# CHAPTER 1
# VARIATIONAL SOLUTION FOR PARABOLIC EQUATION

We present the theory of variational solutions for uniformly elliptic parabolic equations as well as for abstract evolution equations associated to an operator satisfying Gårding's inequality.

Contents



## 1. Introduction

In this chapter we will focus on the existence (and uniqueness) issue of a solution $f = f(t,x)$ to the (linear) evolution PDE of "parabolic type"

$$\partial_t f = \mathcal{L} f \quad \text{on} \quad (0,\infty) \times \mathbb{R}^d, \tag{1.1}$$

where $\mathcal{L}$ is typically the elliptic operator

$$(\mathcal{L} f)(x) = \Delta f(x) + b(x) \cdot \nabla f(x) + c(x)\, f(x), \tag{1.2}$$

that we complement with an initial condition

$$f(0,x) = f_0(x) \quad \text{in} \quad \mathbb{R}^d. \tag{1.3}$$

Here $t \ge 0$ stands for the *"time"* variable, $x \in \mathbb{R}^d$ stands (for instance) for the *"position"* variable, $d \in \mathbb{N}^*$.

In order to develop the variational approach for the equation (1.1)-(1.2), we assume that

$$f_0 \in L^2(\mathbb{R}^d) =: H, \ \ \text{which is an Hilbert space,}$$

and that the coefficients satisfy

$$b, c \in L^\infty(\mathbb{R}^d). \tag{1.4}$$

The **main result** we will present in this chapter is the existence and uniqueness of a weak (variational) global solution (which sense will be specified below)

$$f \in X_T := C([0,T); H) \cap L^2(0,T;V) \cap H^1(0,T;V'), \tag{1.5}$$

to the evolution equation (1.1)-(1.2)-(1.3) and to similar evolution equations, where here $T \in (0,\infty)$, $V := H^1(\mathbb{R}^d)$ and thus $V' := H^{-1}(\mathbb{R}^d)$. We mean variational solution because the space of "test functions" is the same as the space in which the solution lives. It also refers to the associated stationary problem which is of "variational type".

The existence of solutions issue is tackled by following a scheme of proof that we will repeat for all the other evolution equations that we will consider in the next chapters.

(1) We look for **a priori estimates** by performing (formal) differential and integral calculus.

(2) We deduce a possible natural **functional space** in which lives a solution and we propose a **definition of a solution**, that is a (weak) sense in which we may understand the evolution equation.

(3) We state and prove the associated **existence and uniqueness theorem**. For the existence proof we typically argue as follows: we introduce a *"regularized problem"* for which we are able to construct a solution and we are allowed to rigorously perform the calculus leading to the *"a priori estimates"*, and then we pass to the limit in the sequence of regularized solutions. The proof of the uniqueness is often more subtil: most of the time we need a regularization trick in order to justify the computations.

## 2. A priori estimates and weak solution

We explain first how we may obtain *"a priori estimates"* for solutions to the parabolic equation (1.1)-(1.2)-(1.3) with coefficients satisfying (1.4). We mean *"a priori estimates"* because we do not try in this first step to establish the estimates with full mathematical rigor but we rather try to perform formally some reasonable and usual computations (typically: derivation, integration, summation, ...) or, equivalently, we a priori assume that the functions or solutions considered are nice (smooth, rapidly decaying, ...) so that the performed manipulations are licit. This step is fundamental in order to bring out what kind of information is reasonable to hope for. Of course, in some next steps, these bounds will have to be justified.

We denote by $|\cdot| = |\cdot|_H$ the Hilbert norm in $H = L^2(\mathbb{R}^d)$ and $(\cdot,\cdot) = (\cdot,\cdot)_H$ its scalar product. We also define

$$V := H^1(\mathbb{R}^d) \subset H \subset V' := H^{-1}(\mathbb{R}^d), \tag{2.1}$$

the first space being endowed with its usual $H^1$ Hilbert norm denoted by $\|\cdot\| = \|\cdot\|_V$ and we denote by $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{V',V}$ the associated duality product. In particular, $\langle u, v\rangle = (u, v)$ for any $u \in H$ and $v \in V$. We recall the definition (1.2) of the operator $\mathcal{L}$ and we first observe that for any nice function $f = f(x)$ and any $\alpha \in (0, 1)$, we have

$$\begin{aligned} \langle \mathcal{L}f, f\rangle &:= \int_{\mathbb{R}^d} (\Delta f + b\cdot\nabla f + c\, f) f \\ &= -\int |\nabla f|^2 + \int b f\cdot\nabla_x f + \int c\, f^2 \\ &\le -\alpha\|f\|_V^2 + \operatorname{ess\,sup}\big(\alpha + \frac{1}{4\alpha}\|b\|^2_{L^\infty} + c\big)|f|^2_H, \end{aligned}$$

where we have used the Green-Ostrogradski divergence formula for the first term in the second line, and next in the third line, the Cauchy-Schwarz inequality in $L^2(\mathbb{R}^d)$ and the Young inequality $uv \le \alpha u^2/2 + v^2/(2\alpha)$, $\forall\, u, v \ge 0$. In other words, because of the hypothesis (1.4) on the coefficients, the operator $\mathcal{L}$ satisfies the following *"coercive+dissipative"* estimate[1] (or $-\mathcal{L}$ satisfies a *"Gårding's inequality"*)

$$\langle \mathcal{L}f, f\rangle \le -\alpha \|f\|_V^2 + \kappa\, |f|_H^2, \quad \forall\, f \in V, \tag{2.2}$$

for some $\alpha > 0$ and $\kappa \in \mathbb{R}$. Now, for a (nice) solution $f = f(t,x)$ to the parabolic equation (1.1)-(1.2)-(1.3)-(1.4), we compute

$$\frac{1}{2}\frac{d}{dt}|f(t)|_H^2 = \int (\partial_t f) f = \langle \mathcal{L}f, f\rangle \le -\alpha \|f(t)\|_V^2 + \kappa\, |f(t)|_H^2,$$

and, thanks to the Gronwall lemma, we deduce

$$|f(T)|_H^2 + 2\alpha \int_0^T \|f(s)\|_V^2\, ds \le e^{2\kappa T}\, |f_0|_H^2, \quad \forall\, T. \tag{2.3}$$

In order to reformulate this information obtained on $f$, we introduce the two following functional spaces. On the one hand, we note $f \in L^\infty(0,T;H)$ if $f \in L^2(\mathcal{U})$, $\mathcal{U} := (0,T) \times \mathbb{R}^d$, is such that there exists $C \in [0,\infty)$ satisfying

$$\|f(t,\cdot)\|_{L^2(\mathbb{R}^d)} \le C, \ \text{ for a.e. } t \in (0,T), \tag{2.4}$$

and we define

$$\|f\|_{L^\infty(0,T;H)} := \inf\{C \in [0,\infty) \text{ such that (2.4) holds}\}.$$

On the other hand, we define

$$\mathscr{H} = \mathscr{H}_T := L^2(0,T;V) := \{f \in L^2(\mathcal{U});\ \nabla_x f \in L^2(\mathcal{U})\}$$

that we endowed with the Hilbert norm defined by

$$\|f\|_{\mathscr{H}}^2 = \|f\|_{L^2(0,T;V)}^2 := \int_0^T \|f(s)\|_V^2\, ds, \quad \forall\, f \in \mathscr{H}.$$

From (2.3), we have thus established

$$f \in L^\infty(0,T;H) \cap L^2(0,T;V). \tag{2.5}$$

It is worth emphasizing here that we sometimes switch our viewpoint by considering either the function $f = f(t,x)$ as a function of both variables $(t,x) \in \mathcal{U}$ or as a functional mapping $f : [0,T] \to H$ or $V$ defined by $[f(t)](x) = f(t,x)$. Rather than developing the arguments in the more abstract functional mapping (what we will do in Section **??** below), we adopt here an in-between viewpoint. We will always keep in mind the two variables setting (what make possible to use the usual Lebesgue, distributional and Sobolev theories when necessary) and frequently adopt the functional mapping notation. In this framework, we may move from one point of view to the other thanks to the Fubini theorem.

It is worth emphasizing at this point that for two (nice) functions $f = f(x)$ and $g = g(x)$, we may compute

$$\langle \mathcal{L}f, g\rangle := \int_{\mathbb{R}^d} (\Delta f + b\cdot\nabla f + c\, f) g,$$

[1]We commonly say that (the bilinear form associated to) $-\mathcal{L}$ is coercive if (2.2) holds with $\alpha > 0$ and $\kappa = 0$, and that $\mathcal{L} - \kappa$ is dissipative if (2.2) holds with $\alpha = 0$ and $\kappa \in \mathbb{R}$. Our assumption (2.2) is then more general than a coercivity condition (on $-\mathcal{L}$) but less general than a dissipativity condition (on $\mathcal{L}$).

so that

$$\langle \mathcal{L} f, g\rangle = -\int_{\mathbb{R}^d} \nabla f \cdot \nabla g + \int_{\mathbb{R}^d} (b\cdot \nabla_x f) g + \int_{\mathbb{R}^d} c\, f\, g, \tag{2.6}$$

thanks to the Green-Ostrogradski divergence formula, and thus clearly

$$|\langle \mathcal{L} f, g\rangle| \le M \|f\|_V \|g\|_V,$$

for a constant $M > 0$, thanks to the Cauchy-Schwarz inequality in $L^2(\mathbb{R}^d)$ and because of the hypothesis (1.4) on the coefficients. A possible choice is $M := 1 + \|b\|_{L^\infty} + \|c\|_{L^\infty}$. In other words, taking (2.6) as a definition of $\mathcal{L}$, we have

$$\mathcal{L} : V \to V'$$

is a linear and bounded operator with

$$\forall\, f \in V, \quad \|\mathcal{L} f\|_{V'} = \sup_{g\in B_V} \langle \mathcal{L} f, g\rangle \le M \|f\|_V. \tag{2.7}$$

On the other hand, coming back to a nice solution $f = f(t,x)$ to the parabolic equation (1.1)-(1.2)-(1.3), we may multiply (1.1) by a test function $\varphi \in C^1_c([0,T)\times \mathbb{R}^d)$, and integrating by part, we have

$$\begin{aligned} -\int_{\mathbb{R}^d} f_0 \varphi(0) - \int_{\mathcal{U}} f \partial_t \varphi &= \int_{\mathcal{U}} \varphi\, \partial_t f = \int_{\mathcal{U}} \varphi\, \mathcal{L} f \\ &= -\int_{\mathcal{U}} \nabla f \cdot \nabla \varphi + \int_{\mathcal{U}} (b\cdot \nabla f + cf)\varphi. \end{aligned}$$

That formulation gives a first meaningful (distributional) sense to a solution to the equation under the sole assumption $f \in L^2(0,T;V)$. Equivalently (by a density $C^1_c(\mathbb{R}^d) \subset H^1(\mathbb{R}^d)$ argument), we may write

$$-(f_0, \varphi(0)) - \int_0^T (f, \varphi')dt = \int_0^T \langle \mathcal{L} f, \varphi\rangle dt, \tag{2.8}$$

for any $\varphi \in C^1_c([0,T);V)$. We emphasize again that in the last term we use the definition (2.6) of $\mathcal{L}$. We also define

$$[\varphi'(t)](x) = [\frac{d}{dt}\varphi](x) := (\partial_t \varphi)(t,x)$$

in order to emphasize the functional mapping viewpoint.

**Definition 2.1.** *For any given $f_0 \in H$, $T > 0$, we say that*

$$f = f(t) \in L^2(0,T;V)$$

*is a* **weak solution** *to the Cauchy problem associated to the parabolic equation* (1.1)-(1.2)-(1.3) *on the time interval* $[0,T)$ *if it satisfies the weak formulation* (2.8) *for any $\varphi \in C^1_c([0,T);V)$. We say that $f$ is a global weak solution if it is a weak solution on $[0,T)$ for any $T > 0$.*

We now want to strengthen the notion of solution by improving the functional spaces, and more precisely by reducing the space of solutions and enlarging the space of test functions. We observe that for any $f, g \in \mathscr{H} = L^2(0,T;V)$, we have

$$\int_0^T \langle \mathcal{L} f, g\rangle\, dt \le M \int_0^T \|f\|_V \|g\|_V\, dt \le M \|f\|_{\mathscr{H}} \|g\|_{\mathscr{H}},$$

where we have first used (2.7) and next the Cauchy-Schwarz inequality in $L^2(0,T)$ (we may also directly use the Cauchy-Schwarz inequality in $L^2(\mathcal{U})$ on the very definition of the RHS term). For any weak solution $f$ to the parabolic equation (1.1)-(1.2)-(1.3), we may write

$$\langle \partial_t f, \varphi \rangle_{\mathcal{D}'(\mathcal{U}),\mathcal{D}(\mathcal{U})} = -\int_0^T (f, \varphi')\, dt = \int_0^T \langle \mathcal{L} f, \varphi \rangle\, dt,$$

for any $\varphi \in \mathcal{D}(\mathcal{U})$, and we thus have

$$\langle \partial_t f, \varphi \rangle_{\mathcal{D}'(\mathcal{U}),\mathcal{D}(\mathcal{U})} \le C \|\varphi\|_{\mathscr{H}}, \tag{2.9}$$

for the constant $C := M\|f\|_{\mathscr{H}}$. Adopting the functional mapping viewpoint, we deduce (and denote)

$$f' = \mathcal{L} f \in L^2(0,T;V') := \mathscr{H}' = (L^2(0,T;V))'.$$

More concretely, we have

$$\mathscr{H}' = \{F_0 + \sum_{i=1}^d \partial_{x_i} F_i;\ F_i \in L^2(\mathcal{U})\}.$$

Inspired from the usual definition of Sobolev spaces (for real valued functions), we use the shorthand

$$f \in H^1(0,T;V'), \tag{2.10}$$

for notifying that $f$ satisfies the estimate (2.9) and for later reference we denote

$$\|f'\|_{L^2(0,T;V')} := \inf\{C \text{ such that (2.9) holds}\}.$$

**Definition 2.2.** *For any given $f_0 \in H$, $T > 0$, we say that*

$$f \in X_T := C([0,T];H) \cap L^2(0,T;V) \cap H^1(0,T;V') \tag{2.11}$$

*is a* **variational solution** *to the Cauchy problem associated to the parabolic equation* (1.1)-(1.3) *on the time interval* $[0,T)$ *if it is a solution in the following weak sense*

$$(f(t), \varphi(t)) = (f_0, \varphi(0)) + \int_0^t \big\{ \langle \mathcal{L} f(s), \varphi(s) \rangle + \langle \varphi'(s), f(s) \rangle \big\}\, ds, \tag{2.12}$$

*for any $\varphi \in X_T$ and any $0 \le t \le T$. We say that $f$ is a global solution if it is a solution on $[0,T]$ for any $T > 0$.*

We will establish that a weak solution $f$ automatically satisfies $f \in X_T$, what is a consequence of the continuous embedding $L^2(0,T;V) \cap H^1(0,T;V') \subset C([0,T];H)$ discussed below (see section 3.2) and the additional estimate (2.10). The first term in (2.12) is then well-defined because of the condition $f, \varphi \in C([0,T];H)$. Under the sole assumption $f, \varphi \in X_T$, a possible definition of the last term is

$$\int_0^t \langle \varphi'(s), f(s) \rangle\, ds := \langle \varphi', f \rangle_{\mathscr{H}_t', \mathscr{H}_t},$$

see also the Section **??** for an alternative point of view.

**Theorem 2.3.** *With the above definition and assumptions, for any $f_0 \in H$, there exists a unique global variational solution to the Cauchy problem* (1.1)-(1.2)-(1.3)-(1.4)*, and this one satisfies* (2.3).

## 3. Proof of Theorem 2.3

This section is devoted to the proof of Theorem 2.3 which is split into four steps.

**Step 1.** We first establish the existence of a weak solution $f \in L^2(0,T;V)$.

**Step 2.** We next prove that $f \in X_T$.

**Steps 3 & 4.** We finally establish that $f$ is a variational solution from what we immediately deduce the uniqueness and the a posteriori estimate (2.3).

3.1. **On the existence of a weak solution.** Introducing an approximation scheme and next using a weak compactness argument in the Hilbert space $L^2(0,T;V)$, we establish that there exists a function $f \in L^2(0,T;V)$ satisfying the weak formulation (2.8).

*Step 1.* For a given $f_0 \in H$ and $\varepsilon > 0$, we seek $f_1 \in V$ such that

$$f_1 - \varepsilon \mathcal{L} f_1 = f_0. \tag{3.1}$$

We introduce the bilinear form $a : V \times V \to \mathbb{R}$ defined by

$$a(u,v) := (u,v) - \varepsilon \, \langle \mathcal{L} u, v \rangle.$$

Thanks to the assumptions made on $\mathcal{L}$, we have

$$|a(u,v)| \le |u| \, |v| + \varepsilon \, M \, \|u\| \, \|v\|,$$

and

$$a(u,u) \ge |u|^2 + \varepsilon \, \alpha \, \|u\|^2 - \varepsilon \, \kappa \, |u|^2 \ge \varepsilon \, \alpha \, \|u\|^2, \tag{3.2}$$

whenever $\varepsilon \, \kappa < 1$, what we assume from now on. On the other hand, the mapping $v \in V \mapsto (f_0, v)$ is a linear and continuous form. We may thus apply the Lax-Milgram theorem which implies

$$\exists! \, f_1 \in V, \qquad (f_1, v) - \varepsilon \langle \mathcal{L} f_1, v \rangle = (f_0, v), \quad \forall \, v \in V.$$

*Step 2.* We fix $\varepsilon > 0$ as in the preceding step and we build by induction the sequence $(f_k)$ in $V \subset H$ defined by the family of equations

$$\frac{f_{k+1} - f_k}{\varepsilon} = \mathcal{L} \, f_{k+1}, \qquad \forall \, k \ge 0. \tag{3.3}$$

Observe that from the identity

$$(f_{k+1}, f_{k+1}) - \varepsilon \, \langle \mathcal{L} f_{k+1}, f_{k+1} \rangle = (f_k, f_{k+1}),$$

we deduce (that is (3.2) again)

$$|f_{k+1}|^2 + \varepsilon \, \alpha \, \|f_{k+1}\|^2 - \varepsilon \, \kappa \, |f_{k+1}|^2 \le |f_k| \, |f_{k+1}| \le \frac{1}{2} |f_k|^2 + \frac{1}{2} |f_{k+1}|^2,$$

and then

$$|f_{k+1}|^2 + 2\varepsilon\alpha \, \|f_{k+1}\|^2 \le (1 - 2\varepsilon \, \kappa)^{-1} \, |f_k|^2, \quad \forall \, k \ge 0.$$

Thanks to the discrete version of the Gronwall lemma, we get

$$|f_n| + 2\alpha \sum_{k=1}^{n} \varepsilon \|f_k\|^2 \le (1 - 2\varepsilon \, \kappa)^{-n} |f_0| \le e^{2\kappa\varepsilon n} \, |f_0|, \quad \forall \, n \ge 1.$$

We now fix $T > 0$, $n \in \mathbb{N}^*$, and we define

$$\varepsilon := T/n, \quad t_k = k \, \varepsilon, \quad f^\varepsilon(t) := f_k \text{ on } [t_k, t_{k+1}).$$

The last estimate writes then

$$2\alpha \int_0^T \|f^\varepsilon\|_V^2 \, dt \le (e^{2\kappa T} + 2\alpha\varepsilon) \, |f_0|^2. \tag{3.4}$$

*Step 3.* Consider a test function $\varphi \in C_c^1([0,T);V)$ and define $\varphi_k := \varphi(t_k)$, so that $\varphi_n = \varphi(T) = 0$. Multiplying the equation (3.3) by $\varphi_k$ and summing up from $k = 0$ to $k = n-1$, we get

$$-(\varphi_0, f_0) - \sum_{k=1}^{n} (\varphi_k - \varphi_{k-1}, f_k) = \sum_{k=0}^{n} \varepsilon \, \langle \mathcal{L} f_{k+1}, \varphi_k \rangle = \sum_{k=1}^{n} \varepsilon \, \langle \mathcal{L} f_k, \varphi_{k-1} \rangle.$$

Introducing the two functions $\varphi^\varepsilon, \varphi_\varepsilon : [0,T) \to V$ defined by

$$\varphi^\varepsilon(t) := \varphi_{k-1} \quad \text{and} \quad \varphi_\varepsilon(t) := \frac{t_{k+1} - t}{\varepsilon} \, \varphi_{k-1} + \frac{t - t_k}{\varepsilon} \, \varphi_k \qquad \text{for} \quad t \in [t_k, t_{k+1}),$$

in such a way that

$$\varphi_\varepsilon'(t) = \frac{\varphi_k - \varphi_{k-1}}{\varepsilon} \qquad \text{for} \quad t \in (t_k, t_{k+1}),$$

the above equation also writes

$$-\langle \varphi(0), f_0 \rangle - \int_\varepsilon^T (\varphi_\varepsilon', f^\varepsilon) \, dt = \int_0^T \langle \mathcal{L} f^\varepsilon, \varphi^\varepsilon \rangle \, dt. \tag{3.5}$$

On the one hand, from (3.4) and the fact that $L^2(0,T;V)$ is a Hilbert space, we know that up to the extraction of a subsequence, there exists $f \in L^2(0,T;V)$ such that $f^\varepsilon \rightharpoonup f$ weakly in $L^2(0,T;V)$. On the other hand, from the above construction, we have $\varphi_\varepsilon' \to \varphi'$ and $\varphi_\varepsilon \to \varphi$ both uniformly in $L^\infty(0,T;V)$ (using that $\varphi$ and $\varphi'$ belong to $C([0,T];V)$ and thus are uniformly continuous). We may then pass to the limit as $\varepsilon \to 0$ in (3.5) and we get (2.8).

3.2. **About the functional space.** As a consequence of the general argument leading to (2.10) and of the following result, any weak solution belongs in fact to the space $X_T$.

**Lemma 3.1.** *The following inclusion*

$$L^2(0,T;V) \cap H^1(0,T;V') \subset C([0,T];H) \tag{3.6}$$

*holds true. Moreover, for any $g \in X_T$ and $t_1, t_2 \in [0,T]$, there holds*

$$|g(t_2)|^2 = |g(t_1)|^2 + 2 \int_{t_1}^{t_2} \langle g', g \rangle \, ds. \tag{3.7}$$

*Proof of Lemma 3.1.* We first establish (3.6) thanks to a regularization trick and using in a fundamental way that $C([0,T];H)$ is a Banach space. A similar regularization argument allows us to establish (3.7) as well.

*Step 1.* Let us consider $g \in L^2(0,T;V) \cap H^1(0,T;V') \subset L^2(\mathcal{U})$. We define the function $\bar g = g$ on $[0,T]$, $\bar g = 0$ on $\mathbb{R}\backslash[0,T]$, next for a mollifier $\rho : \mathbb{R} \to \mathbb{R}$ with compact support included in $(-1,-1/2)$, we define the approximation to the identity sequence $(\rho_\varepsilon)$ by setting $\rho_\varepsilon(t) := \varepsilon^{-1}\rho(\varepsilon^{-1} t)$ and finally the sequence $g_\varepsilon(t) := \bar g *_t \rho_\varepsilon$ where $*$ stands for the usual convolution operator on $\mathbb{R}$. We fix $\tau \in (0,T)$ and we assume $0 < \varepsilon < T - \tau$. For any $t \in (0,\tau)$, because of the support condition

$$\operatorname{supp} \rho_\varepsilon(t - \cdot) \subset [t + \varepsilon/2, t + \varepsilon] \subset [\varepsilon/2, \tau + \varepsilon] \subset (0,T),$$

we have $s \mapsto \rho_\varepsilon(t-s) \in \mathcal{D}(0,T)$ and

$$g_\varepsilon(t,\cdot) = \int_{\mathbb{R}} \rho_\varepsilon(t-s)\,\bar g(s,\cdot)\,ds = \int_0^T \rho_\varepsilon(t-s)\,g(s,\cdot)\,ds.$$

We observe that $g_\varepsilon \in C^1(\mathbb{R};H)$, $g_\varepsilon \to g$ strongly in $V$ a.e. on $[0,T]$ and in $L^2(0,T;V)$ from standard convolution results for real values measurable functions. We similarly observe that

$$\begin{aligned}\partial_t g_\varepsilon &= \int_{\mathbb{R}} \partial_t \rho_\varepsilon(t-s)\,\bar g(s)\,ds \\ &= -\int_0^T (\partial_s \rho_\varepsilon(t-s))\,g(s)\,ds \\ &= \int_0^T \rho_\varepsilon(t-s)\,(\partial_t g)(s)\,ds = \rho_\varepsilon *_t (\overline{\partial_t g}).\end{aligned}$$

By assumption, we have $\partial_t g = F + \mathrm{div}_x G$ with $F, G \in L^2(\mathcal{U})$, so that

$$\partial_t g_\varepsilon = \rho_\varepsilon *_t \bar F + \mathrm{div}_x(\rho_\varepsilon *_t \bar G) \to F + \mathrm{div}_x G = \partial_t g$$

in the sense of $L^2(0,T;H^{-1}(\mathbb{R}^d))$.

*Step 2.* We observe that for $t \mapsto u(t) \in C^1((0,T);H)$ and because $h \mapsto |h|_H^2$ is $C^1(H;\mathbb{R})$, we have $t \mapsto |u(t)|_H^2$ is $C^1((0,T);\mathbb{R})$ and

$$\frac{d}{dt}|u(t)|_H^2 = 2(u'(t),u(t))_H = 2\langle u'(t),u(t)\rangle_{V',V}.$$

We fix $\tau \in (0,T)$ and $\varepsilon, \varepsilon' \in (0, T-\tau)$, and the above computation gives

$$\frac{d}{dt}|g_\varepsilon(t) - g_{\varepsilon'}(t)|^2 = 2\,\langle g'_\varepsilon - g'_{\varepsilon'}, g_\varepsilon - g_{\varepsilon'}\rangle,$$

so that for any $t_1, t_2 \in [0,\tau]$, we have

$$|g_\varepsilon(t_2) - g_{\varepsilon'}(t_2)|^2 = |g_\varepsilon(t_1) - g_{\varepsilon'}(t_1)|^2 + 2\int_{t_1}^{t_2} \langle g'_\varepsilon - g'_{\varepsilon'}, g_\varepsilon - g_{\varepsilon'}\rangle ds. \tag{3.8}$$

Since $g_\varepsilon \to g$ a.e. on $[0,\tau]$ in $V \subset H$, we may fix $t_1 \in [0,\tau]$ such that

$$g_\varepsilon(t_1) \to g(t_1) \quad \text{in } H. \tag{3.9}$$

As a consequence of (3.8), (3.9) and the convergences $g_\varepsilon \to g$ in $L^2(0,\tau;V)$ and $g'_\varepsilon \to g'$ in $L^2(0,\tau;V')$, so that $(\|g'_\varepsilon\|_{\mathscr{H}'_\tau})$ is a bounded sequence, we have

$$\limsup_{\varepsilon,\varepsilon'\to 0} \sup_{[0,\tau]} |g_\varepsilon(t) - g_{\varepsilon'}(t)|_H^2 \le \lim_{\varepsilon,\varepsilon'\to 0} \|g'_\varepsilon - g'_{\varepsilon'}\|_{\mathscr{H}'_\tau}\,\|g'_\varepsilon - g'_{\varepsilon'}\|_{\mathscr{H}_\tau} = 0.$$

We thus deduce that $(g_\varepsilon)$ is a Cauchy sequence in $C([0,\tau];H)$, and then $g_\varepsilon$ converges in $C([0,\tau];H)$ to a limit $\tilde g \in C([0,\tau];H)$. That proves $g = \tilde g$ a.e. and thus $g \in C([0,\tau];H)$ (up to modifying $g$ on a set of zero Lebesgue measure). We prove similarly that $g \in C([\tau,T];H)$ for any $\tau \in (0,T)$ and thus $g \in C([0,T];H)$.

*Step 3.* Similarly as for (3.8), we have

$$|g_\varepsilon(t_2)|_H^2 = |g_\varepsilon(t_1)|_H^2 + 2\int_{t_1}^{t_2} \langle g'_\varepsilon, g_\varepsilon\rangle ds,$$

and passing to the limit $\varepsilon \to 0$, we get (3.7). □

3.3. **Weak solutions are variational solutions.** We establish the equivalence between several formulations of solutions to the parabolic equation (1.1)-(1.2)-(1.3).

**Lemma 3.2.** *Consider the operator $\mathcal{L}$ defined by (1.2), an initial datum $f_0 \in H$ and a function $f \in L^2(0,T;V)$. There is equivalence between :*
*(1) $f$ is a variational solution in the sense of Definition 2.2;*
*(2) $f$ is a weak solution in the sense of Definition 2.1;*
*(3) $f$ satisfies*

$$f' = \mathcal{L} f \text{ in } V', \quad f(0) = f_0 \text{ in } H,$$

*in the sense*

$$0 = f_0 \psi(0) + \int_0^T (g\psi' + \mathcal{L} g \psi) dt \quad in \quad V', \tag{3.10}$$

*for any $\psi \in C^1_c([0,T);\mathbb{R})$;*
*(4) $f$ satisfies*

$$\frac{d}{dt}\langle f, h\rangle = \langle \mathcal{L} f, h\rangle \text{ in } \mathcal{D}'(0,T), \quad \langle f(0), h\rangle = (f_0, h),$$

*for any $h \in V$.*

*Proof of Lemma 3.2. Step 1.* Condition (1) clearly implies condition (2). If $f$ satisfies (2) then we obtain (3) by particularizing $\varphi = \psi(t) h$, $h \in V$, $\psi \in C^1_c([0,T);\mathbb{R})$, in the formulation (2.8) in order to get

$$\begin{aligned} 0 &= (f_0\psi(0), h) + \int_0^T (f\psi', h) dt = \int_0^T \langle \mathcal{L} f \psi, h, \rangle dt, \\ &= \Big\langle f_0 \psi(0) + \int_0^T f\psi' dt + \int_0^T \mathcal{L} f \psi dt, h \Big\rangle, \end{aligned}$$

by linearity, what is nothing but (3.10). Condition (4) is nothing but the distributional formulation of the first above identity so that it is equivalent to (3). We prove now that (3) implies (2) and next (1) in several steps.

*Step 2.* We observe first that $f$ satisfies (2.8) with $\varphi = \psi(t) h$, $h \in V$, $\psi \in C^1_c([0,T);\mathbb{R})$. Take $\varphi \in C^1_c([0,T);V)$ and define the piecewise affine function

$$\chi_k(t) := \sum_{j=1}^k \mathbf{1}_{[t_{j-1},t_j)} \frac{k}{T} \{\varphi'(t_{j-1})(t_j - t) + \varphi'(t_j)(t - t_{j-1})\},$$

with $t_j := jT/k$. Next, choosing $\chi \in C^1_c([0,T)$, $\chi \equiv 1$ on $\operatorname{supp}\varphi$, let us define

$$\varphi_k(t) := \Big(\varphi(0) + \int_0^t \chi_k(s)\, ds\Big)\chi(t) = \sum_{j=0}^k \psi_j(t) h_j,$$

with $h_j := \varphi'(t_j)$ and $\psi_j \in C^1_c([0,T)$. Because the weak formulation (2.8) is linear in the test functions, $f$ satisfies (2.8) with the choice of test function $\varphi_k$, that is

$$-(f_0, \varphi_k(0)) - \int_0^T (f, \varphi_k') dt = \int_0^T \langle \mathcal{L} f, \varphi_k\rangle dt. \tag{3.11}$$

Because $\varphi' \in C([0,T];V)$, we have $\chi_k \to \varphi'$ uniformly as $k \to \infty$, and thus $\varphi_k \to \varphi$ and $\varphi_k' \to \varphi'$ also in $C([0,T];V)$. We may thus pass to the limit (3.11) as $k \to \infty$ and we obtain that $f$ also satisfies (2.8) with the test function $\varphi$, so that (2) holds.

*Step 3.* Because of the discussion leading to (2.10), we have $f \in H^1(0,T;V')$, and thus $f \in X_T$ thanks to Lemma 3.1. Assume now $\varphi \in C_c([0,T);H) \cap L^2(0,T;V) \cap H^1(0,T;V')$. We define $\varphi_\varepsilon(t) := \varphi *_t \rho_\varepsilon$ for a mollifier $(\rho_\varepsilon)$ associated to $\rho$ with compact support included in $(-1,-1/2)$ so that from Step 1 in the proof of Lemma 3.1, $\varphi_\varepsilon \in C^1_c([0,T);V)$ for any $\varepsilon > 0$ small enough and

$$\varphi_\varepsilon \to \varphi \quad \text{in} \quad C([0,T];H) \cap L^2(0,T;V)$$

and weakly in $H^1(0,T;V')$. Writing the weak formulation (2.8) for $\varphi_\varepsilon$ and passing to the limit $\varepsilon \to 0$, we get that the same weak formulation (2.8) holds true for $\varphi$.

*Step 4.* Assume finally that $\varphi \in X_T$. We fix $\chi \in C^1(\mathbb{R})$ such that $\operatorname{supp}\chi \subset (-\infty,0)$, $\chi' \le 0$, $\chi' \in C_c(]-1,0[)$ and $\int_{-1}^0 \chi' = -1$, and we define $\chi_\varepsilon(t) := \chi((t-T)/\varepsilon)$ so that $\varphi_\varepsilon := \varphi\chi_\varepsilon \in C_c([0,T);H)$ and $\chi_\varepsilon \to \mathbf{1}_{[0,T]}$ a.e., $\chi'_\varepsilon \to -\delta_T$ in $\mathcal{D}'(\mathbb{R})$ as $\varepsilon \to 0$. Equation (2.8) for the test function $\varphi_\varepsilon$ writes

$$-(f_0,\varphi(0)) - \int_0^T \chi'_\varepsilon(\varphi,f)\,ds = \int_0^T \chi_\varepsilon\big\{\langle \mathcal{L}f,\varphi\rangle + \langle \varphi', f\rangle\big\}\,ds,$$

and we obtain the variational formulation (2.12) for $t_1 = 0$ and $t_2 = T$ by passing to the limit $\varepsilon \to 0$ in the above equation. □

3.4. **A posteriori estimate and uniqueness of the variational solution.** Taking $\varphi = f \in X_T$ as a test function in the variational formulation (2.12). From Lemma 3.1, we deduce

$$\begin{aligned}
\frac12|f(t)|_H^2 - \frac12|f_0|_H^2 &= |f(t)|_H^2 - |f_0|_H^2 - \int_0^t \langle f'(s), f(s)\rangle\,ds \\
&= \int_0^t \langle \mathcal{L}f, f\rangle\,ds \\
&\le \int_0^t (-\alpha\,\|f\|_V^2 + \kappa\,|f|_H^2)\,ds,
\end{aligned}$$

where we have used (3.7) at the first line, the variational formulation (2.12) at the second line and the *"coercive+dissipative"* estimate (2.2) on $\mathcal{L}$ at the last line. We then obtain (2.3) as an **a posteriori estimate** thanks to the Gronwall.

Let us prove now the **uniqueness of the variational solution** $f$ associated to a given initial datum $f_0 \in H$. In order to do so, we consider two variational solutions $g$ and $f$ associated to the same initial datum. Since the parabolic equation (1.1)-(1.3) is linear, or more precisely, the variational formulation (2.12) is linear in the solution, the function $g - f$ satisfies the same variational formulation (2.12) but associated to the initial datum $g_0 - f_0 = 0$. The a posteriori estimate (2.3) then holds for $g - f$ thanks to the previous step and implies that $g - f = 0$.

## 4. Parabolic equations with time dependent coefficients

In this section, we present a variant of the Lax-Milgram theorem that we use for getting an alternative proof Theorem 2.3 and then extending Theorem 2.3 to a parabolic equation with time dependent coefficients. We finally deal with the (nonlinear) McKean-Vlasov equation.

4.1. **A variant of the Lax-Milgram theorem.** We consider a Hilbert space $\mathscr{H}$ endowed with a scalar product $(\cdot,\cdot)$ and the associated norm $|\cdot|$. We consider next a subspace $\Phi \subset \mathscr{H}$ endowed with a pre-Hilbertian scalar product $(\!(\cdot,\cdot)\!)$ and the associated norm $\|\cdot\|$ such that

$$|\varphi| \le C\|\varphi\|, \quad \forall\, \varphi \in \Phi. \tag{4.1}$$

We finally consider a bilinear form $\mathcal{E} : \mathscr{H} \times \Phi \to \mathbb{R}$ such that

$$\forall\, \varphi \in \Phi,\ \exists\, C_\varphi \ge 0, \quad |\mathcal{E}(f,\varphi)| \le C_\varphi |f|,\ \forall\, f \in \mathscr{H}, \tag{4.2}$$

$$\exists\, \alpha > 0,\ \forall\, \varphi \in \Phi, \quad \mathcal{E}(\varphi,\varphi) \ge \alpha\|\varphi\|^2. \tag{4.3}$$

**Theorem 4.1.** *For any linear and continuous form $\ell : \Phi \to \mathbb{R}$, meaning that*

$$|\ell(\varphi)| \le C\|\varphi\|, \quad \forall\, \varphi \in \Phi, \tag{4.4}$$

*there exists at least one $f \in \mathscr{H}$ such that*

$$\mathcal{E}(f,\varphi) = \ell(\varphi), \quad \forall\, \varphi \in \Phi. \tag{4.5}$$

*Proof of Theorem 4.1.* For a fixed $\varphi \in \Phi$, the mapping $f \mapsto \mathcal{E}(f,\varphi)$ is a linear and continuous form on $\mathscr{H}$, so that there exists $A\varphi \in \mathscr{H}$ such that

$$\mathcal{E}(f,\varphi) = (f, A\varphi), \quad \forall\, f \in \mathscr{H},\ \varphi \in \Phi, \tag{4.6}$$

and $A : \Phi \to \mathscr{H}$ is a linear mapping. Because of (4.3), $A$ is one-to-one (injection). On the linear subspace $\mathscr{G} := A\Phi \subset \mathscr{H}$, we may then define the inverse linear mapping $B := A^{-1} : \mathscr{G} \to \Phi$. Using (4.6), (4.3) and (4.1), for any $g \in \mathscr{G}$, we have

$$\alpha\|Bg\|^2 \le \mathcal{E}(Bg,Bg) = (Bg,g) \le |Bg||g| \le C\|Bg\||g|,$$

from what we immediately deduce that $B$ is bounded with norm $\|B\| \le C/\alpha$. Defining $\overline{\mathscr{G}}$ the closure of $\mathscr{G}$ in $\mathscr{H}$ (for the norm $|\cdot|$) and $\hat\Phi$ the completion of $\Phi$ for the norm $\|\cdot\|$, we may uniquely extend $B$ as $\bar B : \bar{\mathscr{G}} \to \hat\Phi$, $\bar B_{|\mathscr{G}} = B$. We may also uniquely extend $\ell$ as a linear and continuous form $\bar\ell$ on $\hat\Phi$. The equation (4.5) becomes

$$(f, A\varphi) = \bar\ell(\varphi), \quad \forall\, \varphi \in \Phi,$$

or equivalently

$$(f,\psi) = \bar\ell(\bar B\psi), \quad \forall\, \psi \in \bar{\mathscr{G}}. \tag{4.7}$$

From the Riesz-Frechet representation Theorem in $\bar{\mathscr{G}}$ and because $\bar\ell \circ \bar B$ is a linear and continuous mapping on $\bar{\mathscr{G}}$, there exists a unique $f \in \bar{\mathscr{G}}$ solution to (4.7), and this one provides a solution to (4.5). When $\bar{\mathscr{G}} \ne \mathscr{H}$, the problem (4.5) has a family of solutions given by $\{f\} + \bar{\mathscr{G}}^\perp$. □

4.2. **An alternative proof of Theorem 2.3.** We consider the parabolic equation (1.1)-(1.2)-(1.3)-(1.4) with same notations and we additionally assume

$$\sup c \le -\frac{1}{2} - \frac{1}{2}\|b\|^2_{L^\infty}. \tag{4.8}$$

This additional assumption will be removed in the next section. We define the Hilbert space $\mathscr{H} := L^2(0,T;H^1(\mathbb{R}^d))$ endowed with its usual norm and the pre-Hilbert space $\Phi := C^2_c([0,T) \times \mathbb{R}^d)$ endowed with the norm $\|\cdot\|$ defined by

$$\|\varphi\|^2 := \int_0^T \|\varphi(t,\cdot)\|^2_{H^1(\mathbb{R}^d)} dt + \|\varphi(0,\cdot)\|^2_{L^2(\mathbb{R}^d)}.$$

We also define the bilinear form

$$\mathcal{E}(f,\varphi) := \int_{\mathcal{U}} (\nabla f \cdot \nabla \varphi - (b \cdot \nabla f + cf)\varphi - f \partial_t \varphi) \, dxdt,$$

with always $\mathcal{U} := (0,T) \times \mathbb{R}^d$, and the linear form

$$\ell(\varphi) := \int_{\mathbb{R}^d} \varphi(0,\cdot) f_0 \, dx.$$

We observe that

$$\mathcal{E}(\varphi,\varphi) = \int_{\mathcal{U}} (|\nabla \varphi|^2 - \nabla \varphi \cdot b \, \varphi - c\varphi^2) dxdt + \frac{1}{2} \int_{\mathbb{R}^d} \varphi(0,x)^2 dx \geq \frac{1}{2} \|\varphi\|^2,$$

where we have used the Young inequality and the condition (4.8) in order to get the last inequality, that $\mathcal{E}$ also satisfies (4.2) and that $\ell$ satisfies (4.4). From Theorem 4.1, we know that there exists $f \in \mathscr{H}$ satisfying (4.5), or un other words

$$\int_{\mathcal{U}} (\nabla f \cdot \nabla \varphi - (b \cdot \nabla f + cf)\varphi - f \partial_t \varphi) \, dxdt = \int_{\mathbb{R}^d} \varphi(0,\cdot) f_0 \, dx$$

for any $\varphi \in C^2_c([0,T) \times \mathbb{R}^d)$. Because $C^2_c([0,T) \times \mathbb{R}^d) \subset C^1_c([0,T); H^1(\mathbb{R}^d))$ with dense embedding, we deduce that $f$ is in fact a weak-solution in the sense of Definition 2.1. We have recovered the conclusions established in Section 3.1 and we conclude the proof of Theorem 2.3 by using the next steps presented in Section 3.

4.3. **A time dependent variant of Theorem 2.3.** We consider the parabolic equation

$$\partial_t f = \mathcal{L} f := \operatorname{div}(A \nabla f) + \operatorname{div}(af) + b \cdot \nabla f + cf + \mathfrak{F}, \tag{4.9}$$

where $A_{ij}$, $a_i$, $b_i$ and $c$ are times dependent coefficients and where $A_{ij}$ is uniformly elliptic in the sense that

$$\forall\, t \in (0,T), \ \forall\, x \in \mathbb{R}^d, \ \forall\, \xi \in \mathbb{R}^d \quad A_{ij}(t,x)\, \xi_i \xi_j \geq \nu\, |\xi|^2, \quad \nu > 0. \tag{4.10}$$

**Theorem 4.2** (J.-L. Lions)**.** *Assume that*

$$A, \, a, \, b, \, c \in L^\infty((0,T) \times \mathbb{R}^d) \tag{4.11}$$

*and that $A$ satisfies the uniformly elliptic condition* (4.10)*. For any $f_0 \in L^2(\mathbb{R}^d)$ and $\mathfrak{F} \in L^2(\mathcal{U})$, there exists a unique variational solution to the Cauchy problem associated to* (4.9) *in the sense that*

$$f \in X_T := C([0,T); L^2) \cap L^2(0,T; H^1) \cap H^1(0,T; H^{-1}),$$

*such that for any $\varphi \in X_T$ and any $t \in (0,T)$ there holds*

$$\begin{aligned} \int_{\mathbb{R}^d} f(t)\varphi(t) \, dx &= \int_{\mathbb{R}^d} f_0 \varphi(0) \, dx + \int_0^t \int_{\mathbb{R}^d} (\mathfrak{F}\varphi + f \partial_t \varphi) \, dxds \\ &+ \int_0^t \int_{\mathbb{R}^d} \{(b \cdot \nabla f + cf)\, \varphi - (A \nabla f + af) \cdot \nabla \varphi\} \, dxds. \end{aligned} \tag{4.12}$$

*Proof of Theorem 4.2. Step 1.* We proceed similarly as in the alternative proof of Theorem 2.3 in Section 4.2 and in particular we define $\mathscr{H}$ and $\Phi$ in the same way. We now define the bilinear form on $\mathscr{H} \times \Phi$ by

$$\mathcal{E}(f,\varphi) := \int_{\mathcal{U}} ((A \nabla f + af) \cdot \nabla \varphi - (b \cdot \nabla f + cf)\varphi - f \partial_t \varphi) \, dxdt$$

and the linear form on $\Phi$ by

$$\ell(\varphi) := \ell(\varphi) := \int_{\mathcal{U}} \mathfrak{F}\, \varphi \, dxdt + \int_{\mathbb{R}^d} \varphi(0,\cdot) f_0 \, dx.$$

We additionally first assume that

$$\sup c \le -\min(\frac{1}{2}, \frac{\nu}{2}) - \frac{1}{2\nu} \|a-b\|^2_{L^\infty}. \tag{4.13}$$

In that case, we may observe that

$$\begin{aligned} \mathcal{E}(\varphi,\varphi) &= \int_{\mathcal{U}} (A\nabla\varphi\cdot\nabla\varphi + \nabla\varphi\cdot(a-b)\,\varphi - c\varphi^2) dxdt + \frac{1}{2}\int_{\mathbb{R}^d} \varphi(0,x)^2 dx \\ &\ge \min(\frac{1}{2}, \frac{\nu}{2}) \|\varphi\|^2, \end{aligned}$$

that $\mathcal{E}$ also satisfies (4.2) and that $\ell$ satisfies (4.4). Exactly as in Section 4.2, we deduce the existence of a weak solution $f \in \mathscr{H}$ to the parabolic equation (4.9) with the help of Theorem 4.1 and we next conclude the proof of Theorem 4.2 by following the same steps as those presented in Sections 3.2, 3.3 and 3.4.

*Step 2.* We do not assume anymore (4.13). We define $c_\lambda := c - \lambda$ with $\lambda > 0$ large enough in such a way that $c_\lambda$ satisfies the additional condition (4.13) and $\mathfrak{F}_\lambda := e^{-\lambda t}\mathfrak{F}$. We may apply the first step with the choice of functions $A$, $a$, $b$, $c_\lambda$, $f_0$, $\mathfrak{F}_\lambda$, and we thus obtain the existence and uniqueness of a variational solution $g \in X_T$ to the modified equation

$$\partial_t g + \lambda g = \operatorname{div}(A\,\nabla g) + \operatorname{div}(ag) + b\cdot\nabla g + cg + e^{-\lambda t}\mathfrak{F} \ \text{ in } \ \mathcal{U}, \tag{4.14}$$

with initial condition $g(0,\cdot) = f_0$. For any $\varphi \in X_T$, choosing $\phi := e^{\lambda t}\varphi \in X_T$ as a test function in the variational formulation of (4.14), we immediately deduce that $f := e^{\lambda t} g$ satisfies (4.12). □

4.4. **The weak maximum principle.** We establish that the linear parabolic equation (4.9) satisfies a weak maximum principle.

**Theorem 4.3** (Weak maximum principle)**.** *When $0 \le f_0 \in L^2(\mathbb{R}^d)$, the associated variational solution $f \in X_T$ to the parabolic equation* (4.9)-(4.10)-(4.11) *satisfies $f \ge 0$.*

*Proof of Theorem 4.3.* We split the proof into four steps.

*Step 1.* We claim that for any $g \in X_T$ and any function $\beta \in C^2(\mathbb{R})$ such that $\beta(0) = \beta'(0) = 0$, $\beta'' \in L^\infty$, there holds

$$\int_{\mathbb{R}^d} \beta(g_t) - \int_{\mathbb{R}^d} \beta(g_0) = \int_0^t \langle g'_s, \beta'(g_s)\rangle_{V',V}\, ds, \tag{4.15}$$

for any $t \in (0,T)$. For proving that fact, we take up again the arguments (and the notations) presented in Step 1 of the proof of Lemma 3.1. More precisely, we consider the regularized sequence $g_\varepsilon = g *_t \rho_\varepsilon$ for which we know that $g_\varepsilon \in C^1([0,T]; H^1)$, $g_\varepsilon \to g$ in $C([0,T;L^2)$ and $\partial_t g_\varepsilon \to \partial_t g$ in $L^2(0,t;H^{-1})$. Using the dominated convergence theorem of Lebesgue, we also observe that

$$\beta(g_\varepsilon) \to \beta(g), \quad \beta'(g_\varepsilon) \to \beta'(g), \quad \nabla\beta'(g_\varepsilon) \to \nabla\beta'(g),$$

the first convergence holding in $C([0,T];L^2)$ and the two last convergences holding in $L^2(\mathcal{U})$. It is worth emphasizing that for proving the last convergence we use the classical chain rule $\nabla\beta'(g) = \beta''(g)\nabla g$ for $g \in H^1(\mathbb{R}^d)$. On the other hand, from the chain rule for smooth functions, we have

$$\int_{\mathbb{R}^d} \beta(g_{\varepsilon t}) - \int_{\mathbb{R}^d} \beta(g_{\varepsilon 0}) = \int_0^t\int_{\mathbb{R}^d} \partial_s g_\varepsilon \beta'(g_s) = \int_0^t \langle g'_s, \beta'(g_s)\rangle_{V',V}\, ds.$$

We conclude to (4.15) by passing to the limit as $\varepsilon \to 0$ in the above identity.

*Step 2.* We claim that for any $f \in X_T$ variational solution to the parabolic equation (4.9) and any $\varphi \in L^2(0,T;V)$, there holds

$$\int_0^t \langle f'_s, \varphi_s\rangle_{V',V}\, ds = \int_0^t \langle \mathcal{L} f_s, \varphi_s\rangle_{V',V}\, ds. \tag{4.16}$$

On the one hand, for $f, \varphi \in C^1([0,T],V)$, the integration by part formula gives

$$\int_{\mathbb{R}^d} f_t\varphi_t - \int_{\mathbb{R}^d} f_0\varphi_0 - \int_0^t\int_{\mathbb{R}^d} f_s\varphi'_s = \int_0^t\int_{\mathbb{R}^d} f'_s\varphi_s.$$

By the density $C^1([0,T],V) \subset X_T$, we deduce that

$$\int_{\mathbb{R}^d} f_t\varphi_t - \int_{\mathbb{R}^d} f_0\varphi_0 - \int_0^t \langle \varphi'_s, f_s\rangle ds = \int_0^t \langle f'_s, \varphi_s\rangle ds$$

holds for any $f, \varphi \in X_T$. Using the variational formulation of equation (4.9), we deduce that (4.16) holds for any $\varphi \in X_T$. We conclude that (4.16) holds for any $\varphi \in L^2(0,T;V)$ by the density $X_T \subset L^2(0,T;V)$.

*Step 3.* We consider $f \in X_T$ a variational solution to the parabolic equation (4.9) and a function $\beta$ as defined in step 1. Observing that $\beta'(f) \in L^2(0,T;V)$, we may use (4.15) and (4.16) together with $g = f$ and $\varphi = \beta'(f)$, and we obtain

$$\int_{\mathbb{R}^d} \beta(f_t) - \int_{\mathbb{R}^d} \beta(f_0) = \int_0^t \langle \mathcal{L} f_s, \beta'(f_s)\rangle_{V',V}\, ds,$$

or in other words, for any $t \in (0,T)$, we have

$$\int_{\mathbb{R}^d} \beta(f_t) - \int_{\mathbb{R}^d} \beta(f_0) = \int_0^t\int_{\mathbb{R}^d} \big(-(A\nabla f + af)\cdot\nabla\beta'(f) + (b\cdot\nabla f + cf)\beta'(f)\big).$$

*Step 4.* We now rather prove that $f_0 \le 0$ implies $f \le 0$ on $\mathcal{U}$ what is another formulation of the maximum principle because of the linearity of the equation. We make the fundamental observation

$$\int_{\mathbb{R}^d} g_+^2 = 0 \text{ iff } g \le 0, \tag{4.17}$$

for $g \in L^2(\mathbb{R}^d)$. By a classical density argument, we may in fact take $\beta(s) = s_+^2$ in the last formula established in Step 3, what gives

$$\begin{aligned}\int_{\mathbb{R}^d} (f_t)_+^2 - \int_{\mathbb{R}^d} (f_0)_+^2 \quad &= \quad \int_0^t\int_{\mathbb{R}^d} \big(-(A\nabla f_+ + af_+)\cdot\nabla f_+ + (b\cdot\nabla f_+ + cf_+)f_+\big).\\ &\le \quad \int_0^t\int_{\mathbb{R}^d} \big(\frac{1}{2\nu}|a|^2 + \frac{1}{2\nu}|b|^2 + c\big) f_+^2,\end{aligned}$$

where we use $ff_+ = f_+^2$, $f\nabla f_+ = f_+\nabla f = f_+\nabla f_+$ and $A\nabla f\cdot\nabla f = A\nabla f_+\cdot\nabla f$ in the first line and the Young inequality in the second line. Using that $f_0 \le 0$, the equivalence (4.17) and the boundedness assumption (4.11), we have

$$\int_{\mathbb{R}^d} (f_t)_+^2 \le \big(\frac{1}{2}\|a\|^2_{L^\infty} + \frac{1}{2}\|b\|^2_{L^\infty} + \|c_+\|_{L^\infty}\big) \int_0^t \int_{\mathbb{R}^d} f_+^2.$$

Thanks to the Gronwall lemma, we deduce that $\|(f_t)_+\|_{L^2} = 0$ so that $f_t \le 0$ for any $t \in (0,T)$ thanks to equivalence (4.17) again. □

4.5. **The McKean-Vlasov equation.** In this section, we consider the nonlinear McKean-Vlasov equation

$$\partial_t f = \mathcal{L}_f f := \Delta f + \mathrm{div}(a_f f), \quad f(0) = f_0, \tag{4.18}$$

with

$$a_f := a * f, \quad a \in L^\infty(\mathbb{R}^d)^d, \tag{4.19}$$

and we aim to prove the following existence and uniqueness result. We define

$$H = L^2_k := \big\{ f \in L^2(\mathbb{R}^d);\ \|f\|^2_{L^2_k} := \int f^2 \langle x \rangle^{2k}\, dx < \infty \big\},$$

with $\langle x \rangle^2 := 1 + |x|^2$, and

$$V = H^1_k := \big\{ f \in L^2_k(\mathbb{R}^d);\ \nabla f \in L^2_k(\mathbb{R}^d) \big\}.$$

**Theorem 4.4.** *For any $0 \le f_0 \in H := L^2_k$, $k > d/2$, there exists a unique global variational solution $f$ to the McKean-Vlasov equation* (4.18)*, and more precisely $f \in X_T$, for any $T > 0$, where $X_T$ is defined thanks to* (2.11) *with the choices $H := L^2_k$ and $V := H^1_k$.*

*Proof of Theorem 4.4. Step 1. A priori estimates.* Integrating in the $x$ variable a nice solution $f$ to the McKean-Vlasov equation (4.18) and using the Green-Ostrogradski divergence formula, we have

$$\frac{d}{dt} \int f dx = \int \mathrm{div}(\dots)\, dx = 0,$$

so that the mass (the integral) is conserved. On the other hand, multiplying the equation by $f_+$ and integrating in the $x$ variable, we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt} \int f_+^2 &= -\int |\nabla f_+|^2 - \int \nabla f_+ \cdot a_f f_+ \\ &\le \frac{1}{4}\|a_f\|^2_{L^\infty} \int f_+^2, \end{aligned}$$

by using the Young inequality and assuming $a_f \in L^\infty$. Thanks to to the Gronwall Lemma, we deduce that $f(t)_+ = 0$ if $f_{0+} = 0$, what is equivalent to the fact that the equation is positivity preserving: $f(t) \ge 0$ if $f_0 \ge 0$. The two previous properties together imply

$$\|f(t)\|_{L^1} \le \|f_0\|_{L^1} \quad \forall t \ge 0, \tag{4.20}$$

with equality if $f_0 \ge 0$. We finally multiply the equation by $f\langle x \rangle^{2k}$ and we integrate in the $x$ variable, in order to obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt} \int f^2 \langle x \rangle^{2k} &= -\int |\nabla f|^2 \langle x \rangle^{2k} + \frac{1}{2}\int f^2 \Delta \langle x \rangle^{2k} \\ &\quad - \int f(a_f \cdot \nabla f)\langle x \rangle^{2k} - \int f^2 a_f \cdot \nabla \langle x \rangle^{2k}, \end{aligned}$$

by performing several integration by parts. Using the Young inequality in order to get ride of the third term, we get

$$\frac{d}{dt}\int f^2\langle x\rangle^{2k} \le -\int |\nabla f|^2\langle x\rangle^{2k} + \int f^2(\Delta\langle x\rangle^{2k} + |a_f|^2\langle x\rangle^{2k} - 2a_f\cdot\nabla\langle x\rangle^{2k}).$$

From (4.19) and (4.20), we have

$$\|a_f\|_{L^\infty} \le \|a\|_{L^\infty}\|f_0\|_{L^1},$$

and therefore

$$\Delta\langle x\rangle^{2k} + |a_f|^2\langle x\rangle^{2k} - 2a_f\cdot\nabla\langle x\rangle^{2k} \le C_0\langle x\rangle^{2k},$$

for a constant $C_0 := C_0(k,\|a\|_{L^\infty}\|f_0\|_{L^1})$. Together with the Gronwall lemma, we deduce

$$\|f(t)\|^2_{L^2_k} + \int_0^t \|\nabla f\|^2_{L^2_k}\,ds \le e^{C_0 t}\,\|f_0\|^2_{L^2_k} \quad \forall\, t\ge 0. \tag{4.21}$$

That last estimate is strong enough for defining variational solutions as in the case of a linear parabolic equation at least when $L^2_k\subset L^1$, which means $k>d/2$.

*Step 2.* We observe that for $g\in L^2_k$ and $f\in V=H^1_k$, the formula

$$\langle \mathcal{L}_g f, h\rangle := -\int_{\mathbb{R}^d} (\nabla f + a_g f)\cdot\nabla(h\langle x\rangle^{2k})\,dx, \quad \forall\, h\in V,$$

defines a linear form on $V$. Repeating the same computations as for the proof of (4.21), we have

$$\langle \mathcal{L}_g f, f\rangle \le -\|\nabla f\|^2_{L^2_k} + C_0\|f\|^2_{L^2_k}, \quad \forall\, f\in H^1_k,$$

with $C_0 := C_0(k,\|a\|_{L^\infty}\|g\|_{L^1})$. We consider now

$$g\in\mathscr{C} := \{h\in C([0,T];L^2_k);\ \|h(t)\|_{L^1}\le\|f_0\|_{L^1}\}$$

and the linear time depending problem

$$\partial_t f = \mathcal{L}_g f := \Delta f + \operatorname{div}(a_g f), \quad f(0)=f_0.$$

It is worth emphasizing that $a_f\in L^\infty((0,T)\times\mathbb{R}^d)$. We apply J.-L. Lions Theorem 4.2 which implies that there exists a unique variational solution $f\in X_T$, and more precisely

$$(f(t),\varphi(t))_H = (f_0,\varphi(0))_H + \int_0^t \big\{\langle \mathcal{L}_{g(s)} f(s),\varphi(s)\rangle + \langle \varphi'(s), f(s)\rangle\big\}\,ds,$$

for any $\varphi\in X_T$ and any $0\le t\le T$. Choosing $\varphi := \chi_M\,\langle x\rangle^{-2k}$ as a test function in the above variational formulation, with $\chi_M(x) := \chi(x/M)$, $\chi\in\mathcal{D}(\mathbb{R}^d)$, $\mathbf{1}_{B(0,1)}\le \chi\le \mathbf{1}_{B(0,2)}$, we deduce

$$\int f(t)\chi_M = \int f_0\chi_M - \int_{\mathbb{R}^d}(\nabla f + a_g f)\cdot\nabla\chi_M\,dx.$$

Using that $f(t),f_0\in L^2_k\subset L^1$, $0\le\chi_M\nearrow 1$, $f,\nabla f\in L^2(0,T;L^2_k)\subset L^1_{tx}$ and $\|\nabla\chi_M\|_{L^\infty}\to 0$, we may pass to the limit $M\to\infty$, and we (rigorously) obtain the same mass conservation (4.20) for the solution to this linear equation. Because $f_0\ge 0$, we have $f(t)\ge 0$, and thus $f\in\mathscr{C}$.

*Step 3.* From the previous step, we have built a mapping $\mathscr{C} \to \mathscr{C}$, $g \mapsto f$. For $g_1, g_2 \in \mathscr{C}$, we consider the associated solutions $f_1, f_2 \in \mathscr{C} \cap X_T$ and we define $f := f_2 - f_1$, $g := g_2 - g_1$. We observe that

$$\partial_t f = \Delta f + \operatorname{div}(a_{g_1} f) + \operatorname{div}(a_g f_2), \quad f(0) = 0.$$

Adapting the $L^2_k$ estimate established in Step 1, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int f^2 \langle x\rangle^{2k} &= -\int |\nabla f|^2 \langle x\rangle^{2k} + \frac{1}{2}\int f^2 \Delta\langle x\rangle^{2k} - \int f(a_{g_1}\cdot\nabla f)\langle x\rangle^{2k} \\ &\quad -\int f^2 a_{g_1}\cdot\nabla\langle x\rangle^{2k} - \int f_2 a_g\cdot(\nabla f\langle x\rangle^{2k} + f\nabla\langle x\rangle^{2k}) \\ &\le \frac{1}{2}\int f^2\Delta\langle x\rangle^{2k} + \frac{1}{2}\int f^2 |a_{g_1}|^2\langle x\rangle^{2k} - \int f^2 a_{g_1}\cdot\nabla\langle x\rangle^{2k} \\ &\quad + \frac{1}{2}\int f_2^2 |a_g|^2\langle x\rangle^{2k} + \frac{1}{2}\int (f_2^2|a_g|^2 + f^2|\nabla\langle x\rangle^{2k}|), \end{aligned}$$

where we have used three times the Young inequality. Using (4.21) and the fact that $g_1 \in \mathscr{C}$, we deduce

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int f^2\langle x\rangle^{2k} &\lesssim (1 + \|a\|^2_{L^\infty}\|g_1\|^2_{L^1})\int f^2\langle x\rangle^{2k} + \|a\|^2_{L^\infty}\|g\|^2_{L^1}\int f_2^2\langle x\rangle^{2k} \\ &\lesssim (1 + \|a\|^2_{L^\infty}\|f_0\|^2_{L^1})\int f^2\langle x\rangle^{2k} + \|a\|^2_{L^\infty}\|g\|^2_{L^1} e^{C_0 t}\|f_0\|^2_{L^2_k} \\ &= \frac{C_1}{2}\int f^2\langle x\rangle^{2k} + \|g\|^2_{L^1}\frac{C_2}{2}e^{C_0 t}, \end{aligned}$$

with $C_i := C_i(k, \|a\|_{L^\infty}, \|f_0\|_{L^1})$. Thanks to the Gronwall lemma, we finally obtain

$$\begin{aligned} \sup_{[0,T]}\|f\|^2_{L^2_k} &\le \int_0^T \|g\|^2_{L^1} C_2 e^{C_0 s + C_1(t-s)}\,ds \\ &\le C_2 e^{(C_0+C_1)T}\,T\sup_{[0,T]}\|g\|^2_{L^1}, \end{aligned}$$

and, because $L^2_k \subset L^1$,

$$\sup_{[0,T]}\|f\|^2_{L^2_k} \le \frac{1}{2}\sup_{[0,T]}\|g\|^2_{L^2_k},$$

for $T > 0$ small enough. The Banach-Picard contraction mapping theorem tells us that there exists a unique $f \in \mathscr{C} \cap X_T$ variational solution to the nonlinear McKean-Vlasov equation. Iterating the above process we get a unique global solution. □

4.6. **Aubin-Lions Lemma and application.** We present first a simple but typical version of Aubin-Lions Lemma.

**Lemma 4.5.** *Consider a sequence* $(f_n)$ *of functions satisfying* $f_n \in C([0,T];L^2\mathbb{R}^d))$ *and*

$$\partial_t f_n - \Delta f_n = F_n + \mathrm{div}(G_n) \quad \textit{in} \quad \mathcal{D}'((0,T)\times\mathbb{R}^d),$$

*with*

$$(f_n) \textit{ is bounded in } Y_T := L^\infty(0,T;L^2_\eta(\mathbb{R}^d)) \cap L^2(0,T;H^1(\mathbb{R}^d)),\ \eta > 0;$$
$$(F_n),\ (G_n) \textit{ are bounded in } L^2((0,T)\times B_R),\ \forall\, R > 0.$$

*Then, there exists* $f \in Y_T$ *and a subsequence* $(f_{n_k})$ *such that*

$$f_{n_k} \to f \textit{ strongly in } L^2((0,T)\times\mathbb{R}^d)) \textit{ and weakly in } Y_T.$$

*Proof of Lemma 4.5.* *Step 1.* We introduce a sequence of mollifiers $(\rho_\varepsilon)$, that is $\rho_\varepsilon(x) := \varepsilon^{-d}\rho(\varepsilon^{-1}x)$ with $0 \le \rho \in \mathcal{D}(\mathbb{R}^2)$, $\langle\rho\rangle = 1$. We observe that

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^d} f_n(t,y)\,\rho_\varepsilon(x-y)\,dx = \int_{\mathbb{R}^d}(f_n\,\Delta\rho_\varepsilon + F_n\rho_\varepsilon - G_n\cdot\nabla\rho_\varepsilon)\,dy,$$

where the RHS term is bounded in $L^2((0,T)\times B_R)$ uniformly in $n$ for any fixed $\varepsilon > 0$. We also clearly have

$$\nabla_x \int_{\mathbb{R}^d} f_n(t,y)\,\rho_\varepsilon(x-y)\,dx = -\int_{\mathbb{R}^d} f_n \nabla_y \rho_\varepsilon(x-y)\,dy,$$

where again the RHS term is bounded in $L^2((0,T)\times B_R)$ uniformly in $n$ for any fixed $\varepsilon > 0$. In other words, $f_n * \rho_\varepsilon$ is bounded in $H^1((0,T)\times B_R)$. We finally observe that

$$\begin{aligned}
&\sup_{[0,T]}\int (f_{n_k} * \rho_\varepsilon)^2 \langle x\rangle^\eta dx \\
&\qquad \le \sup_{[0,T]}\iint (f_{n_k}(x))^2 \rho_\varepsilon(y)\langle x-y\rangle^\eta dxdy \\
&\qquad \le \sup_n \sup_{[0,T]}\int (f_n(x))^2\langle x\rangle^\eta dx \int \rho(y)\langle y\rangle^\eta dy < \infty.
\end{aligned}$$

Thanks to the Rellich-Kondrachov Theorem, we get that (up to the extraction of a subsequence) $(f_n * \rho_\varepsilon)_n$ is strongly convergent in $L^2((0,T)\times\mathbb{R}^d)$. Thanks to boundedness assumption on $(f_n)$, we may extract a second subsequence $(f_{n_k})$ such that $f_{n_k} \rightharpoonup f$ weakly in $Y_T$ for some $f \in Y_T$. We then have $f_{n_k} * \rho_\varepsilon \to f * \rho_\varepsilon$ weakly in $Y_T$. Coming back to the previous strong compactness result, we also have

$$f_{n_k} * \rho_\varepsilon \to f * \rho_\varepsilon \text{ strongly in } L^2((0,T)\times\mathbb{R}^d) \text{ as } k \to \infty.$$

*Step 2.* Now, we observe that

$$\begin{aligned}
\int_{(0,T)\times\mathbb{R}^d} |g - g*\rho_\varepsilon|^2\,dxdt &= \int_{(0,T)\times\mathbb{R}^2}\Big|\int_{\mathbb{R}^d}(g(t,x)-g(t,x-y))\rho_\varepsilon(y)\,dy\Big|\,dxdt \\
&= \int_{(0,T)\times\mathbb{R}^d}\Big|\int_{\mathbb{R}^d}\int_0^1 \nabla_x g(t,z_s)\cdot y\rho_\varepsilon(y)\,dsdy\Big|^2\,dxdt,
\end{aligned}$$

with $z_s := x + sy$ thanks to a Taylor expansion. As a consequence, we have

$$\begin{aligned}\int_{(0,T)\times\mathbb{R}^d} |g - g * \rho_\varepsilon|^2 \, dxdt &\leq \int_{(0,T)\times\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^1 |\nabla_x g(t, z_s)|^2 |y|^2 \rho_\varepsilon(y) \, dsdydxdt \\ &\leq \varepsilon^2 \int_{(0,T)\times\mathbb{R}^d} |\nabla_x g(t,z)|^2 dtdz \int_{\mathbb{R}^d} |\zeta|^2 \rho(\zeta) \, d\zeta,\end{aligned}$$

where we have used the Jensen inequality and two changes of variables. We conclude that $f_{n_k} \to f$ in $L^2((0,T)\times\mathbb{R}^d)$ by writing

$$f_{n_k} - f = (f_{n_k} - f_{n_k} * \rho_\varepsilon) + (f_{n_k} * \rho - f * \rho) + (f * \rho_\varepsilon - f)$$

and using the previous convergence and estimates. □

We give now a typical example of application of the Aubin-Lions lemma that we illustrate on the McKean-Vlasov equation. It is worth emphasizing that more than the result in itself, it is the strategy which is interesting because a small variation around the same ideas allows one to establish the existence of Leray solution to the Navier-Stokes equation in any dimension $d \geq 2$.

Let us then consider $a_n \in L^\infty(\mathbb{R}^d)$ and $f_{n,0} \in L^2_k(\mathbb{R}^d)$, $k > d/2$ and the associated unique solution $f_n \in X_T$ to the McKean-Vlasov equation

$$\partial_t f_n = \Delta f_n + \mathrm{div}((a_n * f_n) f_n), \quad f_n(0) = f_{n,0}, \tag{4.22}$$

as built in Theorem 4.4.

**Proposition 4.6.** *Assume that $f_{n,0} \rightharpoonup f_0$ weakly in $L^2_k$ and $a_n \rightharpoonup a_n$ weakly in $L^\infty$. Then $f_n \to f$ in $L^2((0,T)\times\mathbb{R}^d)$ (for instance), where $f \in X_T$ is the unique solution to the McKean-Vlasov equation* (4.22) *associated to the interaction kernel $a$ and to the initial datum $f_0$.*

*Proof of Proposition 4.6.* Because of Theorem 4.4, we know that

$$\begin{aligned}&(f_n) \text{ is bounded in } Y_T := L^\infty(0,T; L^2_k(\mathbb{R}^d)) \cap L^2(0,T; H^1(\mathbb{R}^d)),\\ &((a_n * f_n) f_n) \text{ is bounded in } L^2((0,T)\times\mathbb{R}^d).\end{aligned}$$

We may thus apply the Aubin-Lions Lemma 4.5 and we deduce that there exists $f \in Y_T$ and a subsequence $(f_{n_k})$ such that

$$f_{n_k} \to f \text{ strongly in } L^2((0,T)\times\mathbb{R}^d)) \text{ and weakly in } Y_T.$$

For $\varphi \in C^1_c([0,T)\times\mathbb{R}^d)$, the weak formulation of (4.22) writes

$$\begin{aligned}&\int_0^T \int_{\mathbb{R}^d} f_{n_k} (\partial_t \varphi + \Delta\varphi) dxdt + \int_{\mathbb{R}^d} f_{0,n_k} \varphi(0,\cdot) dxdt \\ &\quad = \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f_{n_k}(t,x) f_{n_k}(t,x) a_{n_k}(x-y) \cdot \nabla\varphi(y) dxdydt.\end{aligned}$$

Using the weak convergence in the LHS term and a standard weak-strong convergence trick in order to deal with the RHS term, we immediately deduce that we may pass to the limit $k \to \infty$ in the above formulation and thus that $f$ is a weak solution to the McKean-Vlasov equation (4.22) associated to the interaction kernel $a$ and to the initial datum $f_0$. Because of Lemma 3.2 and Theorem 4.4, $f$ is in fact the unique variational solution and by uniqueness of the limit it is the whole sequence $(f_n)$ which converges toward $f$. □