CHAPTER 2: DE GIORGI-NASH-MOSER THEORY AND BEYOND FOR PARABOLIC EQUATIONS - PART 2

Contents

I write in red color the additional arguments with respect to the class room.

1. Introduction

We mainly consider the parabolic equation

$$\frac{\partial f}{\partial t} = \operatorname{div}(A\nabla f) \quad \text{in } (0,\infty)\times\mathbb{R}^d, \qquad f(0,\cdot) = f_0 \quad \text{in } \mathbb{R}^d, \tag{1.1}$$

with a measurable, bounded and strictly elliptic matrix A, namely A satisfies (in the sense of quadratic forms) $\nu I \le A(x) \le \nu^{-1} I$ for any $x \in \mathbb{R}^d$ and for some $\nu > 0$. The establish the ultracontractivity estimate

$$\|f(t,.)\|_{L^p} \le \frac{C_{r,d}}{t^{\frac{d}{2}(\frac{1}{q}-\frac{1}{p})}} \|f_0\|_{L^q}, \tag{1.2}$$

for any $t > 0$ and $p, q \in [1,\infty]$, $p \ge q$, where $r \in [1,\infty]$ is defined by the relation $1/p = 1/q + 1/r - 1$.

2. De Giorgi-Nash-Moser and ultracontractivity

We recall the interpolation inequality

$$\|g\|_{L^{q_\theta} L^{r_\theta}} \le \|g\|^{\theta}_{L^{q_0} L^{r_0}} \|g\|^{1-\theta}_{L^{q_1} L^{r_1}}, \tag{2.1}$$

where

$$\frac{1}{q_\theta} = \frac{\theta}{q_0} + \frac{1-\theta}{q_1}, \quad \frac{1}{r_\theta} = \frac{\theta}{r_0} + \frac{1-\theta}{r_1}, \quad \theta \in [0,1].$$

We observe the general and fundamental fact: if $\beta : \mathbb{R} \to \mathbb{R}$ is a function and f is a solution to the heat equation, there holds

$$\partial_t \beta(f) - \Delta \beta(f) = -\beta''(f)|\nabla f|^2.$$

In particular, if β is convex, $g := \beta(f)$ is a subsolution to the heat equation in the sense that it satisfies

$$\partial_t g - \Delta g \le 0. \tag{2.2}$$

3. De Giorgi Holder continuity argument

We give another proof of (a localized variant of) (1.2) by mainly modifying the second step in the proof presented in Section 2. For $c \in \mathbb{R}$ fixed, chosing $\beta(s) := (s-c)_+$, the function $g := \beta(f)$ is a subsolution of the heat equation in the sense of (2.2). Multiplying the equation (2.2) by $g\phi^2$ for $0 \le \phi \in \mathcal{D}(\mathcal{U})$, $\mathcal{U} := (0,T) \times \mathbb{R}^d$, and integrating in the space and time variables, we obtain

$$
\begin{aligned}
&\frac{1}{2}\|(f(t)-c)_+\phi\|^2_{L^2} - \frac{1}{2}\|(f(s)-c)_+\phi\|^2_{L^2} \\
&\quad \le -\int_s^t \int \nabla (f-c)_+ \cdot \nabla((f-c)_+\phi^2)\, dx d\tau \\
&\quad = -\int_s^t \|\nabla (f-c)_+\phi\|^2_{L^2}\, d\tau + \int_s^t \int \phi \nabla (f-c)_+ \cdot (f-c)_+ \nabla \phi \, dx d\tau,
\end{aligned}
$$

for any $0 \le s < t \le T$.

Choising now $0 < r < R$, $\phi(x) = \phi_0(|x|)$, $\phi_0(0) = 1$ and $\phi_0' = -(R-r)^{-1}\mathbf{1}_{[r,R]}$ on $\mathbb{R}_+$, we deduce

$$
\begin{aligned}
&\int_{B_r} (f(t)-c)_+^2 dx + \int_s^t \int_{B_r} |\nabla (f-c)_+|^2 \, dx d\tau \\
&\le \int_{B_R} (f(s)-c)_+^2 dx + \frac{1}{(R-r)^2} \int_s^t \int_{B_R} |(f-c)_+|^2 \, dx d\tau.
\end{aligned}
$$

Taking the mean value in $s \in (t_0, t_1)$ with $t_0 < t_1 < t$, we have

$$
\begin{aligned}
&(t_1 - t_0) \int_{B_r} (f(t)-c)_+^2 dx + (t_1 - t_0) \int_{t_1}^t \int_{B_r} |\nabla (f-c)_+|^2 \, dx ds \\
&\le \Big(1 + \frac{t-t_0}{(R-r)^2}\Big) \int_{t_0}^t \int_{B_R} |(f-c)_+|^2 \, dx ds.
\end{aligned}
$$

Using the Sobolev inequality and the interpolation inequality (2.1), we finally get

$$
\|(f-c)_+\|^2_{L^p((t_1,T)\times B_r)} \le C\Big(\frac{1}{t_1 - t_0} + \frac{1}{(R-r)^2}\Big) \|(f-c)_+\|^2_{L^2((t_0,T)\times B_R)}, \tag{3.3}
$$

with $p := 2(1 + 2/d)$

We shall use the following elementary result.

Lemma 3.1. *If $(v_j)_{j\ge 0}$ satisfies $0 \le v_j \le C^j v_{j-1}^\alpha$ for any $j \ge 1$ and $v_0 < C^{-\frac{\alpha}{(\alpha-1)^2}}$ for some $C > 0$ and $\alpha > 1$, then $v_j \to 0$ as $j \to \infty$.*

Proof of Lemma 3.1. We write recursively

$$
v_j \le C^{\psi(j)} v_0^{\alpha^j}, \quad \psi(j) := j + \alpha(j-1) + \dots + \alpha^{j-1}.
$$

We next observe that

$$
\psi(j) = \alpha^{j-1} \sum_{i=1}^{j} i \alpha^{-(i-1)} \le \alpha^{j-1} \Psi(\alpha^{-1}),
$$

with

$$
\Psi(x) := \sum_{i=0}^{\infty} i x^{i-1} = \frac{1}{(1-x)^2}, \quad \forall\, x \in (0,1).
$$

That last identity comes from the fact that $\Psi = \Phi'$ with

$$\Phi(x) := \sum_{i=0}^{\infty} x^i = \frac{1}{1-x}.$$

We have thus

$$v_j \le (\widetilde{C} v_0)^{\alpha^j}, \quad \widetilde{C} := C^{\frac{\alpha}{(\alpha-1)^2}},$$

from what we immeditaly conclude. □

We establish now some local gain of regularity for a solution f to the parabolic equation (1.1) in a region $(t_0, t_1) \times \Omega \subset \mathbb{R} \times \mathbb{R}^d$. For the sake of simplicity, we will rather consider the equation set in the region

$$Q_r := (-r, 0) \times B_r$$

and during the proof we will only consider a solution to the heat equation (**??**). The generalisation to general domain and general parabolic equation is not difficult by scaling/change of variables/mere translation and by repeating the arguments, it is left to the reader.

Lemma 3.2 (first De Giorgi lemma)**.** *Let f be a solution to the parabolic equation* (1.1) *in Q_1. There holds*

$$\|f_+\|_{L^\infty(Q_{1/2})} \le 1/2 \quad \textit{if} \quad \|f_+\|_{L^2(Q_1)} \le \delta,$$

for some constant $\delta > 0$ which only depends on the dimension $d \ge 3$.

Remark 3.3. *(1) An alternative formulation is that there exists a constant $C > 0$ such that any solution satisfies*

$$\|f_+\|_{L^\infty(Q_{1/2})} \le C \, \|f_+\|_{L^2(Q_1)}.$$

(2) In particular, we recover the same global ultracontractivity estimate $L^2 \to L^\infty$ as yet established in the previous sections. It is however worth emphasizing that in the present argument we only use $L^2 \to L^p$ bound and we do not use an interpolation with a global growth estimate in another Lebesgue space L^q.

Proof of Lemma 3.2. We define the sequence of (increasing) time and (decreasing) radius

$$T_k := -\tfrac{1}{2}(1 + 2^{-k}), \quad r_k := \tfrac{1}{2}(1 + 2^{-k}),$$

the sequence of (decreasing) cylinder and (increasing) truncation barrier

$$\mathcal{Q}_k := (T_k, 0) \times B(0, r_k), \quad c_k := \tfrac{1}{2}(1 - 2^{-k}),$$

and the sequence of energy

$$\mathcal{E}_k := \int_{\mathcal{Q}_{r_k}} f_k^2 \, dxdt, \quad f_k := (f - c_k)_+.$$

Using the estimate (2.1) with $c = c_k$, $T = 0$, $t_0 = T_k$, $t_1 = T_{k+1}$, $r = r_k$ and $R = r_{k+1}$, we get

$$\|f_k\|^2_{L^p(\mathcal{Q}_{k+1})} \le C\Big(2^{k+2} + 2^{2(k+2)}\Big) \|f_k\|^2_{L^2(\mathcal{Q}_k)}. \tag{3.4}$$

We next observe that

$$\{f_{k+1} > 0\} = \{f - c_{k+1} > 0\} = \{f - c_k > \frac{1}{2^{k+2}}\} = \{f_k > \frac{1}{2^{k+2}}\},$$

so that from Tchebychev inequality

$$|\{f_{k+1} > 0\} \cap \mathcal{Q}_{k+1}| \le 2^{2(k+2)} \int_{\mathcal{Q}_{k+1}} f_k^2 dxdt \le 2^{2(k+2)} \|f_k\|^2_{L^2(\mathcal{Q}_k)}. \tag{3.5}$$

Using the Holder inequality and the two estimates (3.4) and (3.5), we obtain

$$\begin{aligned} \mathcal{E}_{k+1} &\le \|f_{k+1}\|^2_{L^p(\mathcal{Q}_{k+1})} |\{f_{k+1} > 0\} \cap \mathcal{Q}_{k+1}|^{2/p'} \\ &\le C 2^{2k+5} \|f_k\|^2_{L^2(\mathcal{Q}_k)} \big(2^{2(k+2)} \|f_k\|^2_{L^2(\mathcal{Q}_k)}\big)^{2/p'}. \end{aligned}$$

Recalling that $p' = 1 + d/2$, we thus conclude with

$$\mathcal{E}_{k+1} \le M^k \mathcal{E}_k^{1+\frac{2}{d+2}}, \quad \forall\, k \ge 1,$$

for some constant $M > 1$. Choosing $\delta > 0$ small enough, we deduce from Lemma 3.1 that $\mathcal{E}_k \to \infty$ as $k \to \infty$ and in particular

$$\mathcal{E}_\infty := \int_{\mathcal{Q}_\infty} (f - c_\infty)^2_+ dxdt = 0,$$

with $\mathcal{Q}_\infty = Q_{1/2}$ and $c_\infty = 1/2$. That precisely means that $f \le 1/2$ on $Q_{1/2}$. □

We now drastically improve the above L^∞ estimate by establishing a Holder continuity result.

Theorem 3.4. *Let*

$$f \in L^\infty(t_0, T; L^2(\Omega)) \cap L^2(t_0, T; H^1(\Omega))$$

be a variational solution to the parabolic equation (1.1) *in* $(t_0, T) \times \Omega$. *There exists* $\alpha \in (0, 1)$ *such that for any* $\mathcal{O} \subset\subset \Omega$ *and any* $t_1 \in (t_0, T)$ *there holds*

$$f \in C^\alpha((t_1, T) \times \mathcal{O}).$$

The proof is split into several intermediate results. The first step is the already established first De Giorgi Lemma. The second argument is an intermediate value result. We start with stating the De Giorgi isoperimetric inequality which is a kind of quantitative version of the fact that a function in H^1 has no jump discontinuity.

Lemma 3.5 (De Giorgi isoperimetric inequality)**.** *Consider a function* g *on* B_1 *such that* $\|\nabla g_+\|^2_{L^2(B_1)} \le C_0$ *and denote*

$$A := \{g \le 0\} \cap B_1, \; C := \{g \ge \tfrac{1}{2}\} \cap B_1, \; D := \{0 < g < \tfrac{1}{2}\} \cap B_1.$$

Then the following inequality holds true

$$C_0 |D| \ge C_d (|C| |A|^{1-\frac{1}{d}})^2,$$

for a constant C_d *which only depends on* d.

Proof of Lemma 3.5. We set $h := (g \wedge 1/2)_+$ and observe that $\nabla h = \nabla g_+ \mathbf{1}_{0<g<1/2}$. For $x \in A$ and $y \in C$, we write

$$\begin{aligned} 1/2 &= h(y) - h(x) = \int_0^\infty \nabla h(x + t(y-x)) \cdot (y-x) dt \\ &\le \int_0^{|y-x|} |\nabla h|(x + s\sigma) ds, \quad \sigma_y := (y-x)/|y-x|. \end{aligned}$$

Integrating this inequality all over $y \in C$, we get

$$\begin{aligned} |C|/2 &\leq \int_C \int_0^{|y-x|} |\nabla h|(x + s\omega_y) ds dy \\ &\leq \int_{B_1} \int_0^{\infty} |\nabla h|(x + s\sigma_y) ds dy \\ &= \int_0^1 r^{d-1} \int_{S^{d-1}} \int_0^{\infty} |\nabla h|(x + s\sigma) ds d\sigma dr \\ &= c_d \int_{S^{d-1}} \int_0^{\infty} s^{d-1} |\nabla h|(x + s\sigma) \frac{1}{s^{d-1}} ds d\sigma, \\ &= c_d \int_{B_1} |\nabla h|(y) \frac{1}{|x-y|^{d-1}} dy, \end{aligned}$$

where we have extended the integration along the whole ray coming from x in the direction σ_y in the second line and we have used that in the last integration the function does not depend on r in the last line. Integrating in $x \in A$, we find

$$|A||C|/2 \leq c_d \int_{B_1} |\nabla h|(y) \Big(\int_A \frac{dx}{|x-y|^{d-1}} \Big) dy.$$

Among all A with same measure $|A|$ the integral in x is maximized by the ball of radius $|A|^{1/d}$ centered in y so that

$$\int_A \frac{dx}{|x-y|^{d-1}} \leq |A|^{1/d}.$$

Using that bound and the Cauchy-Schwarz inequality, we deduce

$$\begin{aligned} |A||C|/2 &\lesssim \|\nabla h\|_{L^2} |\{0 < h < 1/2\} \cap B_1|^{1/2} |A|^{1/d} \\ &\lesssim \|\nabla g_+\|_{L^2} |D|^{1/2} |A|^{1/d}, \end{aligned}$$

from what we immediately conclude. □

An analogous version of Lemma 3.5 for a solution to a parabolic equation can be formulated as follows. We denote

$$\widetilde{Q} := (-3/2, -1) \times B_1.$$

Lemma 3.6 (Parabolic intermediate value)**.** *Consider a solution f to the parabolic equation (1.1) in Q_2 such that $f \leq 1$ on Q_2 and denote*

$$A := \{f \geq 1/2\} \cap Q_1, \ C := \{f \leq 0\} \cap \widetilde{Q}, \ D := \{0 < f < 1/2\} \cap (Q_1 \cup \widetilde{Q}).$$

If $|A| \geq \delta_* > 0$ and $|C| \geq |\widetilde{Q}|/2$, there holds

$$|D| \geq \eta_*,$$

for some constant $\eta_* = \eta_*(d, \delta_*) > 0$ non constructive but independent of f.

Proof of Lemma 3.6. We assume by contradiction that the conclusion of the lemma is wrong. We can then find a sequence (f_k) such that with obvious notations

$$|A_k| \geq \delta_*, \quad |C_k| \geq |\widetilde{Q}|/2, \quad |D_k| \leq 1/k.$$

Multiplying the equation by $f_k \phi_1^2$, $\phi_1 \in \mathcal{D}(B_2)$, $\mathbf{1}_{B(0,r_1)} \leq \phi_1 \leq 1$, $r_1 \in (1,2)$, and after integrating, we have

$$\frac{1}{2} \int f_k(t_1)^2 \phi_1^2 + \int_{t_0}^{t_1} \int |\nabla (f_k \phi_1)|^2 = \frac{1}{2} \int f_k(t_0)^2 \phi_1^2 + \int_{t_0}^{t_1} \int f_k^2 |\nabla \phi_1|^2,$$

for any $t_0, t_1 \in I := (-2,0)$. For $\phi_2 \in \mathcal{D}(B_2)$, $B(0,1) \subset \operatorname{supp}\phi_2 \subset B(0,r_1)$, observing that $\nabla(f_k\phi_2) = \nabla(f_k\phi_1\phi_2) = \nabla(f_k\phi_1)\phi_2 + f_k\nabla\phi_2$, we thus have $(f_k\phi_2)$ is bounded in $L^2(I; H^1_0(B_2))$. Similarly, observing that

$$
\begin{aligned}
(\Delta f_k)\phi_2 &= \Delta(f_k\phi_2) - 2\nabla f_k \cdot \nabla\phi_2 - f_k\Delta\phi_2 \\
\Delta(f_k\phi_2) &= (\Delta(f_k\phi_1))\phi_2 + 2\nabla(f_k\phi_1)\cdot\nabla\phi_2 + f_k\phi_1\Delta\phi_2 \\
(\nabla f_k)\phi_1 &= \nabla(f_k\phi_1) - f_k\nabla\phi_1,
\end{aligned}
$$

we thus have

$$\partial_t(f_k\phi_2) = (\Delta(f_k\phi_1))\phi_2 + 2f_k\nabla\phi_1\cdot\nabla\phi_2$$

with second term bounded in $L^2(I; H^{-1}(B_2)$, and we may use the Aubin-Lions lemma in order to deduce that there exists satisfying $-1 \le f \le 1$ and $f_k\phi_2 \to f\phi_2$ (up to a subsequence) in $L^2(I\times B_2)$ and thus $f_k \to f$ in $L^2(\tilde Q \cup Q_1)$. In particular, using Tchebychev inequality, we have

$$\lim_{k\to\infty} |\{|f_k - f| \ge \varepsilon\} \cap I \times B_1| = 0, \quad \forall\, \varepsilon > 0.$$

• Now, we observe that if $\varepsilon \le f \le 1/2 - \varepsilon$ for some $\varepsilon > 0$, then either $0 < f_k < 1/2$ or $f_k \ge 1/2$ and thus $|f - f_k| \ge \varepsilon$, so that

$$
\begin{aligned}
&|\{\varepsilon \le f \le 1/2 - \varepsilon\} \cap (Q_1 \cup \widetilde{Q})| \le \\
&\qquad \le |\{|f - f_k| \ge \varepsilon\} \cap (Q_1 \cup \widetilde{Q})| + |\{0 < f_k < 1/2\} \cap (Q_1 \cup \widetilde{Q})| \\
&\qquad \le |\{|f - f_k| \ge \varepsilon\} \cap (Q_1 \cup \widetilde{Q})| + 1/k.
\end{aligned}
$$

That last sequence converges to 0 as $k \to \infty$, and thus

$$|\{\varepsilon \le f \le 1/2 - \varepsilon\} \cap (Q_1 \cup \widetilde{Q})| = 0, \quad \forall\, \varepsilon > 0,$$

so that also

$$|\{0 < f < 1/2\} \cap (Q_1 \cup \widetilde{Q})| = 0,$$

by passing to the limit $\varepsilon \to 0$. Because $f(t,\cdot) \in H^1(B_1)$ for a.e. $t \in (-2,0)$, the isoperimetric Lemma 3.5 tells us that either

$$f(t,\cdot) \le 0 \ \text{ in } \ B_1 \quad \text{or} \quad f(t,\cdot) \ge 1/2 \ \text{ in } \ B_1. \tag{3.6}$$

• Next, in the same way, if $f_k \le 0$, then either $f \le \varepsilon$ or $f > \varepsilon$ and thus $|f - f_k| > \varepsilon$. We deduce

$$|\widetilde{Q}|/2 \le |\{f_k \le 0\} \cap \widetilde{Q}| \le |\{|f - f_k| > \varepsilon\} \cap \widetilde{Q}| + |\{f \le \varepsilon\} \cap \widetilde{Q}|.$$

Passing first to the limit $k \to \infty$ and next to the limit $\varepsilon \to 0$, we find

$$|\widetilde{Q}|/2 \le |\{f \le 0\} \cap \widetilde{Q}|.$$

That in particular implies that there exists $t_0 \in (-3/2, -1)$ such that $f(t_0,\cdot) \le 0$ on some non negligible subset of B_1. But, because of (3.6), that means that $f(t_0,\cdot) \le 0$ on B_1. We then fix $0 \le \phi_1 \in \mathcal{D}(B_1)$ with L^2 norm equal to 1 and we write the energy estimate

$$
\begin{aligned}
\frac12 \int_{B_1} f_+^2(t)\phi_1^2 &= -\int_{t_0}^t\!\!\int_{B_1} |\nabla(f_+\phi_1)|^2 + \frac12\int_{B_1} f_+^2(t_0)\phi_1^2 + \int_{t_0}^t\!\!\int_{B_1} f_+^2|\nabla\phi_1|^2 \\
&\le C(t - t_0),
\end{aligned}
$$

for $t > t_0$. Because of this estimate, the second alternative in (3.6) is not achieved for $t - t_0$ small enough. That implies that $f_+(t) = 0$ for $t - t_0$ small enough, and thus by a continuation argument for any $t \in (t_0, 0)$. In particular, we have

$$f \le 0 \ \text{ in } \ Q_1. \tag{3.7}$$

• Last, if $f_k \ge 1/2$, then either $f \ge 1/2 - \varepsilon$ or $f < 1/2 - \varepsilon$ and thus $|f - f_k| > \varepsilon$. We deduce

$$\delta_* \le |\{f_k \ge 1/2\} \cap Q_1| \le |\{|f - f_k| > \varepsilon\} \cap Q_1| + |\{f \ge 1/2 - \varepsilon\} \cap Q_1|.$$

Passing first to the limit $k \to \infty$ and next to the limit $\varepsilon \to 0$, we find

$$\delta_* \le |\{f \ge 1/2\} \cap Q_1|.$$

That is in contradiction with (3.7). □

Gathering the first De Giorgi Lemma 3.2 and the intermediate value Lemma 3.6, we deduce the following oscillation or second De Giorgi Lemma.

Lemma 3.7 (Oscillation)**.** *There exists $0 < \lambda < 1$ such that for any solution g to the parabolic equation (1.1) in Q_2 such that $-1 \le g \le 1$ on Q_2 and $|\{g \le 0\} \cap \widetilde{Q}| \ge |\widetilde{Q}|/2$, we have*

$$g \le 1 - \lambda \ \text{ on } \ Q_{1/2}.$$

Proof of Lemma 3.7. From the very definition on a variational solution, we have $C_0 := \|\nabla_x g\|_{L^2(Q_1 \cup \widetilde{Q})} < \infty$. We define the sequence

$$g_k := 2^k [g - (1 - 2^{-k})].$$

We may observe that for any k, we also have $g_k \le 1$, $|\{g_k \le 0\} \cap \widetilde{Q}| \ge \mu := |\widetilde{Q}|/2$ and $\|\nabla_x g_{k+}\|_{L^2(Q_{7/4})} \le C_0$. For that last estimate, for some $\phi \in \mathcal{D}(B_2)$ such that $\phi \ge \mathbf{1}_{B_1}$, we may indeed compute

$$\frac{1}{2} \frac{d}{dt} \int g_{k+2}^2 \phi^2 = - \int |\nabla g_{k+}|^2 \phi^2 + \frac{1}{2} \int g_{k+}^2 \Delta \phi^2$$

and thus

$$\int_{Q_{7/4}} |\nabla g_{k+}|^2 \le \int_{Q_2} |\nabla g_{k+}|^2 \phi^2 \le C_\phi \sup_{[-2,0]} \int_{B_2} g_{k+}^2 \le C'_\phi.$$

We assume that for some $k_0 \ge 1$ and any $k \in \{1, \cdots, k_0\}$, we have

$$\int_{Q_1} (g_{k+1})_+^2 \, dxdt \ge \delta^2, \tag{3.8}$$

where we recall that $\delta > 0$ has been defined in the first De Giorgi Lemma 3.2. From the very definition of (g_k), we have

$$\{g_k \le 1/2\} = \{g_{k+1} \le 0\}. \tag{3.9}$$

We deduce that

$$|\{g_k \ge 1/2\} \cap Q_1| \ge |\{g_{k+1} \ge 0\} \cap Q_1| \ge \int_{Q_1} (g_{k+1})_+^2 \, dxdt \ge \delta^2,$$

where we have used the fact that $g_{k+1} \le 1$ in the second inequality. Applying Lemma 3.5, we know that there exists $\eta > 0$ independent of k such that

$$|\{0 < g_k < 1/2\} \cap (Q_1 \cup \widetilde{Q})| \ge \eta.$$

Using (3.9) again and repeatedly the above lower bound, we have

$$\begin{aligned} |Q_1 \cup \widetilde{Q}| &\geq |\{g_{k+1} \leq 0\} \cap (Q_1 \cup \widetilde{Q})| \\ &\geq |\{g_k \leq 0\} \cap (Q_1 \cup \widetilde{Q})| + |\{0 < g_k < 1/2\} \cap (Q_1 \cup \widetilde{Q})| \\ &\geq k\eta, \end{aligned}$$

which provide a finite bound on k_0. For the first $k = k_0$ such that (3.8) fails, we have $\|(g_{k_0+1})_+\|_{L^2(Q_1)} \leq \delta$, and thus $g_{k_0+1} \leq 1/2$ in $Q_{1/2}$ from the first De Giorgi Lemma 3.2. Rescaling back to g gives the result with $\lambda := 2^{-k_0-2}$. □

Proof of Theorem 3.4. Step 1. Assume first f defined in Q_2. We write

$$g := \frac{2}{\operatorname{osc}_{Q_2} f}\Big(f - \frac{\sup f + \inf f}{2}\Big),$$

so that $-1 \leq g \leq 1$ on Q_2. We have either

$$|\{g \leq 0\} \cap \widetilde{Q}| \geq |\widetilde{Q}|/2 \quad \text{or} \quad |\{g \geq 0\} \cap \widetilde{Q}| \geq |\widetilde{Q}|/2.$$

In the first case, we apply Lemma 3.7 to g and we deduce $g \leq 1 - \lambda$ on $Q_{1/2}$. In the second case, we apply Lemma 3.7 to $-g$ and we deduce $g \geq -1 + \lambda$ on $Q_{1/2}$. In both cases, we conclude with $\operatorname{osc}_{Q_{1/2}} g \leq 2 - \lambda$. Hence, we have

$$\operatorname{osc}_{Q_{1/2}} f \leq (1 - \lambda/2) \operatorname{osc}_{Q_2} f.$$

Step 2. We come to the general case and we assume f defined in $\mathcal{U}$. Take $y_0 \in \mathcal{U}$ and $d_0 := \min(d(y_0, \mathcal{U}^c), 1)$. We define

$$\widetilde{f}(y) := f(y_0 + \frac{d_0}{4} y_0) \quad \text{on } Q_2$$

and recursively

$$\widetilde{f}_1 = \widetilde{f}, \quad \widetilde{f}_k(y) = \widetilde{f}_{k-1}(y/4), \ k \geq 2.$$

Applying the first Step to $\widetilde{f}_k$ gives

$$\operatorname{osc}_{Q_{1/2}} \widetilde{f}_k \leq \vartheta \operatorname{osc}_{Q_2} \widetilde{f}_k,$$

with $\vartheta := 1 - \lambda/2 \in (0,1)$, and thus

$$\operatorname{osc}_{Q_{1/4^k}} \widetilde{f} \leq \vartheta^k \operatorname{osc}_{Q_2} \widetilde{f} \leq 2\vartheta^k \|f\|_{L^\infty(\mathcal{U})}.$$

In other words, we have

$$\sup_{4^{-k-1} \leq |y - y_0| \leq 4^{-k}} |\widetilde{f}(y) - \widetilde{f}(y_0)| \leq (4^\alpha \vartheta)^k |y_0 - y|^\alpha \|f\|_{L^\infty(\mathcal{U})}$$

by choosing $\alpha := -\log \theta / \log 4$. We have established that $\widetilde{f}$ is α-Holder near y_0, and thus also f on $\mathcal{U}$. □

4. Parabolic equations in a L^1 framework

In this section, we are interested with the evolution equation

$$\partial_t f = \Delta f + b\cdot\nabla f \ \text{ in } \ (0,T)\times\mathbb{R}^d, \qquad f(0,\cdot) = f_0 \ \text{ in } \ \mathbb{R}^d, \tag{4.1}$$

with initial datum $f_0 \in L^p$, $1 \le p < \infty$, $p \ne 2$. For further references, we note

$$\mathscr{L} f := \partial_t f - \Delta f - b\cdot\nabla f, \quad \mathscr{L}^*\varphi := -\partial_t\varphi - \Delta\varphi + \mathrm{div}(b\varphi),$$

$\mathscr{B}$ the set of functions $\beta \in C^2(\mathbb{R})$ such that β'' has compact support

$$T_k(s) := \max(\min(s,k),-k), \quad \theta_k(s) := \min((|s|-k)_+,1)$$

and $\mathcal{U} := [0,T)\times\mathbb{R}^d$.
In order to simplify the presentation, we consider the case $0 \le f_0 \in L^1(\mathbb{R}^d)$. In that case, the main result writes as follows.

Theorem 4.1. *We assume $b \in L^\infty$, $\mathrm{div} b \in L^\infty$. For any $0 \le f_0 \in L^1(\mathbb{R}^d)$, there exists a unique function $f \in C([0,T); L^1(\mathbb{R}^d))$ such that*

$$\nabla T_K(f) \in L^2(\mathcal{U}),\ \forall\, K>0, \quad \|\nabla\theta_n(f)\|_{L^2(\mathcal{U})} \to 0 \ \text{ as } \ n\to\infty \tag{4.2}$$

which is a renormalized solution to (4.1), that is

$$\int_0^T \int_{\mathbb{R}^d} \{\beta(f)\mathscr{L}^*\varphi + \beta''(f)|\nabla f|^2\varphi\} = \int_{\mathbb{R}^d} \beta(f_0)\varphi(0,\cdot)dx, \tag{4.3}$$

for any $\varphi \in \mathcal{D}(\mathcal{U})$ and $\beta \in \mathscr{B}$.

It is worth emphasizing that because $\beta \in \mathscr{B}$, we have $\mathrm{supp}\,\beta'' \subset [-K,K]$ for some $K > 0$ and thus

$$\beta''(g)\,|\nabla g|^2 = \beta''(g)\,\mathbf{1}_{|g|\le K}|\nabla g|^2 = \beta''(g)\,|\nabla T_K(g)|^2. \tag{4.4}$$

Together with (4.2), that implies that the second term in (5.6) makes sense.

4.1. **A priori estimates.** We quickly review the available estimates.

• Multiplying the equation by f^{p-1}, we have

$$\frac{1}{2}\partial_t f^p + 2\frac{p-1}{p}|\nabla f^{p/2}|^2 = \frac{1}{2}\Delta f^p + b\cdot\nabla f^{p/2} f^{p/2},$$

so that we cannot kill uniformly in $p > 1$ the last term at the RHS by the last term as the LHS (since this one vanishes in the limit $p \to 1$). Anyway, integrating, we have

$$\frac{d}{dt}\int \frac{f^p}{p} + \int 4\frac{p-1}{p^2}|\nabla f^{p/2}|^2 = \int (-\mathrm{div} b)\frac{f^p}{p}.$$

From the Gronwall lemma, we deduce

$$\|f(t,\cdot)\|_{L^p} \le e^{\frac{1}{p}\|\mathrm{div} b\|_{L^\infty} t}\|f_0\|_{L^p}, \quad \forall\, t \ge 0. \tag{4.5}$$

• From De Giorgi, Nash, Moser analysis in Section 2, we know that

$$\|f(t,\cdot)\|_{L^p} \le C_{p,T} t^{-(1-1/p)d/2}\|f_0\|_{L^1}, \quad \forall\, t \in (0,T). \tag{4.6}$$

Indeed, multiplying by $f\varphi^2$, with $\varphi \in \mathcal{D}((0,T))$, we have similarly

$$2\int_0^T \varphi^2 \int |\nabla f|^2 = \int_0^T (\varphi^2)' \int f^2 + \int_0^T \varphi^2 \int (-\mathrm{div} b) f^2.$$

We may then argue as in Section 2 using (4.5) for $p = 1$ and $p = 2^*$ and we obtain (4.6) with $p = 2^*$.

• Repeating the estimates presented in Section 2, we also obtain

$$\nabla f \in L^q_{\rm loc}(\mathcal{U}), \quad \forall\, q \in \big[1, \frac{d+2}{d+1}\big),$$

what follows from adapting Boccardo-Gallouet argument to the present situation. More precisely, we formally have

$$\partial_t \beta(f) = -\beta''(f)|\nabla f|^2 + \Delta\beta(f) + b\cdot\nabla\beta(f)$$

with the choice $\beta''(f) = \mathbf{1}_{M\le f\le M+1}$, $\beta(0)=\beta'(0)=0$, which implies

$$\frac{d}{dt}\int \beta(f) + \int |\nabla\theta_M(f)|^2 = \int (-{\rm div} b)\beta(f).$$

Defing $\kappa := \|{\rm div} b\|_{L^\infty}$ and using the Gronwall lemma, we deduce

$$\int_{\mathcal{U}} |\nabla\theta_M(f)|^2 \le e^{\kappa T}\int \beta(f_0) \le e^{\kappa T}\int |f_0|\mathbf{1}_{|f_0|\ge M} \to 0,$$

as $M\to\infty$, which is nothing but (4.2).

• In fact, using the De Giorgi, Nash, Moser estimate (4.6) with $p=2$ combined with the usual energy estimate, we deduce

$$\int_t^T\int |\nabla f|^2 dxds \le C e^{\kappa T} t^{-d/4}\|f_0\|_{L^1}, \quad \text{for any} \quad 0<t<T. \tag{4.7}$$

For further reference, we recall the following result established in the Chapter 1.

Lemma 4.2. *With the usual notations, assume that $g\in X_T$ is a weak solution to the parabolic equation*

$$\partial_t g = \Delta g + b\cdot\nabla g + \mathscr{G},$$

with $b\in L^2_{\rm loc}$ and $\mathscr{G}\in L^1_{\rm loc}$. For any $\beta\in\mathscr{B}$ such that $\beta(0)=0$, the function $\beta(g)\in X_T$ and it satisfies

$$\partial_t\beta(g) = \Delta\beta(g) - \beta''(g)|\nabla g|^2 + b\cdot\nabla\beta(g) + \beta'(g)\mathscr{G}.$$

4.2. **Existence of a renormalized solution.**

For $0\le f_0\in L^1$, we introduce the sequence $f_{0n} := f_0\wedge n \in L^1\cap L^2$ and the associated variational solution $f_n\in X_T$. We may justify all the previous estimate on f_n, in particular (f_n) is a Cauchy sequence in $C([0,T];L^1)$ and converges to a limit $f\in C([0,T];L^1)$. Passing to the limit $n\to\infty$, we obtain that f satisfies the estimates listed in the above paragraph 4.1 and it is a weak (in the distributional sense) solution to the parabolic equation (4.1). Because of (4.7) and Lemma 4.2, we know that f is a renormalized solution on (t,T), namely

$$\int_t^T\int_{\mathbb{R}^d} \beta(f)(\mathscr{L}^*\varphi) - \beta''(f)|\nabla f|^2\varphi = \int_{\mathbb{R}^d}[\beta(f)\varphi](T,\cdot) - \int_{\mathbb{R}^d}[\beta(f)\varphi](t,\cdot)$$

for any $\varphi\in\mathcal{D}(\mathbb{R}^d)$, $\beta\in\mathscr{B}$ and $t\in(0,T)$. Using that $\beta(f)\in C([0,T];L^1)$ and $\beta''(f)|\nabla f|^2\in L^1(\mathcal{U})$, we may pass to the limit $t\to 0$ and we deduce

$$\int_0^T\int_{\mathbb{R}^d} \beta(f)(\mathscr{L}^*\varphi) - \beta''(f)|\nabla f|^2\varphi = \int_{\mathbb{R}^d}[\beta(f)\varphi](T,\cdot) - \int_{\mathbb{R}^d}\beta(f_0)\varphi(0,\cdot)$$

for any $\varphi\in\mathcal{D}(\mathbb{R}^d)$ and $\beta\in\mathscr{B}$.

4.3. **Uniqueness.** Let us consider now two renormalized solutions f_1 and f_2 to the parabolic equation (4.1) with the same initial datum f_0 and let us prove that $f_1 = f_2$. For a given function $S \in \mathscr{B}$, we thus have

$$\partial_t S(f_i) = -S''(f_i)|\nabla f_i|^2 + \Delta S(f_i) + b \cdot \nabla S(f_i), \quad S(f_i)(0) = S(f_0).$$

More precise, taking $S'' := -\mathbf{1}_{[-n-1,-n]} + \mathbf{1}_{[n,n+1]}$, $S(0) = 0$, $S'(0) = 1$, we see that $S' \in L^\infty$ so that $S(f_i) \in X_T$ (with usual definition). We next define

$$f := S(f_2) - S(f_1), \quad \mathfrak{F} := S''(f_1)|\nabla f_1|^2 - S''(f_2)|\nabla f_2|^2,$$

and renormalizing the resulting equation thanks to Lemma 4.2, we get

$$\partial_t \beta(f) = -\beta''(f)|\nabla f|^2 + \Delta\beta(f) + b \cdot \nabla\beta(f) + \beta'(f)\mathfrak{F}, \quad \beta(f)(0) = 0,$$

for any $\beta \in \mathscr{B}$ such that $\beta(0) = 0$. Choosing $\beta \in \mathscr{B}$ such that $\beta(0) = 0$, each term involved in the above equation is in $L^1(\mathcal{U})$ and we may integrate it over $\mathcal{U}$, what implies

$$\int_{\mathbb{R}^d} \beta(f_T) = -\int_{\mathcal{U}} \beta''(f)|\nabla f|^2 + \int_{\mathcal{U}} \beta(f)\mathrm{div} b + \int_{\mathcal{U}} \beta'(f)\mathfrak{F}.$$

Assuming further that β is convex, we have

$$\int_{\mathbb{R}^d} \beta(f_T) \le \int_{\mathcal{U}} \beta(f)\mathrm{div} b + \int_{\mathcal{U}} \beta'(f)\mathfrak{F}.$$

More specifically, for β we choose $\beta''_\varepsilon = \frac{1}{\varepsilon}\mathbf{1}_{[-\varepsilon,\varepsilon]}$ and $\beta_\varepsilon(0) = \beta'_\varepsilon(0) = 0$, in such a way that $|\beta'(s)| \le 1$ for any $\varepsilon > 0$ and $\beta_\varepsilon(s) \to |s|$ as $\varepsilon \to 0$. Passing to the limit $\varepsilon \to 0$ in the last estimate, we get

$$\int_{\mathbb{R}^d} |f_T| \le \int_{\mathcal{U}} |f|\mathrm{div} b + \int_{\mathcal{U}} (|S''(f_1)||\nabla f_1|^2 + |S''(f_2)||\nabla f_2|^2).$$

With the above choice of $S = S_n$, we know that the last integral converges to 0 as $n \to \infty$ from the very definition of a renormalized solution, and we may thus pass to the limit $n \to \infty$ in the last equation in order to get

$$\int_{\mathbb{R}^d} |f_T| \le \|\mathrm{div} b\|_{L^\infty} \int_{\mathcal{U}} |f|.$$

We conclude that $f = 0$ thanks to the Gronwall lemma. We have thus established the uniqueness part in Theorem 4.1.

5. The fundamental solution to a parabolic equation

In this section, we are interested with the evolution equation

$$\partial_t f = \mathrm{div}(A\nabla f) \ \text{ in } \ (0,T) \times \mathbb{R}^d, \qquad f(0,\cdot) = \delta_{x_0} \ \text{ in } \ \mathbb{R}^d. \tag{5.1}$$

For further references, we note

$$\mathscr{L} f := \partial_t f - \mathrm{div}(A\nabla f), \quad \mathscr{L}^*\varphi := -\partial_t\varphi - \mathrm{div}(A^T\nabla\varphi).$$

Theorem 5.1. *We assume $A \in L^\infty$, $A \ge \nu I$, $\nu > 0$. For any $x_0 \in \mathbb{R}^d$, there exists a unique function $F = F(t, x; x_0) \ge 0$ such that*

$$\|F(t,\cdot)\|_{L^\infty} \le C(T) t^{-d/2}, \quad \forall\, t \in (0,T); \tag{5.2}$$

$$\|F(t,\cdot)\|_{L^1} \le 1, \quad \forall\, t \in (0,T); \tag{5.3}$$

$$\|\nabla F\|_{L^2((t,T)\times\mathbb{R}^d)} \le C(T), \quad \forall\, t \in (0,T), \tag{5.4}$$

$$\|\nabla F\|_{L^q(\mathcal{U})} \le C(T,q), \quad \forall\, q \in [1,q^*),\ q^* > 1, \tag{5.5}$$

which is a weak solution to (4.1)*, that is*

$$\int_0^T \int_{\mathbb{R}^d} \{f \partial_t \varphi - \nabla \varphi \cdot A \nabla f\} = \int_{\mathbb{R}^d} \varphi(0,\cdot) \delta_{x_0}(dx), \tag{5.6}$$

for any $\varphi \in \mathcal{D}(\mathcal{U})$ *.*

5.1. **Existence.** The proof is very similar to the proof of Theorem 4.1. The last a priori estimate comes from Boccardo-Gallouët result.

5.2. **Uniqueness.** Let us consider now two solutions F_1 and F_2 to the parabolic equation (5.1). The difference $f := F_2 - F_1$ is a weak solution to the parabolic equation with vanishing initial datum

$$\partial_t f = \mathrm{div}(A \nabla f) \ \text{ in } \ (0,T) \times \mathbb{R}^d, \qquad f(0,\cdot) = 0 \ \text{ in } \ \mathbb{R}^d. \tag{5.7}$$

On the other hand, we know from (5.4) that it is a variational solution on $(t,T) \times \mathbb{R}^d$, and we may thus write

$$\int_{\mathbb{R}^d} \psi_T f_T + \int_t^T \int_{\mathbb{R}^d} (-f \partial_t \psi + \nabla \psi \cdot A \nabla f) = \int_{\mathbb{R}^d} \psi_t f_t,$$

for any $\psi \in W^{1,\infty}([0,T] \times \mathbb{R}^d)$. For $\phi \in L^1 \cap L^\infty$, we define the solution $\varphi \in X_T$ to the backward problem

$$-\partial_t \varphi = \mathrm{div}(A^T \nabla \varphi) \ \text{ in } \ (0,T) \times \mathbb{R}^d, \qquad \varphi(T,\cdot) = \phi \ \text{ in } \ \mathbb{R}^d. \tag{5.8}$$

We define $\varphi^\varepsilon = \varphi *_x \rho_\varepsilon$ for a mollifer (ρ_ε). Observing that

$$\varphi^\varepsilon, \nabla \varphi^\varepsilon, \partial_t \varphi^\varepsilon = (-\mathrm{div}(A^T \nabla \varphi)) * \rho_\varepsilon \in L^\infty(\mathcal{U}),$$

we may take $\psi = \varphi^\varepsilon$ in the above variational formulation and we get

$$\begin{aligned} \Big[\int_{\mathbb{R}^d} \varphi^\varepsilon f\Big]_t^T &= \int_t^T \int_{\mathbb{R}^d} (\{f(-\mathrm{div}(A^T \nabla \varphi)) * \rho_\varepsilon - \nabla \varphi^\varepsilon \cdot A \nabla f) \\ &= \int_t^T \int_{\mathbb{R}^d} (\nabla f^\varepsilon \cdot A^T \nabla \varphi + \nabla \varphi^\varepsilon \cdot A \nabla f), \end{aligned}$$

with $f^\varepsilon := f * \check\rho_\varepsilon$, $\check\rho_\varepsilon(x) := \rho_\varepsilon(-x)$. Using that $\nabla f^\varepsilon \to \nabla f$ and $\nabla \varphi^\varepsilon \to \nabla \varphi$ in $L^2((t,T) \times \mathbb{R}^d)$, as well as $\varphi^\varepsilon_s \to \varphi_s$ in $L^2(\mathbb{R}^d)$ for $s = t, T$, we may pass to the limit $\varepsilon \to 0$ in the previous equation and we conclude that

$$\int_{\mathbb{R}^d} \phi f_T = \int_{\mathbb{R}^d} \varphi_t f_t, \quad \forall t > 0.$$

From (5.1) and (5.5), we formally have

$$\frac{d}{dt} \int \langle x \rangle^\vartheta F = \int \vartheta \langle x \rangle^{\vartheta - 1} \frac{x}{|x|} \cdot A \nabla F \in L^q(0,T),$$

for some $q \in (1, q^*)$ by choosing $1 - \vartheta > 0$ small enough and using the Holder inequality. We deduce that

$$\int \langle x \rangle^\vartheta F_t \le 1 + C t^{1/q'} \le C_T, \quad \forall t \in (0,T), \tag{5.9}$$

by using the Holder inequality again, what provides an additional a priori estimate. For any $\varphi \in C_b(\mathbb{R}^d)$, we may write

$$\begin{aligned} \int F(t)\varphi - \int F(s)\varphi &= \int F(t)(\varphi - \varphi^{\varepsilon,M}) + \int F(t)\varphi^{\varepsilon,M} - \int F(s)\varphi^{\varepsilon,M} \\ &\quad + \int F(s)(\varphi^{\varepsilon,M} - \varphi), \end{aligned}$$

with $\varphi^{\varepsilon,M} := (\varphi\chi_M) * \rho^s_\varepsilon \in \mathcal{D}(\mathbb{R}^d)$ using the usual notation for the truncations χ_M and the mollifiers (ρ_ε). Because $F \in L^\infty(0,T;L^1_\vartheta(\mathbb{R}^d))$ and $\varphi^{\varepsilon,M} \to \varphi$ in $L^\infty_{-\vartheta}$, the two extremal terms are small uniformly in $s,t \in [0,T]$ for any convenient choices of $\varepsilon, M > 0$. From the very definition of weak solution, we know that $F \in C([0,T];\mathcal{D}'(\mathbb{R}^d))$, so that the middle term is small for $|t-s|$ small enough. We deduce that $F \in C([0,T];(C_b(\mathbb{R}^d))')$, in particular $F_t \rightharpoonup \delta_{x_0}$ weakly in $(C_b(\mathbb{R}^d))'$ as $t \to 0$. Gathering this information with the De Giorgi-Nash regularity estimate $\varphi \in C_b([0,T/2] \times \mathbb{R}^d)$, we obtain that

$$\int_{\mathbb{R}^d} \phi f_T = \lim_{t\to 0} \int_{\mathbb{R}^d} \varphi_t f_t = 0.$$

Because $\phi \in L^1 \cap L^\infty$ is arbitrary, we deduce that $f_T = 0$ for any $T > 0$, and that concludes the uniqueness of the fundamental solution.

6. Refined bound on the fundamental solution

In this section, we are interest in the fundamental solution to the parabolic equation (1.1), namely to the solution Γ to

$$\frac{\partial \Gamma}{\partial t} = \operatorname{div}(A\nabla\Gamma) \quad \text{in } (0,\infty)\times\mathbb{R}^d, \qquad \Gamma_0 = \delta_{x_0} \quad \text{in } \mathbb{R}^d, \tag{6.1}$$

We first consider a smooth, positive and fast decaying initial datum f_0, the solution f to the associated heat equation, and for a given $\alpha \in \mathbb{R}^d$, we define $g := f\,e^\psi$, $\psi(x) := \alpha \cdot x$. The equation satisfied by g is

$$\begin{aligned} \partial_t g &= \frac{1}{2} e^\psi \Delta(g\,e^{-\psi}) = \frac12 \Delta g - \nabla\psi\cdot\nabla g + \frac12 |\nabla\psi|^2 g \\ &= \frac12\Delta g - \alpha\cdot\nabla g + \frac12|\alpha|^2 g. \end{aligned}$$

For the L^1 norm, we have

$$\frac{d}{dt}\|g\|_{L^1} = \frac12 \alpha^2\,\|g\|_{L^1},$$

and then $\|g(t,.)\|_{L^1} = e^{\alpha^2 t/2}\,\|g_0\|_{L^1}$ for any $t \ge 0$. For the L^2 norm and thanks to the Nash inequality, we have

$$\begin{aligned} \frac{d}{dt}\|g\|^2_{L^2} &= -\|\nabla g\|^2_{L^2} + \alpha^2\,\|g\|^2_{L^2} \\ &\le -K_0\,e^{-2\alpha^2 t/d}\,\|g\|^{2(1+2/d)}_{L^2} + \alpha^2\,\|g\|^2_{L^2}, \end{aligned}$$

with $K_0 := C_N\,\|g_0\|^{-4/d}_{L^1}$. We see that the function $u(t) := e^{-\alpha^2 t}\,\|g(t)\|^2_{L^2}$ satisfies the differential inequality

$$u' \le -K_0\,u^{1+2/d},$$

from what, exactly as in Nash $L^1 \to L^2$ estimate, we deduce

$$\|g(t)\|_{L^2}^2 \, e^{-\alpha^2 t} \le \frac{\|g_0\|_{L^1}^2}{(2/d \, C_N \, t)^{d/2}}, \quad \forall \, t > 0.$$

Denoting by $T(t)$ the semigroup associated to the parabolic equation satisfies by g, the above estimate writes

$$\|T(t) g_0\|_{L^2} \le \frac{C \, e^{\alpha^2 t/2}}{t^{d/4}} \, \|g_0\|_{L^1}, \quad \forall \, t > 0.$$

Because the equation associated to the dual operator is

$$\partial_t h = \frac{1}{2} \Delta h + \alpha \cdot \nabla h + \frac{1}{2} |\alpha|^2 h, \quad h(0) = h_0,$$

the same estimate holds on $T^*(t) h_0 = h(t)$, and we thus deduce

$$\|T(t) g_0\|_{L^\infty} \le \frac{C \, e^{\alpha^2 t/2}}{t^{d/4}} \, \|g_0\|_{L^2}, \quad \forall \, t > 0.$$

Using the trick $T(t) = T(t/2) T(t/2)$, both estimates together give an accurate time depend estimate on the mapping $T(t) : L^1 \to L^\infty$ for any $t > 0$. More precisely and in other words, we have proved that the heat semigroup S satisfies

$$\|(S(t) f_0) \, e^{\psi}\|_{L^\infty} \le \frac{C}{t^{d/2}} \, e^{\alpha^2 t/2} \, \|f_0 \, e^{\psi}\|_{L^1}, \quad \forall \, t > 0.$$

Denoting $F(t,x,y) := (S(t)\delta_x)(y)$ the fundamental solution associated to the heat equation when starting from the Dirac function in $x \in \mathbb{R}^d$, the above estimate rewrites as

$$F(t,x,y) \le \frac{C}{t^{d/2}} \, e^{\alpha \cdot (x-y) - \alpha^2 t/2}, \quad \forall \, t > 0, \forall \, x, y, \alpha \in \mathbb{R}^d.$$

Choosing $\alpha := (x-y)/t$, we end with

$$F(t,x,y) \le \frac{C}{t^{d/2}} \, e^{-\frac{|x-y|^2}{2t}}, \quad \forall \, t > 0, \forall \, x, y \in \mathbb{R}^d.$$

In particular, we immediately deduce

$$M_2(t) := \int_{\mathbb{R}^d} |y|^2 F(t,x,y) dy \le C(t + |x|^2),$$

and we thus recover (5.9) with $\vartheta = 2$.