CHAPTER 2: DE GIORGI-NASH-MOSER THEORY AND BEYOND FOR PARABOLIC EQUATIONS

Contents

I write in blue color what has been taught during the classes.

1. Introduction

Let us consider the parabolic equation

$$\frac{\partial f}{\partial t} = \mathrm{div}(A \nabla f) \quad \text{in } (0,\infty)\times\mathbb{R}^d, \qquad f(0,\cdot) = f_0 \quad \text{in } \mathbb{R}^d, \tag{1.1}$$

with a measurable, bounded and strictly elliptic matrix A, namely A satisfies (in the sense of quadratic forms) $\nu I \le A(x) \le \nu^{-1} I$ for any $x \in \mathbb{R}^d$ and for some $\nu > 0$. The heat equation corresponds to the case $A = \nu I > 0$. In this case and when $\nu = 1/2$, we know that

$$\gamma_t(x) := \frac{1}{(2\pi t)^{d/2}} \exp(-\frac{|x|^2}{2t}) \tag{1.2}$$

is the associated fundamental solution (that it is the unique solution f such that $f(t,\cdot) \rightharpoonup \delta_0$ as $t \to 0$) and for any $f_0 \in L^1(\mathbb{R}^d)$ the solution f to (1.1) satisfies $f \in C^\infty((0,\infty)\times\mathbb{R}^d)$. The main aim of this chapter is to recover part of these results using some techniques which are valid for a general matrix A. However, for the sake of simplicity, we will mainly consider the case $A = \nu I$, with $\nu = 1$ or $1/2$.

1.1. A first glance over the heat equation: a priori estimates.

The section is devoted to the heat equation

$$\frac{\partial f}{\partial t} = \Delta f \quad \text{in } (0,\infty)\times\mathbb{R}^d, \qquad f(0,\cdot) = f_0 \quad \text{in } \mathbb{R}^d. \tag{1.3}$$

We start with formally observing several qualitative properties of the solutions to the heat equation. On the one hand, we have

$$\frac{d}{dt}\int_{\mathbb{R}^d} f(t,x)\,dx = \int_{\mathbb{R}^d} \Delta f\,dx = 0,$$

so that the mass is conserved (by the flow of the heat equation)

$$\langle f(t,\cdot)\rangle := \int_{\mathbb{R}^d} f(t,x)\,dx = \int_{\mathbb{R}^d} f_0\,dx = \langle f_0\rangle, \quad \forall\, t \ge 0.$$

The dispersion/diffusion effect of the heat equation can be revealed through the decay of L^p norms. For instance, we have

$$\frac{d}{dt}\int_{\mathbb{R}^d} f^2\,dx = 2\int_{\mathbb{R}^d} f\Delta f\,dx = -2\int |\nabla f|^2 \le 0, \tag{1.4}$$

for any $t \ge 0$. The same computation gives

$$\frac{d}{dt}\int_{\mathbb{R}^d} f_+^2\,dx = 2\int_{\mathbb{R}^d} f_+\Delta f\,dx = -2\int |\nabla f_+|^2 \le 0,$$

so that

$$\int_{\mathbb{R}^d} (f_+(t,\cdot))^2\,dx = 0, \quad \forall\, t\ge 0, \quad \text{if} \quad \int_{\mathbb{R}^d} (f_{0+})^2\,dx = 0.$$

Equivalently, we have

$$f(t,\cdot) \ge 0, \ \ \forall\, t \ge 0, \quad \text{if} \quad f_0 \ge 0.$$

That means that the equation preservers the positivity, or in other words, the equation (or the associated operator) satisfies a *weak maximum principle*. Coming back to the dispersion/diffusion effect, and more generally than (1.4), for any convex function β, we similarly have

$$\frac{d}{dt}\int_{\mathbb{R}^d} \beta(f)\,dx = \int_{\mathbb{R}^d} \beta'(f)\Delta f\,dx = -\int_{\mathbb{R}^d} \beta''(f)|\nabla f|^2\,dx \le 0, \quad \forall\, t\ge 0,$$

and we thus obtain a large family of Lyapunov functional. In particular, the L^p-norm, for any $p \in [1,\infty]$, falls in this family, and thus

$$\|f(t,\cdot)\|_{L^p} \le \|f_0\|_{L^p}, \quad \forall\, t \ge 0. \tag{1.5}$$

Finally, for a positive solution, the dispersion/diffusion effect of the heat equation can also be brought out through the increasing of moments: we have indeed

$$\frac{d}{dt}\int_{\mathbb{R}^d} f(t,x)\langle x\rangle^k\,dx = \frac12\int_{\mathbb{R}^d} f\Delta\langle x\rangle^k\,dx \ge 0, \quad \forall\, t\ge 0,$$

for $k+d-2 \ge 0$ and $\langle x\rangle^2 := 1+|x|^2$ (since $\Delta\langle x\rangle^k = k\langle x\rangle^{k-4}[(k+d-2)|x|^2+d] \ge 0$). By differentiating the heat equation, we can easily establish some estimates on its smoothing effect. For example, for $f_0 \in H^1(\mathbb{R}^d)$, the associated solution to the heat equation satisfies

$$\partial_t f = \Delta f \quad \text{and} \quad \partial_t \nabla f = \Delta \nabla f$$

from what we deduce

$$\frac{d}{dt}\|f\|_{L^2}^2 = -2\|\nabla f\|_{L^2}^2 \quad \text{and} \quad \frac{d}{dt}\|\nabla f\|_{L^2}^2 = -2\|D^2 f\|_{L^2}^2$$

and then

$$\frac12\frac{d}{dt}\big\{\|f\|_{L^2}^2 + t\|\nabla f\|_{L^2}^2\big\} = -t\|D^2 f\|_{L^2}^2 \le 0, \quad \forall\, t > 0.$$

Integrating in time this differential inequality, we readily obtain that the solution to the heat equation satisfies

$$\|\nabla f(t)\|_{L^2} \le \frac{1}{t^{1/2}}\,\|f_0\|_{L^2}, \quad \forall\, t>0. \tag{1.6}$$

It is worth emphasizing that a similar result as this last estimate (1.6) is available for solutions to the general parabolic equation (1.1) when A is a smooth function, but certainly not in the case when A is only measurable.

1.2. **Heat semigroup representation and L^p estimates.** We consider now the heat equation

$$(1.7) \qquad \frac{\partial f}{\partial t} = \frac{1}{2}\Delta f \quad \text{in } (0,\infty)\times\mathbb{R}^d, \qquad f(0,\cdot) = f_0 \quad \text{in } \mathbb{R}^d.$$

In this very particular case, the solutions to the heat equation are given through the representation formula

$$(1.8) \qquad f(t,.) = \gamma_t * f_0,$$

where the gaussian kernel (γ_t) is defined in (1.2) and $* = *_x$ stands for the convolution operator. The 1/2 in (1.7) is just put in order to get this usual gaussian kernel γ_t (instead of a rescaled version of it). Let us observe that

$$\|\gamma_t\|_{L^p} = \frac{C_{p,d}}{t^{\frac{d}{2}(1-\frac{1}{p})}}, \quad C_{p,d} := \frac{1}{p^{\frac{d}{2p}}(2\pi)^{\frac{d}{2}(1-\frac{1}{p})}},$$

so that from the Young inequality on convolution products, we get the ultracontractivity estimate

$$(1.9) \qquad \|f(t,.)\|_{L^p} \le \frac{C_{r,d}}{t^{\frac{d}{2}(\frac{1}{q}-\frac{1}{p})}}\,\|f_0\|_{L^q},$$

for any $t>0$ and $p,q\in[1,\infty]$, $p\ge q$, where $r\in[1,\infty]$ is defined by the relation $1/p = 1/q+1/r-1$. In particular, choosing $p>q$, we see that $f(t,.)\to 0$ as $t\to\infty$, when $f_0\in L^q$, $q\in[1,\infty)$. Of course, the estimate (1.9) is much more precise and reveals some kind of smoothing (gain of local integrability) effect of the heat equation.

Exercise 1.1.
(1) Show that γ_t provides a fundamental solution to the heat equation (1.7) *and that $\gamma_{t+s} = \gamma_t * \gamma_s$ for any $t,s>0$.*
(2) Show that (1.8) *provides a solution to the heat equation* (1.7) *for any initial datum $f_0\in L^q$, $q\in[1,\infty]$.*
(3) Show that

$$(1.10) \qquad \|\nabla_x\gamma_t\|_{L^r} = \frac{C_{d,r}}{t^{\frac{d}{2}(1-\frac{1}{r})+\frac{1}{2}}}$$

and recover estimate (1.6)*.*
(4) We denote $\mathcal{U} := (0,T)\times\mathbb{R}^d$. For $g:\mathcal{U}\to\mathbb{R}$ (smooth and rapidly decaying) show that

$$(1.11) \qquad f := \gamma *_{t,x} g = \int_0^t \gamma_{t-s} *_x g(s,\cdot)ds$$

provides a solution to the heat equation with source term

$$\partial_t f - \frac{1}{2}\Delta f = g, \quad f(0) = 0.$$

(5) For $g\in L^1(\mathcal{U})$ establish that the solution f to the heat equation with source term given by (1.11) *satisfies $f\in L^p(\mathcal{U})$ for any $1<p<1+2/d$. More generally and more precisely, establish that*

$$\|f\|_{L^p(\mathcal{U})} \lesssim C T^{1-(1+\frac{d}{2})(\frac{1}{q}-\frac{1}{p}-1)}\|g\|_{L^q(\mathcal{U})}, \quad C := \frac{C_{r,d}}{(1-\frac{d}{2}(\frac{1}{q}-\frac{1}{p})r)^{1/r}},$$

under the condition $1 \le q < p$, $(1+\frac{d}{2})(\frac{1}{q}-\frac{1}{p}) < 1$ *and where* $C_{r,d}$ *and* r *are defined in* (1.9).

Taking advantage of (3) and (4) in Exercise 1.1, we establish a variant of (5) (which is also a hint for the proof of (5)). We consider the heat equation with source term

$$\partial_t f - \frac{1}{2}\Delta f = \mathrm{div}_x G, \quad f(0) = 0,$$

with $G \in L^q(\mathcal{U})$, $1 \le q < \infty$. From (1.11), we may write

$$f = \gamma *_{t,x} \mathrm{div}_x G = (\nabla_x \gamma) *_{t,x} G.$$

For $q < p < \infty$ and r defined by $\frac{1}{p} = \frac{1}{q} + \frac{1}{r} - 1$, we next compute

$$\begin{aligned}
\|f\|_{L^p(\mathcal{U})} &= \Big(\int_0^T \Big\| \int_0^t \nabla_x \gamma_{t-s} *_x G_s \, ds \Big\|^p_{L^p(\mathbb{R}^d)} dt \Big)^{1/p} \\
&\le \Big(\int_0^T \Big(\int_0^t \big\| \nabla \gamma_{t-s} *_x G_s \big\|_{L^p(\mathbb{R}^d)} ds \Big)^p dt \Big)^{1/p} \\
&\lesssim \Big\| \int_0^t \frac{1}{(t-s)^{\frac{d}{2}(\frac{1}{q}-\frac{1}{p})+\frac{1}{2}}} \|G_s\|_{L^q(\mathbb{R}^d)} \, ds \Big\|_{L^p(0,T)} \\
&\lesssim \Big\| \frac{1}{s^{\frac{d}{2}(\frac{1}{q}-\frac{1}{p})+\frac{1}{2}}} \Big\|_{L^r(0,T)} \|G\|_{L^q(\mathcal{U})} \\
&\lesssim T^{\frac{1}{2}-(1+\frac{d}{2})(\frac{1}{q}-\frac{1}{p})} \|G\|_{L^q(\mathcal{U})},
\end{aligned}$$

where we have used (1.10) in the third line, the Holder inequality in the fourth line and the condition $\frac{1}{2} > (1+\frac{d}{2})(\frac{1}{q}-\frac{1}{p})$ in the last line (which is true when $p - q > 0$ is small enough). In other words, if $G \in L^q(\mathcal{U})$, there exists $p \in (q, \infty)$ such that $f \in L^p(\mathcal{U})$ and the above estimate holds.

2. De Giorgi-Nash-Moser and ultracontractivity

2.1. **The Nash approach.** We start establishing (1.9) for $p = 2$ and $q = 1$. For that purpose, we first establish the following fundamental functional estimate.

Nash inequality. There exists a constant C_d such that for any $f \in L^1(\mathbb{R}^d) \cap H^1(\mathbb{R}^d)$, there holds

$$\|f\|_{L^2}^{1+2/d} \le C_d \, \|f\|_{L^1}^{2/d} \, \|\nabla f\|_{L^2}. \tag{2.1}$$

Proof of Nash inequality. We write for any $R > 0$

$$\begin{aligned}
\|f\|^2_{L^2} &= \|\hat f\|^2_{L^2} = \int_{|\xi| \le R} |\hat f|^2 + \int_{|\xi| \ge R} |\hat f|^2 \\
&\le c_d \, R^d \, \|\hat f\|^2_{L^\infty} + \frac{1}{R^2} \int_{|\xi| \ge R} |\xi|^2 \, |\hat f|^2 \\
&\le c_d \, R^d \, \|f\|^2_{L^1} + \frac{1}{R^2} \, \|\nabla f\|^2_{L^2},
\end{aligned}$$

and we take the optimal choice for R by setting $R := (\|\nabla f\|^2_{L^2}/c_d \|f\|^2_{L^1})^{\frac{1}{d+2}}$ so that the two terms at the RHS pf the last line are equal. □

Alternative proofs of the Nash inequality (2.1) are presented in **Exercise ??** and **Exercise ??**.

The cornerstone $L^1 - L^2$ estimate. We consider now a solution f to the heat equation (1.3) and we recall that

$$\frac{d}{dt}\int_{\mathbb{R}^d} f(t,x)^2\,dx = -2\int_{\mathbb{R}^d} |\nabla f|^2\,dx, \quad \forall\, t \ge 0, \tag{2.2}$$

and

$$\|f(t,\cdot)\|_{L^1} \le \|f_0\|_{L^1}, \quad \forall\, t \ge 0,$$

from (1.4) and (1.5) with $p = 1$. Putting together that two last equations and the Nash inequality, we obtain the following ordinary differential inequality

$$\frac{d}{dt}\int_{\mathbb{R}^d} f(t,x)^2\,dx \le -K\left(\int_{\mathbb{R}^d} f(t,x)^2\,dx\right)^{\frac{d+2}{d}}, \quad K = C_d\,\|f_0\|_{L^1}^{-4/d}.$$

We finally observe that for any solution u of the ordinary differential inequality

$$u' \le -K\,u^{1+\alpha}, \quad \alpha = 2/d > 0,$$

some elementary computations (as already performed in the first chapter about Gronwall lemma) lead to the inequality

$$u^{-\alpha}(t) \ge \alpha\,K\,t + u_0^\alpha \ge \alpha\,K\,t,$$

from which we conclude that

$$\int_{\mathbb{R}^d} f^2(t,x)\,dx \le C\,\frac{\left(\|f_0\|_{L^1}^{4/d}\right)^{d/2}}{t^{d/2}} = C\,\frac{\|f_0\|_{L^1}^2}{t^{d/2}}. \tag{2.3}$$

That is nothing but the announced estimate (1.9) for $p = 2$ and $q = 1$.

Extension to $L^q - L^p$ estimates. In order to prove the estimate for the full range of exponents, we use a duality and an interpolation argument as follow. We introduce the heat semigroup $S(f)f_0 = f(t)$ associated to the heat equation as well as the dual semigroup $S^*(t)$. We clearly have $S^* = S$ because the Laplacian opeartor is symetric in $L^2(\mathbb{R}^d)$. As a consequence, thanks to (2.3) and for any $f_0 \in L^2(\mathbb{R}^d)$, there holds

$$\begin{aligned}\|S(t)f_0\|_{L^\infty} &= \sup_{\phi\in B_{L^1}} \langle S(t)f_0,\phi\rangle = sup_{\phi\in B_{L^1}}\langle f_0, S(t)\phi\rangle \\ &\le \sup_{\phi\in B_{L^1}} \|f_0\|_{L^2}\,\|S(t)\phi\|_{L^2} \le \|f_0\|_{L^2}\,\frac{C}{t^{d/4}},\end{aligned}$$

which exacly means that $S(t) : L^2 \to L^\infty$ for positive times with norm bounded by $C\,t^{-d/4}$. We deduce

$$\|S(t)\|_{L^1\to L^\infty} \le \|S(t/2)\|_{L^2\to L^\infty}\,\|S(t/2)\|_{L^1\to L^2} \le \frac{C}{t^{d/2}},$$

which establishes (1.9) for $p = \infty$ and $q = 1$. Finally, for any $p \in (1,\infty)$ and using the interpolation inequality

$$\|S(t)f_0\|_{L^p} \le \|S(t)f_0\|_{L^1}^\theta\,\|S(t)f_0\|_{L^\infty}^{1-\theta} \le \|S(t)\|_{L^1\to L^\infty}^{1-\theta}\,\|f_0\|_{L^1} \quad \forall\, t > 0,$$

with $\theta = 1/p$, and that is nothing but (1.9) in the general case.

2.2. **An alternative De Giorgi-Nash proof.** Anticipating with the most classical De Giorgi-Moser approach that we will develop in the next sections, we present here an alternative proof which mixes some arguments coming from Nash argument (the contraction estimate in any L^r spaces) and others coming from the De Giorgi-Moser approach (the use of the Sobolev inequality). We assume here $d \ge 3$ (in order to be able to use the Sobolev inequality).

Multiplying the equation (2.2) by φ^2, with $0 \le \varphi \in C^1([0,T])$, $\varphi(0) = \varphi(T) = 0$, and integration in time, we find

$$2\int_0^T \varphi^2 \int |\nabla f|^2 = \int_0^T (\varphi^2)' \int f^2.$$

Using the Sobolev inequality, we deduce

$$\|\varphi f\|^2_{L^2(0,T;L^{2^*})} \lesssim \int_0^T \varphi(\varphi')_+ \|f\|^2_{L^2} dt \tag{2.4}$$

Directly from (2.4), choosing $\varphi(t) := \varphi_0(t/T)$ and using the decay estimate on the L^p norms (1.5) with $p = 2$ and $p = 2^*$, we get

$$\begin{aligned} T\|\varphi_0\|^2_{L^2(0,1)}\|f_T\|^2_{L^{2^*}} &= \int_0^T \varphi_0(t/T)^2 dt \|f_T\|^2_{L^{2^*}} \\ &\le \int_0^T \varphi(t)^2 \|f_t\|^2_{L^{2^*}} dt \\ &\lesssim \int_0^T \varphi(\varphi')_+ \|f_t\|^2_{L^2} dt \\ &\lesssim \int_0^T \varphi(\varphi')_+ dt \|f_0\|^2_{L^2}. \end{aligned}$$

Observing that

$$\int_0^T \varphi(\varphi')_+ dt = \int_0^T \varphi_0(t/T)(\varphi_0'(t/T)/T)_+ dt = \int_0^1 \varphi_0(s)(\varphi_0')_+(s) ds,$$

we deduce

$$T\|f_T\|^2_{L^{2^*}} \le C_{\varphi_0}\|f_0\|^2_{L^2},$$

what is exactly the decay estimate (1.9) with $p = 2^*$ and $q = 2$. We cannot deduce from that last estimate together with duality and interpolation arguments the whole range of estimates (1.9) with $1 \le q < p \le \infty$. For that reason, we sligthly modify the above proof in the following way.

For the RHS term in (2.4), we write

$$\begin{aligned} \int_0^T \|f\|^2_{L^2}\varphi\varphi'_+ dt &\le \int_0^T \|f\|^{2(1-\theta)}_{L^1}\varphi'_+\varphi^{1-2\theta}\|f\|^{2\theta}_{L^{2^*}}\varphi^{2\theta} dt \\ &\le \Big(\int_0^T \|f\|^2_{L^1}(\varphi'_+\varphi^{1-2\theta})^{\frac{1}{1-\theta}} dt\Big)^{1-\theta}\Big(\int_0^T \|f\|^2_{L^{2^*}}\varphi^2 dt\Big)^{\theta}, \end{aligned}$$

where we have used the interpolation inequality with $1/2 = 1-\theta+\theta/2^*$ in the first line and the Holder inequality in the second line. Coming back to (2.4) and simplifying both sides of the inequality, we obtain

$$\|\varphi f\|^2_{L^2(0,T;L^{2^*})} \lesssim \int_0^T \|f\|^2_{L^1}(\varphi'_+\varphi^{1-2\theta})^{\frac{1}{1-\theta}} dt.$$

Repeating the same argument as for the last pointwise estimate, we get

$$T\|\varphi_0\|^2_{L^2(0,1)}\|f_T\|^2_{L^{2^*}} \le K_\varphi(T)\|f_0\|^2_{L^1},$$

with

$$K_\varphi(T) = \int_0^T (\varphi'_+ \varphi^{1-2\theta})^{\frac{1}{1-\theta}}\,dt.$$

We compute $\theta = d/(d+2)$ and then

$$\begin{aligned} K_\varphi(T) &= \int_0^T (\varphi'_+ \varphi^{1-2\theta})^{\frac{1}{1-\theta}}\,dt. \\ &= \int_0^T (\varphi'_+)^{\frac{d+2}{2}} \varphi^{\frac{2-d}{d}}\,dt. \\ &= T^{-d/2}\int_0^1 (\varphi'_0)_+^{\frac{d+2}{2}} \varphi_0^{\frac{2-d}{d}}\,dt. \end{aligned}$$

The last term is finite when $\varphi_0(s) = s^a(1-s)^a$, with $s^{(a-1)\frac{d+2}{2}+a\frac{2-d}{d}} \in L^1(0,1)$ what is the case when $a > (2/d)/(d/2+2/d)$. All together, we have established

$$T\|f_T\|^2_{L^{2^*}} \lesssim T^{-d/2}\|f_0\|^2_{L^1}.$$

That is again the decay estimate (1.9), but now with $p = 2^*$ and $q = 1$. Taking advantage of that last estimate, we are able to obtain (1.9) for the the full range of exponents $1 \le q < p \le \infty$ by proceeding exactly as in Section 2.1.

2.3. **The De Giorgi fundamental functional estimate.** Let us consider again a solution f to the heat equation (1.3). We integrate in time equation (1.4) in order to get

$$\frac12\int f_t^2 + \int_s^t\int |\nabla f|^2 = \frac12\int f_s^2,$$

for any $0 < s < t$. We fix $0 < t_0 < t_1 < t < T$ and we integrate in $s \in (t_0, t_1)$ the above equation. We obtain

$$(t_1 - t_0)\int f_t^2 + 2(t_1-t_0)\int_{t_1}^t\int |\nabla f|^2 \le \int_{t_0}^T\int f^2.$$

Taking the supremum in $t \in (t_1, T)$ of both terms at the RHS, we deduce

$$\sup_{[t_1,T]}\int f_t^2 + 2\int_{t_1}^T\int |\nabla f|^2 \le \frac{2}{t_1-t_0}\int_{t_0}^T\int f^2,$$

for any $0 \le t_0 < t_1 \le T$. Using the Sobolev inequality

$$\|f\|_{L^{2^*}} \le C_S\|\nabla f\|_{L^2}, \quad \frac{1}{2^*} = \frac12 - \frac1d,$$

we have (in fact) proved

$$\begin{aligned} \|f\|^2_{L^\infty(I_1;L^2)} &\le \frac{1}{t_1-t_0}\|f\|^2_{L^2(I_0;L^2)}, \\ \|f\|^2_{L^2(I_1;L^{2^*})} &\le \frac{C_S^2}{2}\frac{1}{t_1-t_0}\|f\|^2_{L^2(I_0;L^2)}, \end{aligned}$$

where $I_i := [t_i, T]$. We now recall the interpolation inequality

$$\|\Lambda\|_{X;L^{q_\theta}L^{r_\theta}} \le \|\Lambda\|^\theta_{X;L^{q_0}L^{r_0}}\|\Lambda\|^{1-\theta}_{X;L^{q_1}L^{r_1}},$$

for a linear and bounded operator $\Lambda : X \to L^{q_i} L^{r_i}$, $i = 0, 1$, where

$$\frac{1}{q_\theta} = \frac{\theta}{q_0} + \frac{1-\theta}{q_1}, \quad \frac{1}{r_\theta} = \frac{\theta}{r_0} + \frac{1-\theta}{r_1}, \quad \theta \in [0,1].$$

Using this interpolation inequality with θ such that

$$\frac{1}{p} := \frac{1-\theta}{2} = \frac{\theta}{2} + \frac{1-\theta}{2^*},$$

we deduce

$$\|f\|^2_{L^p(I_1;L^p)} \le \frac{C}{t_1 - t_0} \|f\|^2_{L^2(I_0;L^2)}, \quad p := 2(1 + 2/d). \tag{2.5}$$

In order to avoid the use of the above interpolation inequalities we may argue in a slightly different manner. We present the argument in the following exercise on a more general parabolic equation in such a way that it will be clear that we use more than just the arguments of Section 1.2.

Exercise 2.1. *Let us consider a solution f to the parabolic equation* (1.1).

(1) Repeating the first above argument, establish that

$$\nu \int_0^T \varphi^2 \int |\nabla f|^2 \le \int_0^T \varphi'_+ \varphi \int f^2,$$

for any $0 \le \varphi \in \mathcal{D}(0,T)$. (Hint. See also Section 2.5)

(2) Establish that $f\varphi$ satisfies

$$\partial_t (f\varphi) - \Delta (f\varphi) = \operatorname{div}((A - I)\nabla f \varphi) + f\varphi'.$$

Using Exercice 1.1 and the estimate which follows, prove that

$$\|f\varphi\|_{L^p} \le C \|(A-I)\nabla f \varphi\|_{L^2} + C\|f\varphi'\|_{L^2},$$

for some exponent $p = p(d) > 2$ and a constant $C = C(d) > 0$.

(3) Recover (2.5) *by combining (1) and (2).*

2.4. **Moser iterative argument.** We first observe the general and fundamental fact: if $\beta : \mathbb{R} \to \mathbb{R}$ is a function and f is a solution to the heat equation, there holds

$$\partial_t \beta(f) - \Delta \beta(f) = -\beta''(f) |\nabla f|^2.$$

In particular, if β is convex, $g := \beta(f)$ is a subsolution to the heat equation in the sense that it satisfies

$$\partial_t g - \Delta g \le 0. \tag{2.6}$$

For a subsolution $g \ge 0$, we may repeat the arguments of Section 2.3, and we we get in the same manner

$$\|g\|^2_{L^p(\mathcal{U}_{k+1})} \le C \frac{1}{t_{k+1} - t_k} \|g\|^2_{L^2(\mathcal{U}_k)}, \tag{2.7}$$

with $\mathcal{U}_k := I_k \times \mathbb{R}^d$, $I_k := (t_k, T]$ and $0 \le t_k < t_{k+1} < T$.

We consider a solution $f \ge 0$ to the heat equation and we define

$$t_k := \frac{T}{2} - \frac{T}{2^k}, \quad k \ge 1, \quad p_{k+1} := (1 + 2/d) p_k, \quad k \ge 1, \quad p_1 := 2.$$

Applying (2.7) to the subsolution $g := f^{p_k/2}$, we obtain

$$\begin{aligned}\|f\|_{L^{p_{k+1}}(\mathcal{U}_{k+1})} &= \|f^{p_k/2}\|^{2/p_k}_{L^p(\mathcal{U}_{k+1})} \\ &\le \big(C\frac{2^k}{T}\|f^{p_k/2}\|^2_{L^2(\mathcal{U}_k)}\big)^{1/p_k} = \big(C\frac{2^k}{T}\big)^{1/p_k}\|f\|_{L^{p_k}(\mathcal{U}_k)}.\end{aligned}$$

Observing that

$$\sum_{k=1}^\infty \frac{1}{p_k} = \frac{1}{2}\sum_{j=0}^\infty \frac{1}{(1+2/d)^j} = \frac{1}{2} + \frac{d}{4},$$

we deduce that

$$\prod_{k=1}^\infty \big(C\frac{2^k}{T}\big)^{1/p_k} \lesssim T^{-1/2-d/4},$$

and thus

$$\begin{aligned}\|f\|_{L^\infty(\mathcal{U}_\infty)} &\le \liminf_{k\to\infty}\|f\|_{L^{p_k}(\mathcal{U}_k)} \\ &\le \liminf_{k\to\infty}\prod_{j=1}^k\big(C\frac{2^j}{T}\big)^{1/p_j}\|f\|_{L^{p_1}(\mathcal{U}_1)} \\ &\lesssim T^{-1/2-d/4}\|f\|_{L^2(\mathcal{U}_1)}.\end{aligned}$$

Finally, together with the decay of the L^2 norm (1.4) which implies

$$\|f\|_{L^2(\mathcal{U}_1)} \le T^{1/2}\|f_0\|_{L^2},$$

we have thus established

$$\|f_T\|_{L^\infty} \lesssim \frac{1}{T^{d/4}}\|f_0\|_{L^2}. \tag{2.8}$$

Estimate (2.8) is the dual estimate of (2.3). We may thus end the proof of the full range estimate (1.9) by arguing by duality and interpolation exactly as in Section 2.1.

2.5. **De Giorgi argument.** We give another proof of (2.8) by mainly modifying the second step in the proof presented in Section 2.4. For $c \in \mathbb{R}$ fixed, chosing $\beta(s) := (s-c)_+$, the function $g := \beta(f)$ is a subsolution of the heat equation in the sense of (2.6). Multiplying the equation (2.6) by $g\varphi^2$ for $0 \le \varphi \in \mathcal{D}(\mathcal{U})$, $\mathcal{U} := (0,T)\times\mathbb{R}^d$, and integrating in the space and time variables, we obtain

$$\begin{aligned}&\frac{1}{2}\|(f(t)-c)_+\varphi\|^2_{L^2} - \frac{1}{2}\|(f(s)-c)_+\varphi\|^2_{L^2} \\ &\le -\int_s^t\int \nabla(f-c)_+\cdot\nabla((f-c)_+\varphi^2)\,dxd\tau \\ &= -\int_s^t \|\nabla(f-c)_+\varphi\|^2_{L^2}\,d\tau + \int_s^t \varphi\nabla(f-c)_+\cdot(f-c)_+\nabla\varphi\,dxd\tau,\end{aligned}$$

for any $0 \le s < t \le T$.

Choising now $0 < r < R$, $\varphi(x) = \phi(|x|)$, $\phi(0) = 1$ and $\phi' = -(R-r)^{-1}\mathbf{1}_{[r,R]}$ on $\mathbb{R}_+$, we deduce

$$\int_{B_r} (f(t)-c)_+^2 dx + \int_s^t \int_{B_r} |\nabla (f-c)_+|^2 \, dx d\tau$$
$$\le \int_{B_R} (f(s)-c)_+^2 dx + \frac{1}{(R-r)^2} \int_s^t \int_{B_R} |(f-c)_+|^2 \, dx d\tau.$$

We define the parabolic cylinder $Q_r := (T-r^2, T) \times B_r$ for $r > 0$ small enough. We use the above inequality with $R = r_j := \frac{1}{2}(1+2^{-k})$ and $r := r_{k+1}$, so that

$$\int_{B_{r_{j+1}}} (f(t)-c)_+^2 dx + \int_s^t \int_{B_{r_{j+1}}} |\nabla (f-c)_+|^2 \, dx d\tau$$
$$\le \int_{B_{r_j}} (f(s)-c)_+^2 dx + 2^{2(j+2)} \int_s^t \int_{B_{r_j}} |(f-c)_+|^2 \, dx d\tau.$$

We define the parabolic cylinder $Q_r := (0-r^2, 0) \times B_r$ for any $r > 0$ such that $Q_r \subset (T_0, T_1) \times \Omega$. We shall use the above inequality with $R = r_j := \frac{1}{2}(1+2^{-k})$ and $r := r_{k+1}$, so that

Lemma 2.2. *If $(v_j)_{j\ge 0}$ satisfies $0 \le v_j \le C^j v_{j-1}^\alpha$ for any $j \ge 1$ and $v_0 < C^{-\frac{\alpha^2}{(\alpha-1)^2}}$ for some $C > 0$ and $\alpha > 1$, then $v_j \to 0$ as $j \to \infty$.*

Lemma 2.3. *There exists a constant $C = C(d)$ such that*

$$\|u_+\|_{L^\infty(Q_{1/2})} \le C\,(1 + \|u_+\|_{L^2(Q_1)}).$$

Proof. We define

$$V_k := \int_{Q_{r_k}} (u-c_k)^2 dx dt,$$

with $r_k := \frac{1}{2}(1+2^{-k})$ and $c_k := \frac{1}{2}(1-2^{-k})$. We set $p := 2^*/2$. Thanks to the Holder inequality, we have

$$V_k \le \int_{-r_k^2}^0 \Big(\int_{B_{r_k}} (u_t - c_k)_+^p dx \Big)^{2/p} \big| \{u_t - c_k \ge 0\} \cap B_{r_k} \big|^{1-2/p} dt$$

On the one hand, we observe that $\{u_t - c_k \ge 0\} = \{u_t - c_{k-1} \ge 2^{-k-1}\}$, so that

$$\big| \{u_t - c_k \ge 0\} \cap B_{r_k} \big|^{1-2/p} \le \Big(2^{2k+2} \int_{B_{r_k}} (u_t - c_{k-1})_+^2 dx \Big)^{1-2/p}$$
$$\le C^k \Big(\sup_{-r_k^2 < s < 0} \int_{B_{r_k}} (u_s - c_{k-1})_+^2 dx \Big)^{1-2/p}$$

2.6. **An alternative Moser's $L^1 - L^2$ estimate.** We want to prove the same kind of estimate but starting from less integrability condition. For a given function $\beta : \mathbb{R} \to \mathbb{R}$, we compute

$$\frac{1}{2}\partial_t \beta(f)^2 - \frac{1}{2}\Delta \beta(f)^2 - \big(1 + \frac{\beta(f)\beta''(f)}{(\beta'(f))^2}\big) |\nabla \beta(f)|^2 = \beta'(f)\beta(f)(\partial_t f - \Delta f) = 0,$$

from what, with the choice $\beta(s) = s^{p/2}$, $p \ne 0$, we deduce

$$\frac{1}{2}\partial_t f^p - \frac{1}{2}\Delta f^p - 2\frac{1-p}{p} |\nabla f^{p/2}|^2 = 0.$$

When $p > 1$, after multiplication by φ^2 for a function $\varphi \in \mathcal{D}((0,T])$ and integration in time, we find

$$4\frac{p-1}{p}\int_0^T \varphi^2 \int |\nabla(f^{p/2})|^2 + \varphi_T^2 \int f_T^p = \int_0^T 2\varphi\varphi' \int f^p$$

When $p < 1$, after multiplication by φ^2 for a function $\varphi \in \mathcal{D}([0,T))$ and integration, we find

$$4\frac{1-p}{p}\int_0^T \varphi^2 \int |\nabla f^{p/2}|^2 + \varphi_0^2 \int f_0^p = -\int_0^T 2\varphi\varphi' \int f^p \tag{2.9}$$

Starting from $f_0 \in L^1(\Omega)$, $\Omega \subset \mathbb{R}^d$, we choose $p \in (2/2^*, 1)$, and we observe that

$$\int_{\mathcal{U}} f^p \le \Big(\int_{\mathcal{U}} f\Big)^p |\mathcal{U}|^{1-p} \le \|f_0\|_{L^1}^p |\mathcal{U}|^{1-p}.$$

Using the dissipation estimate (2.9) for the L^p (quasi)norm and fixing $T_0 < T_1$, we have

$$\int_{\mathcal{U}_0} |\nabla f^{p/2}|^2 \lesssim \int_{\mathcal{U}_1} f^p \lesssim \|f_0\|_{L^1}^p.$$

Together with the Sobolev imbedding

$$\int_0^{T_0} \|f_t^p\|_{L^{2^*/2}} dt = \int_0^{T_0} \|f_t^{p/2}\|_{L^{2^*}}^2 \, dt \lesssim \int_0^{T_0} \|\nabla(f_t^{p/2})\|_{L^2}^2 dt,$$

we obtain

$$\|f_t^p\|_{L^1 L^{2^*/2}} \lesssim \|f_0\|_{L^1}^p.$$

On the other hand, we have

$$\|f^p\|_{L^\infty L^{1/p}} = \sup_{[0,T_0]} \|f_t\|_{L^1}^p \le \|f_0\|_{L^1}^p.$$

The two last estimate together and a interpolation inequality yield

$$\|f^p\|_{L^2 L^q} \lesssim \|f_0\|_{L^1}^p,$$

with

$$\frac{1}{q} = \frac{1}{2^*} + \frac{p}{2}.$$

When $p > 1 - 2/2^*$ so that $q \le 2$, we deduce

$$\|f\|_{L^{pq}}^p = \|f^p\|_{L^q} \lesssim \|f^p\|_{L^2 L^q} \lesssim \|f_0\|_{L^1}^p.$$

We finally observe that $r := pq > 1$ when $p > 1/(2^*2)$, so that we have established

$$\|f\|_{L^r(\mathcal{U}_1)} \lesssim \|f_0\|_{L^1},$$

for some $r > 1$. We may next use the previous step in order to improve the integrability estimate.

2.7. **Boccardo Gallouët argument.** Let us introduce a familly of new estimates which are the key new argument we use in this section. For a solution f to the heat equation (1.3), we recall that

$$\partial_t \beta(f) - \Delta \beta(f) + \beta''(f)|\nabla f|^2 = 0.$$

Choosing $\beta \in W^{2,\infty}$ the even (and convex) function such that $\beta(0) = \beta'(0) = 0$ and $\beta'' := \mathbf{1}_{[an,a(n+1)]}$, $a > 0$ to be fixed later, and integrating in all variables the previous equation, we deduce

$$\int \beta(f_t)dx + \int_{|f| \in [an,a(n+1)]} |\nabla f|^2 dxds = \int \beta(f_0)dx \le a \int |f_0|dx, \tag{2.10}$$

where we have used that $|\beta'(s)| \le a$ and $\beta(s) \le a|s|$. Using a variant of this new estimate and the classical L^1 non expansive estimate, we will establish the following result.

Proposition 2.4. *Assume $d \ge 3$ and define $r := 1 + 1/d$. For any solution f to the heat equation* (1.3)*, we (at least formally) have*

$$\|f\|_{L^r((T,2T)\times\mathbb{R}^d)} \lesssim T^{\frac{1}{2r}} \|f_0\|_{L^1(\mathbb{R}^d)}, \tag{2.11}$$

for any $T > 0$, and thus (1.9) *holds with $p = r$ and $q = 1$.*

Remark 2.5. *(1) Reciprocally, if f satisfies* (1.9) *with $p = 1 + 1/d$ and $q = 1$, or in other words,*

$$\|f_t\|_{L^{1+1/d}} \lesssim \frac{1}{t^{\frac{1}{2}\frac{d}{d+1}}} \|f_0\|_{L^1}, \quad \forall t > 0,$$

we deduce that

$$\int_0^T \|f_t\|^{1+1/d}_{L^{1+1/d}} dt \lesssim \int_0^T \frac{1}{t^{\frac{1}{2}}} dt \|f_0\|^{1+1/d}_{L^1} \lesssim T^{1/2} \|f_0\|^{1+1/d}_{L^1},$$

what is nothing but (2.11).
(2) Repeating the arguments presented in the previous sections, we easily deduce the ultracontractivity estimate (1.9) *for the full range of exponents $1 \le q \le p \le \infty$. We thus rather focus to the the proof of* (2.11) *in the sequel.*

We consider a solution f to the heat equation (1.3) with initial datum $f_0 \in L^1(\mathbb{R}^d)$. We restrict to the case $f_0 \ge 0$ so that $f \ge 0$. For $\varphi \in C^1_c((0,3T))$ such that $\mathbf{1}_{(T,2T)} \le \varphi \le 1$, we introduce the function $g := f\varphi$ which satisties $g \ge 0$ and the estimate

$$\|g\|_{L^\infty L^1} \le \|f\|_{L^\infty L^1} \le \|f_0\|_{L^1}, \tag{2.12}$$

because of the estimate (1.5) on f. It also satisfies

$$\partial_t \beta(g) - \Delta \beta(g) + \beta''(g)|\nabla g|^2 = f\varphi'\beta'(f\varphi) \quad \text{in} \quad \mathcal{U},$$

with $\beta(g)$ vanishing at times $t = 0$ and $t = 3T$. Integrating in all variables, we obtain

$$\int_{\mathcal{V}_n} |\nabla g|^2 = \int_{\mathcal{U}} f\varphi'\beta'(f\varphi) \le \|\varphi'\|_{L^\infty} a \int_{\mathcal{U}} |f|,$$

where $\mathcal{U} := (0,3T) \times \mathbb{R}^d$ and $\mathcal{V}_n := \{(t,x) \in \mathcal{U}; na \le g(t,x) \le (n+1)a\}$. Gathering that last estimate with (2.12), for any $n \ge 0$, we have

$$\int_{\mathcal{V}_n} |\nabla g|^2 \le \|\varphi'\|_{L^\infty} T a \|f_0\|_{L^1}. \tag{2.13}$$

We assume now $\|\varphi'\|_{L^\infty} \le 4/T$ what is compatible with the support condition. For $n \ge 1$, using the Holder inequality and next the Tchebychev inequality, we then deduce

$$\begin{aligned}\int_{\mathcal{V}_n} |\nabla g| &\le \Big(\int_{\mathcal{V}_n} |\nabla g|^2\Big)^{1/2} |\mathcal{V}_n|^{1/2} \\ &\le 2a^{1/2}\|f_0\|_{L^1}^{1/2} \frac{1}{(an)^{r/2}} \|g\mathbf{1}_{\mathcal{V}_n}\|_{L^r}^{r/2}.\end{aligned}$$

Summing up and using the Cauchy-Schwartz inequality, we have

$$\begin{aligned}\int_{g\ge a} |\nabla g| &\le 2a^{(1-r)/2}\|f_0\|_{L^1}^{1/2} \sum_{n\ge1} \frac{1}{n^{r/2}} \|g\mathbf{1}_{\mathcal{V}_n}\|_{L^r}^{r/2} \\ &\le 2a^{(1-r)/2}\|f_0\|_{L^1}^{1/2} \Big(\sum_{n\ge1} \frac{1}{n^r}\Big)^{1/2} \Big(\sum_{n\ge1} \|g\mathbf{1}_{\mathcal{V}_n}\|_{L^r}^{r}\Big)^{1/2},\end{aligned}$$

and we conclude with

$$\int_{g\ge a} |\nabla g| \lesssim a^{(1-r)/2}\|f_0\|_{L^1}^{1/2}\|g\|_{L^r}^{r/2}, \tag{2.14}$$

because $r > 1$.

On the other hand, the Holder inequalility yields

$$\|g\|_{L^r} \le \|g\|_{L^1}^{\theta}\|g\|_{L^{1^*}}^{1-\theta}, \quad \frac{1}{r} = \theta + \frac{1-\theta}{1^*},$$

with $\theta \in (0,1)$ because $r \in (1, 1^*)$, and thus

$$\|g\|_{L^r(\mathcal{U})}^r \le \|g\|_{L^\infty L^1}^{r\theta} \int_0^{3T} \|g_s\|_{L^{1^*}}^{r(1-\theta)} ds = \|g\|_{L^\infty L^1}^{r\theta}\|g\|_{L^1 L^{1^*}}^{r(1-\theta)},$$

because $r(1-\theta) = 1$. The two conditions on θ yield

$$\theta = \frac{1}{r}\big(1 - \frac{1}{1^*}\big) = \frac{1}{rd} = \frac{1}{d+1},$$

and thus together with (2.12), we get

$$\|g\|_{L^r(\mathcal{U})}^r \le \|f_0\|_{L^1}^{1/d}\|g\|_{L^1L^{1^*}}. \tag{2.15}$$

We next write $g = g \wedge a + (g-a)_+$ and then

$$\begin{aligned}\|g\|_{L^1L^{1^*}} &\le \|g\wedge a\|_{L^1L^{1^*}} + \|(g-a)_+\|_{L^1L^{1^*}} \\ &\lesssim a^{1-1/1^*}T\|g\|_{L^\infty L^1}^{1/1^*} + \|\nabla(g-a)_+\|_{L^1} \\ &\lesssim a^{1-1/1^*}T\|f_0\|_{L^1}^{1/1^*} + a^{(1-r)/2}\|f_0\|_{L^1}^{1/2}\|g\|_{L^r}^{r/2} \\ &\lesssim a^{1/d}T\|f_0\|_{L^1}^{1/1^*} + a^{-1/(2d)}\|f_0\|_{L^1}^{r/2}\|g\|_{L^1L^{1^*}}^{1/2},\end{aligned}$$

where we have used the Sobolev estimate at the second line, the estimates (2.12) and (2.14) at the third line and the interpolation estimate (2.15) at the fourth line. Thanks to the Young inequality

$$\frac{1}{a^{1/(2d)}}\|f_0\|_{L^1}^{r/2}\|g\|_{L^1L^{1^*}}^{1/2} \le \frac{1}{\varepsilon a^{1/d}}\|f_0\|_{L^1}^{r} + \varepsilon\|g\|_{L^1L^{1^*}},$$

we deduce

$$\|g\|_{L^1L^{1^*}} \lesssim a^{1/d}T\|f_0\|_{L^1}^{1/1^*} + \frac{1}{a^{1/d}}\|f_0\|_{L^1}^{1+1/d}.$$

Choosing $a := T^{-d/2}\|f_0\|_{L^1}$, we conclude with

$$\|g\|_{L^1L^{1*}} \lesssim T^{1/2}\|f_0\|_{L^1}.$$

Together with (2.15), we deduce that (2.11) holds. Using the already used trick

$$T^{1/r}\|f_T\|_{L^r} = \Big(\int_T^{2T}\|f_T\|_{L^r}ds\Big)^{1/r} \le \Big(\int_T^{2T}\|f_s\|_{L^r}ds\Big)^{1/r}$$

based on (1.5) for $p = r$ together with (2.11), we finally have

$$\|f_T\|_{L^r} \lesssim \frac{1}{T^{\frac{1}{2r}}}\|f_0\|_{L^1},$$

what is nothing but (1.9) for $p = r$ and $q = 1$.

Proposition 2.6. *Under the same asumptions, we also have*

$$\nabla f \in L^q_{\rm loc}(\mathcal{U}), \quad \forall\, q \in \big[1, \frac{d+2}{d+1}\big).$$

Proof of Proposition 2.6. We start observing that

$$\begin{aligned}\int_{\mathcal{V}_n}|\nabla f|^q &\le \Big(\int_{\mathcal{V}_n}|\nabla f|^2\Big)^{q/2}\Big(\int \mathbf{1}_{\mathcal{V}_n}\Big)^{1-q/2}\\ &\lesssim \Big(\int_{\mathcal{V}_n} f^r\Big)^{1-q/2}\frac{1}{n^{r(1-q/2)}}\end{aligned}$$

by using the Holder inequality in the first line and by using the estimate (2.13) and the Tchebychev inequality in the second line. We deduce

$$\begin{aligned}\int_{|f|\ge 1}|\nabla f|^q &= \sum_{n=1}^\infty\int_{\mathcal{V}_n}|\nabla f|^q\\ &\lesssim \sum_{n=1}^\infty\Big(\int_{\mathcal{V}_n} f^r\Big)^{1-q/2}\frac{1}{n^{r(1-q/2)}}\\ &\lesssim \Big(\sum_{n=1}^\infty\int_{\mathcal{V}_n} f^r\Big)^{1-q/2}\Big(\sum_{n=1}^\infty\frac{1}{n^{r(1-q/2)2/q}}\Big)^{q/2}\end{aligned}$$

by using the Holder inequality in the last line. From Proposition 2.4, we can take $r := 1 + 1/d$ so that the first sum is bounded by $\|f\|^{r(1-q/2)}_{L^r} < \infty$ and that leads to the condition $q < (2d+2)/(2d+1)$ in order that the last sum is finite. For any ball B_R, we have

$$\begin{aligned}\int_{B_R}|\nabla f|^q &= \int_{B_R}|\nabla f|^q\mathbf{1}_{|f|<1} + \int_{B_R}|\nabla f|^q\mathbf{1}_{|f|\ge 1}\\ &\lesssim \int_{B_R}|\nabla f|^2\mathbf{1}_{|f|\le 1} + \int|\nabla f|^q\mathbf{1}_{|f|\ge 1} < \infty\end{aligned}$$

from the very first estimate (2.13) and the above discussion. In fact, repeating the proof of Proposition 2.4, we may establish $f \in L^r$ for any $r \in [1, 1+2/d)$ which leads to the condition $q < (d+2)/(d+1)$. $\square$