Lemma 2.1. Step 1. We first assume that $b \equiv 0$*.* We set $v(t) = u(t) - u_0\, e^{A(t)}$ and we compute

$$\begin{aligned} v(t) &\le \int_0^t a(s)\,u(s)\,ds + u_0\,(1 - e^{A(t)}) \\ &= \int_0^t a(s)\,(v(s) + u_0\, e^{A(s)})\,ds + u_0\,(1 - e^{A(t)}) \\ &= \int_0^t a(s)\,v(s)\,ds. \end{aligned}$$

Because a is not negative, it yields

$$v^+(t) \le \int_0^t a(s)\,v^+(s)\,ds =: w(t). \tag{2.5}$$

The function $w \in W^{1,1}(0,T)$ satisfies

$$w'(t) = a(t)\,v^+(t) \le a(t)\,w(t) \quad \text{on} \quad (0,T),$$

and we may use Lemma 1.1 in order to deduce $w(t) \le w(0) = 0$, next $v(t) \le v^+(t) \le w(t) \le 0$ and the conclusion.

Step 2. We do not assume $b \equiv 0$ *anymore.* We define

$$v(t) := u(t) - u_0 e^{A(t)} - \int_0^t b(s) e^{A(t)-A(s)}\,ds.$$

We observe that we have again

$$v(t) \le \int_0^t a(s)\, v(s)\, ds,$$

and we conclude as in the first step. □

Remark 2.3. *Starting from* (2.5)*, we may conclude directly and without using Lemma 1.1. We observe that*

$$\sup_{t\in[0,T^*]} v^+(t) \le \sup_{s\in[0,T^*]} v^+(s) \int_0^{T^*} a(s)\, ds, \quad \forall\, T^* \in (0,T].$$

Taking $T^* > 0$ *small enough such that* $\int_0^{T^*} a(s)\, ds < 1$*, we deduce*

$$\sup_{t\in[0,T]} v^+(t) \le 0.$$

We conclude that $v \le 0$ *on* $[0,T^*]$*, and next* $v \le 0$ *on* $[0,T]$ *by just iterating the argument.*

We also have the following slightly more general version of Gronwall lemma.

Lemma 2.4 (integral version of Gronwall lemma)**.** *We assume that* $u \in C([0,T);\mathbb{R})$*,* $T \in (0,\infty)$*, satisfies the integral inequality*

$$u(t) \le B(t) + \int_0^t a(s)u(s)\, ds \quad on \quad [0,T), \tag{2.6}$$

for some $B \in C([0,T))$ *and* $0 \le a \in L^1(0,T)$*. Then,* u *satisfies the pointwise estimate*

$$u(t) \le B(t) + \int_0^t a(s)B(s)e^{A(t)-A(s)}\, ds, \quad \forall\, t \in (0,T).$$

Proof of Lemma 2.4. We define

$$v(t) := \int_0^t a(s)u(s)\, ds,$$

which belongs to $W^{1,1}(0,T)$, and we compute

$$v' = a(t)u(t) \le a(t)B(t) + a(t)v(t),$$

because of (2.6) and $a \ge 0$. We conclude thanks to Lemma 1.1 applied with v, a and $b := aB$. □

Exercise 2.5 (a variant of the proof of Lemma 2.4)**.** *Show that, under the hypotheses of Lemma 2.4, the function*

$$v(t) := \int_0^t a(s)u(s)\, ds\, e^{-A(t)} - \int_0^t a(s)B(s)e^{-A(s)}\, ds$$

satisfies $v' \le 0$*, and recover the conclusion of Lemma 2.4.*

Remark 2.6. *1) We may prove* (2.3) *(and thus* (2.2)*) as a corollary of Lemma 2.4. Indeed, we apply Lemma 2.4 with* $B = u_0$ *which implies*

$$\begin{aligned} u(t) &\le u_0 + u_0 e^{A(t)} \int_0^t a(s)e^{-A(s)}\, ds \\ &= u_0 + u_0 e^{A(t)}[1 - e^{-A(t)}], \end{aligned}$$

what is nothing but the desired estimate.

2) More generally, we may see Lemma 2.1 (and thus (2.4)*) as a corollary of Lemma 2.4. Indeed, we apply Lemma 2.4 with*

$$B(t) = u_0 + \int_0^t b(s)\, ds,$$

and arguing similarly as above, we have

$$\begin{aligned} u(t) &\le u_0 + \int_0^t b(s)\, ds + e^{A(t)} \int_0^t a(s)\big[u_0 + \int_0^s b(\tau)\, d\tau\big]e^{-A(s)}\, ds \\ &= u_0 + \int_0^t b(s)\, ds - e^{A(t)} \Big[\big(u_0 + \int_0^s b(\tau)\, d\tau\big)e^{-A(s)}\Big]_0^t \\ &\quad + e^{A(t)} \int_0^t b(s)e^{-A(s)}\, ds. \end{aligned}$$

That last inequality easily simplifies into the desired estimate. □

3. Decay estimates

In this section, we establish some pointwise decay estimates which are relevant as time goes to infinity. We recall that for a positive and continuous function $u : \mathbb{R}_+ \to \mathbb{R}_+$ which satisfies the ordinary differential inequality

$$u' \leq -\lambda\, u, \tag{3.1}$$

for some $\lambda > 0$, the Gronwall Lemma (in its most classical form (0.1) $\Rightarrow$ (0.2)) tells us that u enjoys a decay estimate with exponential rate

$$u(t) \leq u(0) e^{-\lambda t}, \quad \forall\, t \geq 0. \tag{3.2}$$

On the other hand, by integrating in time the differential inequality (3.1) and using that $u(t) \to 0$ as $t \to \infty$ from (3.2), we get

$$\lambda \int_t^\infty u(s)\, ds \leq -\int_t^\infty u'(s)\, ds = u(t).$$

As a matter of fact, the decay estimate (3.2) can be established starting from that last integral inequality.

Lemma 3.1. *Assume that $u : \mathbb{R}_+ \to \mathbb{R}_+$ is decreasing and satisfies*

$$\lambda \int_t^\infty u(s)\, ds \leq\, u(t), \quad \forall t \geq 0, \tag{3.3}$$

for some constant $\lambda > 0$. Then, there exists $C = C(\lambda, u(0)) > 0$, such that

$$u(t) \leq C\, e^{-\lambda t}, \quad \forall t \geq 0. \tag{3.4}$$

Proof of Lemma 3.1. We define

$$v(t) := \int_t^\infty u(s)\, ds \in W^{1,\infty}(0,\infty).$$

From (3.3), we deduce that v satisfies

$$v'(t) = -u(t) \leq -\lambda v(t),$$

and thus from the Gronwall lemma (in its classical form (0.1) $\Rightarrow$ (0.2))

$$v(t) \leq v(0) e^{-\lambda t} \leq \frac{u(0)}{\lambda} e^{-\lambda t}. \tag{3.5}$$

Since u is decreasing, we get $u(t) \leq u(0)$ for $t \in [0,1]$ and

$$u(t)\ \leq\ \int_{t-1}^t u(s)\, ds \leq v(t-1), \quad \forall\, t \geq 1, \tag{3.6}$$

and we conclude by gathering (3.5) and (3.6). □

We now consider a positive solution u to the ordinary differential inequality

$$u' \leq -K\, u^{1+\alpha}, \quad \alpha > 0. \tag{3.7}$$

From (3.7), we classically compute

$$\frac{du}{u^{1+\alpha}} \leq -K dt,$$

and next by time integration

$$-\frac{1}{u^\alpha} + \frac{1}{u_0^\alpha} \le -\alpha K t.$$

Rewriting that last inequality, we get

$$u^{-\alpha}(t) \ge \alpha\, K\, t + u_0^{-\alpha} \ge \alpha\, K\, t,$$

from which we conclude to the decay estimate with polynomial rate

$$u(t) \le \frac{C}{t^{1/\alpha}}, \quad C := (\alpha K)^{-1/\alpha}. \tag{3.8}$$

On the other hand again, integrating in time the differential inequality (3.7) and using that $u(t) \to 0$ as $t \to \infty$ from (3.8), we get

$$K \int_t^\infty u(s)^{1+\alpha}\, ds \le -\int_t^\infty u'(s)\, ds = u(t).$$

Here again, we can recover the decay estimate (3.8) starting from that last integral inequality.

Lemma 3.2. *Assume that $u : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, decreasing and satisfies the integral inequality*

$$K \int_t^\infty u(s)^{1+\alpha}\, ds \le u(t), \quad \forall t \ge 0, \tag{3.9}$$

for some constants $K, \alpha > 0$. Then, u satisfies the pointwise estimate

$$u(t) \le C\, t^{-1/\alpha}, \quad \forall t > 0, \tag{3.10}$$

for some constant $C = C(K, \alpha, ||u||_{L^1}) > 0$.

Proof of Lemma 3.2. We set

$$v(t) := \int_t^\infty u(s)^{1+\alpha}\, ds \in C^1(0,\infty).$$

We deduce from (3.9) that v satisfies the differential inequality

$$v' = -u^{1+\alpha} \le -\left(K v\right)^{1+\alpha},$$

and thus from (3.8) that v satisfies the pointwise estimate

$$v(t) \le \frac{1}{(\alpha K^{1+\alpha})^{1/\alpha} t^{1/\alpha}} \tag{3.11}$$

Since u is decreasing, we obtain

$$\frac{t}{2}\, u(t)^{1+\alpha} \;\le\; \int_{t/2}^t u(s)^{1+\alpha}\, ds \le v(t/2), \tag{3.12}$$

and we conclude by gathering (3.11) and (3.12). □

We end this section presenting several related decay estimates issues.

Lemma 3.3. *Assume $u : \mathbb{R}_+ \to \mathbb{R}_+$ is decreasing and satisfies*

$$\int_0^\infty \varphi'(s) u(s)\, ds \le C_1, \tag{3.13}$$

for a constant $C_1 > 0$ and a nonnegative increasing function φ. Then, there exists $C_2 \ge 0$ such that

$$u(t) \le \frac{C_2}{\varphi(t)}, \quad \forall t > 0. \tag{3.14}$$

Proof of Lemma 3.3. • A non optimal proof is as follow. We make the additional (but not very restricting) assumption that there exist $T > 0$ and $\gamma \in (0,1)$ such that $\varphi(t/2) \le \gamma\varphi(t)$ for any $t \ge T$. As a consequence, we have

$$(1-\gamma)\varphi(t) \le \varphi(t) - \varphi(t/2) =: \Phi(t) \le \varphi(t)$$

for any $t \ge T$. We then compute

$$\Phi(t)u(t) = \int_{t/2}^t \varphi'(s)\,ds\,u(t) \le \int_{t/2}^t \varphi'(s)u(s)\,ds \le C,$$

just using that u is a decreasing function. We conclude thanks to $\Phi(t) \simeq \varphi(t)$.
• A simpler and optimal proof can be handle in the following way. Using again that $u' \le 0$, we have

$$\begin{aligned} \varphi(t)u(t) &\le \varphi(t)u(t) - \int_0^t \varphi(s)u'(s)\,ds \\ &= \varphi(0)u(0) + \int_0^t \varphi'(s)u(s)\,ds \le \varphi(0)u(0) + C_1, \end{aligned}$$

and we get the conclusion with $C_2 := \varphi(0)u(0) + C_1$. □

Alternative proof of Lemma 3.2. We want now to show that we can deduce the polynomial decay directly from a linear analysis. More precisely, we assume that $u : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function satisfying the differential inequality

$$u' \le -Ku^{1+\alpha}, \quad K, \alpha > 0,$$

and we want to prove (3.10). Observing that

$$u \le \ell + \ell^{-\alpha}u^{1+\alpha}, \quad \forall\, \ell > 0, \tag{3.15}$$

we deduce that u satisfies the family of differential inequalities

$$u' \le -\lambda u + K^{-1/\alpha}\lambda^{1+1/\alpha},$$

for any positive function $\lambda = \lambda(t)$. From the classical Gronwall Lemma 1.1, u satisfies the pointwise estimate

$$u(t) \le e^{-\Lambda(t)}u(0) + K^{-1/\alpha}\int_0^t \lambda(s)^{1+1/\alpha}e^{\Lambda(s)-\Lambda(t)}\,ds, \quad \forall\, t > 0,$$

where we have defined the primitive function

$$\Lambda(t) := \int_0^t \lambda(s)\,ds.$$

The choice $\lambda(t) := C/(1+t)$ gives $e^{\Lambda(t)} = (1+t)^C$. We then find

$$u(t) \lesssim \frac{1}{(1+t)^C} + \int_0^t \frac{1}{(1+s)^{1+1/\alpha}}\frac{(1+s)^C}{(1+t)^C}\;ds \lesssim \frac{1}{(1+t)^{1/\alpha}},$$

with the choice $C := 1 + 1/\alpha$, which is nothing but (3.10). □

Exercise 3.4. *Assume that $u : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and satisfies*

$$u' \le -\varphi(u)$$

for some continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$H(0) = -\infty, \quad H(u) := \int_1^u \frac{dw}{\varphi(w)}.$$

Prove that u satisfies the pointwise estimate

$$u(t) \le H^{-1}(H(u(0)) - t).$$

Prove a similar result when $u : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, decreasing and satisfies

$$\int_t^\infty \varphi(u(s))\, ds \le u(t), \quad \forall\, t > 0.$$

4. Nonlinear differential inequality, trap trick and decay

We next present a generalization of the Gronwall lemma to a nonlinear differential inequality framework.

Lemma 4.1 (nonlinear version of Gronwall lemma)**.** *Let $f \in C^1((0,T) \times \mathbb{R})$ and consider $u, v \in C([0,T];\mathbb{R})$ such that*

$$u' \le f(t,u), \quad v' \ge f(t,u), \quad u(0) \le v(0), \tag{4.1}$$

(in a distributional sense). Then $u \le v$ on $[0,T]$.

Proof of Lemma 4.1. We set $w(t) := u(t) - v(t) \in C([0,T])$ and

$$a(t) := \partial_u f(t,u(t)) \text{ if } v(t) = u(t); \quad a(t) := \frac{f(t,u(t)) - f(t,v(t))}{u(t) - v(t)} \text{ otherwise,}$$

and we observe that $a \in C([0,T])$. We compute

$$w' = u' - v' \le f(t,u) - f(t,v) = a\, w, \quad w(0) \le 0.$$

We may apply Lemma 1.1 (or more precisely (1.3)) which implies $w \le 0$ and that gives our conclusion. □

Exercise 4.2. *Establish a integral version of Lemma 4.1. More precisely, consider $f \in C^1((0,T)\times \mathbb{R})$ increasing with respect to the second variable and $u, v \in C([0,T])$ such that*

$$u(t) \le u_0 + \int_0^t f(s,u(s))\, ds, \quad v(t) \ge v_0 + \int_0^t f(s,v(s))\, ds, \quad u_0 \le v_0,$$

and prove that $u \le v$ on $[0,T]$. [Hint. Observe that $w_t := u(t) - v(t)$ satisfies

$$w_t \le L \int_0^t w_s\, ds$$

for some constant L which may depend on u, v and f].

A consequence of Lemma 4.1 is the following useful result.

Lemma 4.3. *Let $0 \le u \in C([0,\infty))$ satisfy (in the sense of distributions)*

$$u' + k_1\, u^{\theta_1} \le k_2\, u^{\theta_2} + k_3,$$

with $\theta_1 > 0$, $\theta_2 \ge 0$, $\theta_2/\theta_1 < 1$, $k_1 > 0$ and k_2, $k_3 \ge 0$. Then, there exists $C_0 = C_0(k_i,\theta_i) \ge 0$ such that

$$\sup_{t\ge 0} u(t) \le \max(C_0, u(0)). \tag{4.2}$$

Assume moreover $\theta_1 > 1$. For any $\tau > 0$, there exists $C_\tau = C_\tau(k_i,\theta_i,\tau) \ge 0$ such that

$$\sup_{t\ge\tau} u(t) \le C_\tau. \tag{4.3}$$

Proof of Lemma 4.3. Step 1. We set $f(u) := k_2\, u^{\theta_2} + k_3 - k_1\, u^{\theta_1}$ for any $u \ge 0$ and we observe that there exists $C_0 > 0$ (large enough) such that $f(u) \le 0 = f(C_0)$ for any $u \ge C_0$. As a consequence, u is a subsolution and $v := \max(u_0, C_0)$ is a supersolution to the ODE $w' = f(w)$ both with same initial datum. We conclude thanks to Lemma 4.1.

Step 2. For (any) $C_0' > C_0$ there exists $k_1' > 0$ such that $f(u) \le -k_1'\, u^{\theta_1}$ for any $u \ge C_0'$. We consider the solution v to the ODE

$$v' = -k_1' v^{\theta_1}, \quad v(0) = u_0.$$

As we have already seen that (3.7) implies (3.8), we immediately deduce that v satisfies

$$v(t) \le (k_1' t)^{-\frac{1}{\theta_1 - 1}}, \quad \forall\, t > 0.$$

We conclude that (4.3) holds for any $\tau > 0$ with $C_\tau := \max(C_0', (k_1'\tau)^{-\frac{1}{\theta_1-1}})$. □

We give now an integral inequality variant of the previous result.

Lemma 4.4. *Assume $0 \le u \in C([0,T])$ satisfies*

$$u(t) \le u_0 + 2\int_0^t c(s)\sqrt{u(s)}\, ds$$

for some $u_0 \ge 0$ and $0 \le c \in L^2(0,T)$. Then

$$u(t) \le \Big(\sqrt{u_0} + \int_0^t c(s)\, ds\Big)^2,$$

and this estimate is sharp.

Proof of Lemma 4.4. For some fixed $\varepsilon > 0$, we define $v \in C^1$ by

$$v(t) := \varepsilon + u_0 + 2\int_0^t c(s)\sqrt{u(s)}\, ds \ge u(t),$$

and we compute

$$v' = 2c(t)\sqrt{u(t)} \le 2c(t)\sqrt{v(t)}.$$

Because $v \ge \varepsilon$, we may integrate the differential inequality

$$\frac{v'}{2\sqrt{v}} \le c$$

and we get

$$\sqrt{v(t)} - \sqrt{v(0)} \le \int_0^t c(s)\, ds.$$

As a consequence, we have

$$u(t) \le v(t) \le \Big(\sqrt{u_0+\varepsilon} + \int_0^t c(s)\,ds\Big)^2,$$

and we conclude by passing to the limit $\varepsilon \to 0$. □

Consider $u \in C^1([0,T^*);\mathbb{R}_+)$ satisfying the differential inequality

$$u' \le -\lambda u + C u^{1+\alpha}, \quad \lambda, C, \alpha > 0,$$

as well as the maximality condition

$$u(t) \to \infty \;\text{ as }\; t \to T^* \;\text{ if }\; T^* < \infty.$$

Under the initial condition $u(0) < (\lambda/C)^{1/\theta}$, the function u is decreasing and thus trapped below $(\lambda/C)^{1/\theta}$ (see Lemma 4.3), so that $T_* = +\infty$. Moreover, $u(t) \to 0$ as $t \to \infty$, and more precisely, for any $\lambda' \in (0,\lambda)$, we may use the Gronwall lemma (in its most classical form (0.1) $\Rightarrow$ (0.2)) which implies that there exists $C_2 = C_2(\lambda', u(0), C)$ such that

$$u \le C_2\, e^{-\lambda t}, \quad \forall\, t \ge 0.$$

An integral inequality variant is the following.

Lemma 4.5. *Assume that $u \in C([0,\infty);\mathbb{R}_+)$ satisfies the integral inequality*

$$u(t) \le C_1 e^{at}\, u_0 + C_2 \int_0^t e^{a(t-s)} u(s)^{1+\alpha}\, \mathrm{d}s, \quad \forall\, t > 0,$$

for some constants $C_1 \ge 1$, $C_2, u_0 \ge 0$, $\alpha > 0$ and $a < 0$. Under the smallness assumption

$$a + (1+1/\alpha) C_2 2^\alpha C_1^\alpha u_0^\alpha < 0,$$

there holds

$$u(t) \le \Big(1 + \frac{C_2 C_1^\alpha u_0^\alpha}{|\alpha a + (1+\alpha) C_2 2^\alpha C_1^\alpha u_0^\alpha|}\Big) C_1\, e^{at}\, u_0, \quad \forall\, t \ge 0.$$

Proof of Lemma 4.5. We fix $A \in (C_1 u_0, 2 C_1 u_0)$ arbitrarily, so that $u(t) \le A$ at least on a small interval, that is for any $t \in [0,\tau]$, $\tau > 0$ small enough, and then the integral inequality implies on the same interval

$$u(t) \le C_1 e^{at}\, u_0 + C_2 2^\alpha C_1^\alpha u_0^\alpha \int_0^t e^{a(t-s)} u(s)\, \mathrm{d}s.$$

Corollary 2.6 applied to the function $t \mapsto u(t) e^{-at}$ and the smallness assumption $a + C_2 2^\alpha C_1^\alpha u_0^\alpha < 0$ imply

$$u(t) \le C_1\, u_0\, e^{(a + C_2 2^\alpha C_1^\alpha u_0^\alpha) t} \le C_1\, u_0 < A$$

on that interval. By a continuation argument, the first above inequality holds on $\mathbb{R}_+$ and then with $A := C_1 u_0$. Next, replacing that first estimate in the integral inequality we started with, we get

$$u(t) \le C_1 e^{at}\, u_0 + C_2 C_1^{1+\alpha} u_0^{1+\alpha} e^{at} \int_0^t e^{(\alpha a + (1+\alpha) C_2 2^\alpha C_1^\alpha u_0^\alpha) s}\, \mathrm{d}s, \quad \forall\, t > 0,$$

from which we immediately conclude. □

5. Discrete inequalities

We present several discrete versions of the Gronwall lemma.

Lemma 5.1 (discrete version of Gronwall lemma)**.** *We consider a real numbers sequence* (u_n) *such that*

$$u_{n+1} \le a_{n+1} u_n + b_{n+1}, \quad \forall\, n \ge 0, \tag{5.1}$$

where (a_n) *and* (b_n) *are two given real numbers sequences and* (a_n) *is furthermore positive. Then*

$$u_n \le A_n u_0 + \sum_{k=1}^{n} A_{k,n} b_k, \quad \forall\, n \ge 0, \tag{5.2}$$

where we have defined

$$A_n := \prod_{k=1}^{n} a_k, \quad A_{k,n} = A_n/A_k = \prod_{i=k+1}^{n} a_i.$$

Proof of Lemma 5.1. We define

$$v_n := A_n u_0 + \sum_{k=1}^{n} A_{k,n} b_k,$$

and we observe that

$$\begin{aligned} v_{n+1} &= A_{n+1} u_0 + \sum_{k=1}^{n+1} A_{k,n+1} b_k \\ &= a_{n+1} A_n u_0 + \sum_{k=1}^{n} a_{n+1} A_{k,n} b_k + b_{n+1} \\ &= a_{n+1} v_n + b_{n+1}. \end{aligned}$$

We then easily check by induction that $u_n \le v_n$ for any $n \ge 0$. □

Some particularly interesting special cases of that discrete Gronwall lemma are

$$u_{n+1} \le a u_n + b_{n+1} \implies u_n \le a^n u_0 + \sum_{k=1}^{n} a^{n-k} b_k, \tag{5.3}$$

$$u_{n+1} \le (1+\alpha) u_n + b_{n+1},\ \alpha, b_{n+1} \ge 0 \implies u_n \le e^{n\alpha} u_0 + e^{n\alpha} \sum_{k=1}^{n} b_k, \tag{5.4}$$

$$u_{n+1} + b_{n+1} \le a u_n,\ a \ge 1, b_{n+1} \ge 0 \implies u_n + \sum_{k=1}^{n} b_k \le a^n u_0. \tag{5.5}$$

Lemma 5.2 (first summing version of Gronwall lemma)**.** *We consider a positive real numbers sequence* (u_n) *such that*

$$u_n \le \sum_{k=0}^{n-1} \alpha_k u_k + B_n, \quad \forall\, n \ge 0, \tag{5.6}$$

where (α_n) *and* (B_n) *are two given real numbers sequences and* (α_n) *is furthermore positive. Then*

$$u_n \le B_n + \sum_{k=0}^{n-1} \alpha_k B_k \mathcal{A}_{k,n-1}, \quad \forall\, n \ge 0, \tag{5.7}$$

with

$$\mathcal{A}_{k,n} = \prod_{i=k+1}^{n} (1+\alpha_i).$$

Proof of Lemma 5.2. We define

$$v_n := \sum_{k=0}^{n} \alpha_k u_k$$

and we observe that from (5.6), we have

$$v_{n+1} - v_n = \alpha_{n+1} u_{n+1} \le \alpha_{n+1} v_n + \alpha_{n+1} B_{n+1}$$

as well as $v_0 = \alpha_0 u_0 \le \alpha_0 B_0$. Applying the first discrete Gronwall Lemma 5.1 to the sequence (v_n), we get

$$v_n \le \sum_{k=0}^{n} \mathcal{A}_{k,n} \alpha_k B_k, \quad \forall\, n \ge 0.$$

We conclude thanks to (5.6) and the definition of (v_n). □

Some particularly interesting formulations of that discrete Gronwall lemma are

$$u_n \le \sum_{k=0}^{n-1} \alpha_k u_k + B \implies u_n \le B \prod_{i=0}^{n-1} (1+\alpha_i) \le B \exp\Big\{ \sum_{i=0}^{n-1} \alpha_i \Big\}, \tag{5.8}$$

$$u_n \le \sum_{k=0}^{n-1} \alpha_k u_k + B_k \implies u_n \le B_n + \sum_{k=0}^{n-1} \alpha_k B_k \exp\Big\{ \sum_{i=k}^{n-1} \alpha_i \Big\}. \tag{5.9}$$

Indeed, in order to establish (5.8), we may use (5.7) with $B_n = B$ and we compute

$$\begin{aligned} u_n &\le B + B \sum_{k=0}^{n-1} \alpha_k \prod_{i=k+1}^{n} (1+\alpha_i) \\ &= B + B \sum_{k=0}^{n-1} \Big\{ \prod_{i=k}^{n} (1+\alpha_i) - \prod_{i=k+1}^{n} (1+\alpha_i) \Big\} \\ &= B + B \Big\{ \prod_{i=0}^{n} (1+\alpha_i) - 1 \Big\}. \end{aligned}$$

Lemma 5.3. *Consider a sequence* (u_n) *of positive real numbers such that*

$$u_n - u_{n-1} \le -K u_n^{1+\alpha}, \quad \forall\, n \ge 1, \tag{5.10}$$

with $\alpha > 0$. *There exists a constant* C *such that*

$$u_n \le C\, n^{-1/\alpha}, \quad \forall\, n \ge 1. \tag{5.11}$$

We present two proofs of Lemma 5.3. The first one uses Lemma 3.2, while the second one follows the alternative proof of Lemma 3.2 presented at the end of section 3.

Proof 1 of Lemma 5.3. We first observe that (u_n) is decreasing and $u_n \to 0$ as $n \to \infty$. We may then sum up (5.10) and obtain

$$K \sum_{k \ge n+1} u_k^{1+\alpha} \le u_n.$$

We define the pointwise affine map u by $u(t) := (1-t-n)u_n + (t-n)u_{k+1}$ for $t \in [k, k+1]$ so that $u(t) \le u_k$ for any $t \ge k$, $u(n) = u_n$ and then

$$K \int_{n+1}^{\infty} u(s)^{1+\alpha}\, ds = K \sum_{k \ge n+1} \int_k^{k+1} u(t)^{1+\alpha} dt \le u(n).$$

As a consequence, we have

$$K \int_{[t]+2}^{\infty} u(s)^{1+\alpha}\, ds \le u([t]+1) \le u(t).$$

On the other hand, we have

$$\frac{1}{2} u(0)^{-\alpha} \int_t^{[t+2]} u(s)^{1+\alpha} ds \le \frac{1}{2} \int_t^{[t+2]} u(s) ds \le u(t).$$

Both together, we deduce that

$$v(t) := \int_t^{\infty} u(s)^{1+\alpha}\, ds \le C u(t)$$

and then

$$v'(t) = -u(s)^{1+\alpha} \lesssim -v^{1+\alpha}.$$

Integrating that ODE, we get

$$v(t) \le C\, \frac{1}{t^{1/\alpha}},$$

from what we deduce

$$\frac{n}{2}\, u_n^{1+\alpha} \;\le\; \int_{n/2}^{n} u(s)^{1+\alpha}\, ds \le v(n/2), \tag{5.12}$$

and that concludes. □

Proof 2 of Lemma 5.3. Starting from (5.10) and using (3.15), we deduce

$$u_{n+1} - u_n \le -\lambda u_{n+1} + K^{-1/\alpha} \lambda^{1+1/\alpha}, \quad \forall\, \lambda > 0,$$

and then

$$u_{n+1} \le (1+\lambda_{n+1})^{-1} u_n + (1+\lambda_{n+1})^{-1} K^{-1/\alpha} \lambda_{n+1}^{1+1/\alpha},$$

for a sequence (λ_n) of positive numbers to be fixed later. Thanks to Lemma 5.1, we have

$$\begin{aligned} u_n \;&\le\; \prod_{k=1}^{n} (1+\lambda_k)^{-1} u_0 + \sum_{k=1}^{n} \prod_{i=k}^{n} (1+\lambda_i)^{-1} K^{-1/\alpha} \lambda_k^{1+1/\alpha} \\ &=\; e^{-\sum_{k=1}^{n} \ln(1+\lambda_k)} u_0 + K^{-1/\alpha} \sum_{k=1}^{n} e^{-\sum_{i=k}^{n} \ln(1+\lambda_i)} \lambda_k^{1+1/\alpha}. \end{aligned}$$

We choose $\lambda_k := 2\beta/k$ with $\beta := 1 + 1/\alpha$ and we use that $\ln(1+\lambda) \ge \lambda/2$ for any $\lambda \in (0,1)$ and thus $\ln(1+\lambda_k) \ge \lambda_k/2$ for any $k \ge k^* = k^*(\alpha)$ in order to get

$$u_n \quad \le \quad e^{-\sum_{k=k^*}^n \lambda_k/2} u_0 + K^{-1/\alpha} \sum_{k=1}^n e^{-\sum_{i=k\vee k^*}^n \lambda_i/2} \lambda_k^{1+1/\alpha}.$$

We observe that

$$\sum_{i=k^*}^n \lambda_i/2 = \sum_{i=1}^n \frac{\beta}{i} \ - \sum_{i=1}^{k^*-1} \frac{\beta}{i} \ge \ln n^\beta - \gamma$$

and

$$\sum_{i=k\vee k^*}^n \lambda_i/2 = \sum_{i=k}^n \frac{\beta}{i} \ - \sum_{i=k}^{(k\vee k^*)-1} \frac{\beta}{i} \ge \ln n^\beta - \ln k^\beta - \gamma,$$

for some constant $\gamma := \gamma(k^*) > 0$. All together, we obtain

$$u_n \le \frac{e^\gamma}{n^\beta} u_0 + K^{-1/\alpha} e^\gamma \sum_{k=1}^n \frac{k^\beta}{n^\beta} \, \frac{(2\beta)^{1+1/\alpha}}{k^{1+1/\alpha}} \lesssim \frac{1}{n^{1/\alpha}},$$

for any $n \ge 1$, because of the definition of β. □

6. Uniqueness

We recall that from (2.2), the following version of Gronwall Lemma holds true.

Lemma 6.1. *We assume that $u \in C([0,T);\mathbb{R}_+)$, $T \in (0,\infty)$, satisfies the integral inequality*

$$u(t) \le \int_0^t \delta(s) u(s)\, ds \quad \text{on} \quad [0,T), \tag{6.1}$$

for some $0 \le \delta \in L^1(0,T)$. Then, $u \equiv 0$.

Two classical corollaries are about the uniqueness of solution to the ODE problem

$$x' = a(t,x), \quad x(0) = x_0. \tag{6.2}$$

Corollary 6.2 (Cauchy-Lipschitz). *Consider $a \in C^1([0,T) \times \mathbb{R}^d;\mathbb{R}^d)$. There exists at most one maximal solution $x \in C^1([0,T^*);\mathbb{R}^d)$, $T^* \in (0,T)$, to the initial value problem* (6.2).

Corollary 6.3 (Osgood). *Assume that*

$$(a(t,x) - a(t,y), x-y) \le L|x-y|^2, \quad \forall\, x,y \in \mathbb{R}^d, \ \forall\, t \in (0,T),$$

for some $L \ge 0$. There exists at most one solution $x \in C^1([0,T);\mathbb{R}^d)$ to the initial value (6.2).

The proofs of corollaries 6.2 and 6.3 are straightforward consequences of Lemma 6.1, or even of the most classical form (0.1) ⇒ (0.2), and they are left to the reader. It happens that we are not able to get the assumptions of Lemma 6.1, but that we may use one of the following two variants/generalizations of Lemma 6.1.

Lemma 6.4 (Nagumo). *Consider* $u \in C^1((0,T);\mathbb{R}_+)$, $T \in (0,\infty)$, *such that*

$$u' \le \frac{\delta(t)}{t} u, \quad \delta(t) \to 0, \quad u(0) = 0.$$

Then $u \equiv 0$.

Proof of Lemma 6.4. Because u is smooth, we have $u(t) \le Lt$ and therefore $u'(t) \le \delta(t) L$. As a consequence, we have

$$u(t) \le \Delta(t) := \int_0^t \delta(s) L \, ds$$

and then

$$v(t) := \frac{u(t)}{t} \le \frac{\Delta(t)}{t} = \frac{L}{t} \int_0^t \delta(s)\, ds \to 0 \ \text{ as } \ t \to 0.$$

On the other hand, we have

$$v'(t) = \frac{u'}{t} - \frac{u}{t^2} \le \frac{u}{t^2}(\delta(t) - 1) \le 0$$

for any $t \in (0,T)$, $T > 0$ small enough. Both together, we deduce

$$\sup_{t \in [\varepsilon, T]} v(t) \le v(\varepsilon) \to 0 \ \text{ as } \ \varepsilon \to 0,$$

and then $u(t) \equiv 0$. □

Lemma 6.5 (Yudovich). *Consider* $u \in C^1((0,T);\mathbb{R}_+)$, $T \in (0,\infty)$, *such that*

$$u(t) \le u_0 + \int_0^t \eta(u(s))\, ds, \tag{6.3}$$

with $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ *a continuous, positive and increasing function such that*

$$\int_0^1 \frac{dz}{\eta(z)} = +\infty.$$

Typical examples are $\eta(z) = z$ *or* $\eta(z) = z(|\log z| + 1)$. *The following alternative holds true:*

- *if* $u_0 = 0$ *then* $u \equiv 0$ *;*
- *if* $u_0 > 0$ *then*

$$m(u_0) \le m(u(t)) + t, \quad m(z) := \int_z^1 \frac{dy}{\eta(y)}. \tag{6.4}$$

Proof of Lemma 6.5. We define

$$X(t) := u_0 + \int_0^t \eta(u(s))\, ds,$$

which is continuous and increasing. We deduce

$$X'(t) = \eta(u(t)) \le \eta(X(t))$$

because $X(t) \ge u(t)$ and η is increasing.

- When $u_0 > 0$, the function X is strictly positive (because $X(t) \ge u_0$) and we may compute

$$-\frac{d}{dt} m(X(t)) = -m'(X(t))\, X'(t) = \frac{X'(t)}{\eta(X(t))} \le 1.$$

We deduce

$$m(X(0)) \le t + m(X(t)),$$

from which we get (6.4) by using that $z \mapsto m(z)$ is decreasing, $X(0) = u_0$ and $X(t) \geq u(t)$.

- When $u_0 = 0$, we assume by contradiction that $u \not\equiv 0$. There exists then $\tau > 0$, that we fix, such that $u(\tau) > 0$. Because u satisfies (6.3) with $u_0 = 0$, it also satisfies

$$u(t) \leq \varepsilon + \int_0^t \eta(u(s))\, ds, \quad \forall\, \varepsilon > 0.$$

The first step implies

$$m(\varepsilon) \leq m(u(\tau)) + \tau < \infty,$$

which is in contradiction with the fact that $m(\varepsilon) \to \infty$ when $\varepsilon \to 0$. □

7. Discussion

The various forms of Gronwall lemmas are classical and belong to folklore. Osgood's criteria in Corollary 6.3 is due to W. F. Osgood (*Beweis der Existenz einer Lösung der Differentialgleichung $\frac{dy}{dx} = f(x, y)$ ohne Hinzunahme der Cauchy-Lipschitz'schen Bedingung*(German) Monatsh. Math. Phys. 9 (1898), no. 1, 331–345). Lemma 6.4 is due to M. Nagumo (*Eine hinreichende Bedingung für die Unität der Lösung von Differential gleichungenerster Ordnung*, Japan. J. Math. 3 (1926), 107–112). Lemma 6.5 is due to V. Yudovich (*Non-stationary flows of an ideal incompressible fluid* (Russian) Ž. Vyčisl. Mat i Mat. Fiz. 3 (1963), 1032–1066).