CHAPTER 3 - THE FOKKER-PLANCK EQUATION, THE POINCARÉ INEQUALITY AND LONGTIME BEHAVIOUR

Contents

In this chapter we present some results about the self-similar behavior of the solutions to the heat equation in large time. Let us emphasize that the method lies on an interplay between evolution PDEs and functional inequalities.

1. Self-similar solutions of the heat equation and the Fokker-Planck equation

We consider the heat equation

$$\frac{\partial f}{\partial t} = \frac{1}{2}\Delta f \quad \text{in } (0,\infty)\times\mathbb{R}^d, \qquad f(0,\cdot) = f_0 \quad \text{in } \mathbb{R}^d. \tag{1.1}$$

We recall that $f(t,.) \to 0$ as $t\to\infty$, and more precisely, that for any $p\in(1,\infty]$ and a constant $C_{p,d}$ the following rate of decay holds:

$$\|f(t,.)\|_{L^p} \le \frac{C_{p,d}}{t^{\frac{d}{2}(1-\frac{1}{p})}}\,\|f_0\|_{L^1} \qquad \forall\, t>0. \tag{1.2}$$

That estimate can be classically prove thanks to the representation formula

$$f(t,.) = \gamma_t * f_0, \quad \gamma_t(x) := \frac{1}{(2\pi t)^{d/2}}\exp\Big(-\frac{|x|^2}{2t}\Big)$$

or by using Nash argument presented in a previous chapter.
It is in fact possible to describe in a more accurate way that the mere estimate (1.2) how the heat equation solution $f(t,.)$ converges to 0 as time goes on. In order to do so, the first step consists in looking for particular solutions to the heat equation that we will discover by identifying some good change of scaling. We thus look for a self-similar solution to (1.2), namely we look for a solution F with particular form

$$F(t,x) = t^\alpha\, G(t^\beta x),$$

for some $\alpha,\beta\in\mathbb{R}$ and a *"self-similar profile"* G. As F must be mass conserving, we have

$$\int_{\mathbb{R}^d} F(t,x)\,dx = \int_{\mathbb{R}^d} F(0,x)\,dx = t^\alpha \int_{\mathbb{R}^d} G(t^\beta\, x)\,dx,$$

and we get from that the first equation $\alpha = \beta\, d$. On the other hand, we easily compute

$$\partial_t F = \alpha\, t^{\alpha-1} G(t^\beta\, x) + \beta\, t^{\alpha-1}\,(t^\beta\, x)\cdot(\nabla G)(t^\beta\, x), \quad \Delta F = t^\alpha\, t^{2\beta}\,(\Delta G)(t^\beta\, x).$$

In order that (1.1) is satisfied, we have to take $2\,\beta + 1 = 0$. We conclude with

$$F(t,x) = t^{-d/2}\, G(t^{-1/2}\, x), \qquad \frac{1}{2}\Delta G + \frac{1}{2}\mathrm{div}(x\, G) = 0. \tag{1.3}$$

We observe (and that is not a surprise!) that a solution $G \in L^1(\mathbb{R}^d) \cap \mathbf{P}(\mathbb{R}^d)$ to (1.3) will satisfy $\nabla G + x\, G = 0$, it is thus unique and given by

$$G(x) := c_0\, e^{-|x|^2/2}, \quad c_0^{-1} = (2\,\pi)^{d/2} \quad \text{(normalized Gaussian function).}$$

To sum up, we have proved that F is our favorite solution to the heat equation: that is the fundamental solution to the heat equation.

Changing of point view, we may now consider G as a stationary solution to the harmonic Fokker-Planck equation (sometimes also called the Ornstein-Uhlenbeck equation)

$$\frac{\partial}{\partial t} g = \frac{1}{2}\, L\, g = \frac{1}{2}\, \nabla \cdot (\nabla g + g\, x) \qquad \text{in } (0,\infty) \times \mathbb{R}^d. \tag{1.4}$$

The link between the heat equation (1.1) and the Fokker-Planck equation (1.4) is as follows. If f is a solution to the Fokker-Planck equation (1.4), some elementary computations permit to show that

$$f(t,x) = (1+t)^{-d/2}\, g(\log(1+t), (1+t)^{-1/2}\, x)$$

is a solution to the heat equation (1.1), with $f(0,x) = g(0,x)$. Reciprocally, if f is a solution to the heat equation (1.1) then

$$g(t,x) := e^{d\,t/2}\, f(e^t - 1, e^{t/2}\, x)$$

solves the Fokker-Planck equation (1.4). The last expression also gives the existence of a solution *in the sense of distributions* to the Fokker-Planck equation (1.4) for any initial datum $f_0 = \varphi \in L^1(\mathbb{R}^d)$ as soon as we know the existence of a solution to the heat equation for the same initial datum (what we get thanks to the usual representation formula for instance).

2. Fokker-Planck equation and Poincaré inequality

2.1. Long time asymptotic behaviour of the solutions to the Fokker-Planck equation.

From now on in this chapter, we consider the Fokker-Planck equation

$$\frac{\partial}{\partial t} f = \mathcal{L}\, f = \Delta f + \nabla \cdot (f\, \nabla V) \quad \text{in } (0,\infty) \times \mathbb{R}^d \tag{2.1}$$

$$f(0,x) = f_0(x) \quad \text{on } \mathbb{R}^d, \tag{2.2}$$

and we assume that the *"confinement potential"* V is the harmonic potential

$$V(x) := \frac{|x|^2}{2} + V_0, \quad V_0 := \frac{d}{2}\, \log 2\pi.$$

We start observing that

$$\frac{d}{dt} \int_{\mathbb{R}^d} f(t,x)\, dx = \int_{\mathbb{R}^d} \nabla_x \cdot (\nabla_x f + f\, \nabla_x V)\, dx = 0,$$

so that the mass (of the solution) is conserved. We also have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt} \int_{\mathbb{R}^d} (f_+)^2\, dx &= \int_{\mathbb{R}^d} f_+ (\Delta f + \mathrm{div}(x f))\, dx \\ &= -\int_{\mathbb{R}^d} |\nabla f_+|^2 - \int_{\mathbb{R}^d} f_+\, x \cdot \nabla f_+\, dx \le \frac{d}{2} \int_{\mathbb{R}^d} (f_+)^2\, dx, \end{aligned}$$

and thanks to the Gronwall lemma, we conclude that the maximum principle holds. Moreover, the function $G = e^{-V} \in L^1(\mathbb{R}^d) \cap \mathbf{P}(\mathbb{R}^d)$ is nothing but the normalized Gaussian function, and since $\nabla G = -G\, \nabla V$, it is a stationary solution to the Fokker-Planck equation (2.1).

Theorem 2.1. *Let us fix $f_0 \in L^p(\mathbb{R}^d)$, $1 \le p < \infty$.*
(1) There exists a unique global solution $f \in C([0,\infty); L^p(\mathbb{R}^d))$ to the Fokker-Planck equation (2.1). This solution is mass conservative

$$\langle f(t,.)\rangle := \int_{\mathbb{R}^d} f(t,x)\,dx = \int_{\mathbb{R}^d} f_0(x)\,dx =: \langle f_0\rangle, \quad \textit{if } f_0 \in L^1(\mathbb{R}^d), \tag{2.3}$$

and the following maximum principle holds

$$f_0 \ge 0 \quad \Rightarrow \quad f(t,.) \ge 0 \quad \forall\, t \ge 0.$$

(2) Asymptotically in large time the solution converges to the unique stationary solution with same mass, namely

$$\|f(t,.) - \langle f_0\rangle\, G\|_E \le e^{-\lambda_P t}\, \|f_0 - \langle f_0\rangle\, G\|_E \quad \textit{as} \quad t \to \infty, \tag{2.4}$$

where $\|\cdot\|_E$ stands for the norm of the Hilbert space $E := L^2(G^{-1})$ defined by

$$\|f\|_E^2 := \int_{\mathbb{R}^d} f^2\, G^{-1}\, dx$$

and λ_P is the best (larger) constant in the Poincaré inequality.

More generally, for any weight function $m : \mathbb{R}^d \to \mathbb{R}_+$, we denote by $L^p(m)$ the Lebesgue space associated to the mesure $m(x)dx$ and by L^p_m the Lebesgue space associated to the norm $\|f\|_{L^p_m} := \|f\,m\|_{L^p}$. We will also write $L^p_k := L^p_m$, when $m := \langle x\rangle^k$.

For the proof of point (1) we refer to Chapter 1 as well as the final remark of Section 1. We are going to give the main lines of the proof of point 2. Because the equation is linear, we may assume in the sequel that $\langle f_0\rangle = 0$.

Using that $G\,G^{-1} = 1$, we deduce that $\nabla V = -G^{-1}\,\nabla G = G \cdot \nabla(G^{-1})$. We can then write the Fokker-Planck equation in the equivalent form

$$\begin{aligned} \frac{\partial}{\partial t} f &= \mathrm{div}_x\big(\nabla_x f + G\,f\,\nabla_x G^{-1}\big) \\ &= \mathrm{div}_x\big(G\,\nabla_x(f\,G^{-1})\big). \end{aligned} \tag{2.5}$$

We then compute

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int f^2\,G^{-1} &= \int_{\mathbb{R}^d} (\partial_t f)\, f\, G^{-1}\,dx = \int_{\mathbb{R}^d} \mathrm{div}_x\left(G\,\nabla_x\left(\frac{f}{G}\right)\right)\frac{f}{G}\,dx \\ &= -\int_{\mathbb{R}^d} G\,\left|\nabla_x \frac{f}{G}\right|^2\,dx. \end{aligned} \tag{2.6}$$

Using the Poincaré inequality established in the next Theorem 2.2 with the choice of function $h := f(t,.)/G$ and observing that $\langle f/G\rangle_G = 0$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int f^2\,G^{-1} \le -\lambda_P \int_{\mathbb{R}^d} G\left(\frac{f}{G}\right)^2 dx = -\lambda_P \int_{\mathbb{R}^d} f^2\,G^{-1}\,dx,$$

and we conclude using the Gronwall lemma.

Theorem 2.2 (Poincaré inequality)**.** *There exists a constant $\lambda_P > 0$ (which only depends on the dimension) such that for any $h \in \mathcal{D}(\mathbb{R}^d)$, there holds*

$$\int_{\mathbb{R}^d} |\nabla h|^2\,G\,dx \ge \lambda_P \int_{\mathbb{R}^d} |h - \langle h\rangle_G|^2\,G\,dx, \tag{2.7}$$

where we have defined

$$\langle h\rangle_\mu := \int_{\mathbb{R}^d} h(x)\,\mu(dx)$$

for any given (probability) measure $\mu \in \mathbf{P}(\mathbb{R}^d)$ and any function $h \in L^1(\mu)$.

We present below three slightly different proofs of this important result.

2.2. **A first proof of the Poincaré inequality.** We split the proof into three steps.

2.2.1. *Poincaré-Wirtinger inequality (in an open and bounded set Ω).*

Lemma 2.3. *Let us denote $\Omega = B_R$ the ball of $\mathbb{R}^d$ with center 0 and radius $R > 0$, and let us consider $\nu \in \mathbf{P}(\Omega)$ a probability measure such that (abusing notations) $\nu, 1/\nu \in L^\infty(\Omega)$. There exists a constant $\kappa \in (0,\infty)$, such that for any (smooth) function f, there holds*

$$\kappa \int_\Omega |f - \langle f\rangle_\nu|^2 \,\nu \le \int_\Omega |\nabla f|^2 \,\nu, \qquad \langle f\rangle_\nu := \int_\Omega f\,\nu, \tag{2.8}$$

and therefore

$$\int_\Omega f^2\,\nu \le \langle f\rangle_\nu^2 + \frac{1}{\kappa}\int_\Omega |\nabla f|^2\,\nu. \tag{2.9}$$

Proof of Lemma 2.3. We start with

$$f(x) - f(y) = \int_0^1 \nabla f(z_t)\cdot(x-y)\,dt, \quad z_t = t\,x + (1-t)\,y.$$

Multiplying that identity by $\nu(y)$ and integrating in the variable $y \in \Omega$ the resulting equation, we get

$$f(x) - \langle f\rangle_\nu = \int_\Omega \int_0^1 \nabla f(z_t)\cdot(x-y)\,dt\,\nu(y)\,dy.$$

Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\int_\Omega (f(x) - \langle f\rangle_\nu)^2\,\nu(x)\,dx &\le \int_\Omega\int_\Omega\int_0^1 |\nabla f(z_t)|^2\,|x-y|^2\,dt\,\nu(y)\,\nu(x)dydx\\ &\le C_1 \int_\Omega\int_\Omega\int_0^{1/2} |\nabla f(z_t)|^2\,dtdy\,\nu(x)dx + C_1\int_\Omega\int_\Omega\int_{1/2}^1 |\nabla f(z_t)|^2\,dtdx\,\nu(y)dy,\end{aligned}$$

with $C_1 := \|\nu\|_{L^\infty}\,\mathrm{diam}(\Omega)^2$. Performing the the changes of variables $(x,y)\mapsto(z,y)$ and $(x,y)\mapsto(x,z)$ and using the fact that $z_t \in [x,y] \subset \Omega$, we deduce

$$\begin{aligned}&\int_\Omega (f(x) - \langle f\rangle_\nu)^2\,\nu(x)\,dx\\ &\le C_1\int_\Omega\int_0^{1/2}\int_\Omega |\nabla f(z)|^2\,\frac{dz}{(1-t)^d}\,dt\,\nu(x)dx + C_1\int_\Omega\int_{1/2}^1\int_\Omega |\nabla f(z)|^2\,\frac{dz}{t^d}\,dt\,\nu(y)dy\\ &\le 2C_1\int_\Omega |\nabla f(z)|^2\,dz.\end{aligned}$$

We have thus established that the Poincaré-Wirtinger inequality (2.8) holds with the constant $\kappa^{-1} := 2\,C_1\,\|1/\nu\|_{L^\infty}$. □

2.2.2. *Weighted L^2 estimate through L^2 estimate on the derivative.*

Proposition 2.4. *There holds*

$$\frac14\int_{\mathbb{R}^d} h^2\,|x|^2\,Gdx \quad\le\quad \int_{\mathbb{R}^d} |\nabla h|^2\,Gdx + \frac{d}{2}\int_{\mathbb{R}^d} h^2\,Gdx,$$

for any $h \in C^1_b(\mathbb{R}^d)$.

Proof of Proposition 2.4. We define $\Phi := -\log G = |x|^2/2 + \log(2\pi)^{d/2}$. For a given function h, we denote $g = hG^{1/2}$, and we expand

$$\begin{aligned}\int_{\mathbb{R}^d} |\nabla h|^2\,G\,dx &= \int_{\mathbb{R}^d} \left|\nabla g\,G^{-1/2} + g\,\nabla G^{-1/2}\right|^2\,G\,dx\\ &= \int_{\mathbb{R}^d}\left\{|\nabla g|^2 + g\nabla g\nabla\Phi + \frac14 g^2|\nabla\Phi|^2\right\}dx,\end{aligned}$$

because $\nabla G^{-1/2} = \frac12\nabla\Phi\,G^{-1/2}$. Performing one integration by part, we get

$$\int_{\mathbb{R}^d}|\nabla h|^2\,G\,dx = \int_{\mathbb{R}^d}|\nabla g|^2\,dx + \int_{\mathbb{R}^d} h^2\,\left(\frac14|\nabla\Phi|^2 - \frac12\Delta\Phi\right)\,Gdx.$$

We conclude by neglecting the first term and computing the second term at the RHS. □

2.2.3. *End of the first proof of the Poincaré inequality.* We split the L^2 norm into two pieces

$$\int_{\mathbb{R}^d} h^2\, G\, dx = \int_{B_R} h^2\, G\, dx + \int_{B_R^c} h^2\, G\, dx,$$

for some constant $R > 0$ to be choosen later. One the one hand, we have

$$\begin{aligned}\int_{B_R} h^2\, G\, dx &\le C_R \int_{B_R} |\nabla h|^2\, G\, dx + \Big(\int_{B_R^c} h\, G\, dx\Big)^2 \\ &\le C_R \int |\nabla h|^2\, G\, dx + \Big(\int_{B_R^c} G\, dx\Big) \int h^2\, G\, dx,\end{aligned}$$

where in the first line, we have used the Poincaré-Wirtinger inequality (2.9) in B_R with

$$\nu := G(B_R)^{-1}\, G_{|B_R}, \quad G(B_R) := \int_{B_R} G\, dx,$$

and the fact that $\langle hG\rangle = 0$, and in the second line, we have used the Cauchy-Schwarz inequality. One the other hand, we have

$$\begin{aligned}\int_{B_R^c} h^2\, G\, dx &\le \frac{1}{R^2}\int_{\mathbb{R}^d} h^2\, |x|^2\, G\, dx \\ &\le \frac{4}{R^2}\int_{\mathbb{R}^d} |\nabla h|^2\, G dx + \frac{2d}{R^2}\int_{\mathbb{R}^d} h^2\, G dx,\end{aligned}$$

by using Proposition 2.4. All together, we get

$$\int_{\mathbb{R}^d} h^2\, G\, dx \le \big(C_R + \frac{4}{R^2}\big)\int_{\mathbb{R}^d} |\nabla h|^2\, G dx + \Big(\frac{2d}{R^2} + \int_{B_R^c} G\, dx\Big)\int h^2\, G\, dx,$$

and we choose $R > 0$ large enough in such a way that the constant in front of the last term at the RHS is smaller than 1. □

2.3. **An second proof of the Poincaré inequality.**

2.3.1. *A Lyapunov condition.* There exists a function W such that $W \ge 1$ and there exist some constants $\theta > 0$, $b, R \ge 0$ such that

$$(L^*W)(x) := \Delta W(x) - \nabla V \cdot \nabla W(x) \le -\theta\, W(x) + b\, \mathbf{1}_{B_R}(x), \qquad \forall\, x \in \mathbb{R}^d, \tag{2.10}$$

where again $B_R = B(0, R)$ denotes the centered ball of radius R. The proof is elementary. We look for W as $W(x) := e^{\gamma\langle x\rangle}$. We then compute

$$\nabla W = \gamma\, \frac{x}{\langle x\rangle}\, e^{\gamma\langle x\rangle} \quad \text{and} \quad \Delta W = \left(\gamma^2 + \gamma\, \frac{d-1}{\langle x\rangle}\right) e^{\gamma\langle x\rangle},$$

and thus

$$\begin{aligned}L^*W = \Delta W - x\cdot\nabla W &= \gamma\, \frac{d-1}{\langle x\rangle}\, W + \left(\gamma^2 - \gamma\, \frac{|x|^2}{\langle x\rangle}\right) W \\ &\le -\theta\, W + b\, \mathbf{1}_{B_R},\end{aligned}$$

with the choice $\theta = \gamma = 1$ and then R and b large enough. □

2.3.2. *End of second the proof of the Poincaré inequality.* We write (2.10) as

$$1 \le -\frac{L^*W(x)}{\theta\, W(x)} + \frac{b}{\theta\, W(x)}\, \mathbf{1}_{B_R}(x), \quad \forall x \in \mathbb{R}^d.$$

For any $f \in C^2_b(\mathbb{R}^d)$, we deduce

$$\int_{\mathbb{R}^d} f^2\, G \le -\int_{\mathbb{R}^d} f^2\, \frac{L^*W(x)}{\theta\, W(x)}\, G + \frac{b}{\theta}\int_{B_R} f^2\, \frac{1}{W}\, G \quad =: T_1 + T_2.$$

On the one hand, we have

$$\begin{aligned}
\theta\, T_1 &= \int \nabla W \cdot \Big\{\nabla\Big(\frac{f^2}{W}\Big)\, G + \frac{f^2}{W}\nabla G\Big\} + \int \frac{f^2}{W}\nabla V\cdot\nabla W\, G \\
&= \int \nabla W\cdot\nabla\Big(\frac{f^2}{W}\Big)\, G \\
&= \int 2\,\frac{f}{W}\,\nabla W\cdot\nabla f\, G - \int \frac{f^2}{W^2}\,|\nabla W|^2\, G \\
&= \int |\nabla f|^2\, G - \int\Big|\frac{f}{W}\,\nabla W - \nabla f\Big|^2\, G \\
&\le \int |\nabla f|^2\, G.
\end{aligned}$$

On the other hand, using the Poincaré-Wirtinger inequality in B_R and the notation

$$G(B_R) := \int_{B_R} G\, dx, \quad \nu_R := G(B_R)^{-1}\, G_{|B_R}, \quad \langle f\rangle_R = \int_{B_R} f\,\nu_R,$$

we have

$$\begin{aligned}
\frac{\theta}{b}\, T_2 &= \int_{B_R} f^2\,\frac{1}{W}\, G \le G(B_R)\int_{B_R} f^2\,\nu_R \\
&\le G(B_R)\Big(\langle f\rangle_R^2 + C_R\int_{B_R}|\nabla f|^2\,\nu_R\Big).
\end{aligned}$$

Gathering the two above estimates, we have shown

$$\int_{\mathbb{R}^d} f^2\, G \le C\Big(\langle f\rangle_R^2 + \int_{\mathbb{R}^d}|\nabla f|^2\, G\Big). \tag{2.11}$$

Consider now $h\in C_b^2$. We know that for any $c\in\mathbb{R}$, there holds

$$\int_{\mathbb{R}^d}(h-\langle h\rangle_G)^2 G \le \phi(c) := \int_{\mathbb{R}^d}(h-c)^2\, G, \tag{2.12}$$

with $\langle h\rangle_G$ defined in (2.8), because ϕ is a polynomial function of second degree which reaches is minimum value in $c_h := \langle h\rangle_G$. More precisely, by mere expantion, we have

$$\phi(c) = \int_{\mathbb{R}^d}(h-\langle h\rangle_G)^2 G\, dx + (c-\langle h\rangle_G)^2.$$

We last define $f := h - \langle h\rangle_R$, so that $\langle f\rangle_R = 0$, $\nabla f = \nabla h$. Using first (2.12) and next (2.11), we obtain

$$\begin{aligned}
\int_{\mathbb{R}^d}(h-\langle h\rangle_G)^2\, G &\le \int_{\mathbb{R}^d}(h-\langle h\rangle_R)^2\, G = \int_{\mathbb{R}^d} f^2\, G \\
&\le C\Big(\langle f\rangle_R^2 + \int_{\mathbb{R}^d}|\nabla f|^2\, G\Big) = C\int_{\mathbb{R}^d}|\nabla h|^2\, G.
\end{aligned}$$

That ends the proof of the Poincaré inequality (2.7). □

2.4. **A third proof of the Poincaré inequality.** From (2.5), introducing the unknown $h := f/G$, we have

$$\begin{aligned}
\partial_t h &= G^{-1}\mathrm{div}(G\nabla h) \\
&= \Delta h - x\cdot\nabla h =: Lh.
\end{aligned}$$

On the one hand, we have

$$h(Lh) = L(h^2/2) - |\nabla h|^2,$$

L is self-adjoint in $L^2(G)$ and $L^*1 = 0$. We then recover the identity (2.6), namely

$$\frac{1}{2}\frac{d}{dt}\int h^2 G\, dx = -\int |\nabla h|^2 G\, dx. \tag{2.13}$$

We fix $h_0\in L^2(G)$ with $\langle h_0 G\rangle = 0$. We accept that $h_T\to 0$ in $L^2(G)$ as $T\to\infty$, what it has been already established during the proofs 1 and 2 or can be established without rate using softer argument (as it will be explained in the chapter about Lyapunov techniques). By time integration of (2.13), we thus have

$$\|h_0\|^2 = -\lim_{T\to\infty}\Big[\|h_t\|^2\Big]_0^T = \lim_{T\to\infty}\int_0^T 2\|\nabla h_t\|^2\,dt,$$

where here and below $\|\cdot\|$ denotes the $L^2(G)$ norm, and therefore

$$\|h_0\|^2 = \int_0^\infty 2\|\nabla h_t\|^2\,dt. \tag{2.14}$$

On the other hand, we compute

$$\begin{aligned}\nabla h\cdot\nabla Lh &= \nabla h\cdot\Delta\nabla h - \nabla h\cdot\nabla(x\cdot\nabla h)\\ &= \Delta(|\nabla h|^2/2) - |D^2h|^2 - |\nabla h|^2 - xDh : D^2h\\ &= L(|\nabla h|^2/2) - |D^2h|^2 - |\nabla h|^2.\end{aligned}$$

We deduce

$$\frac{1}{2}\frac{d}{dt}\int|\nabla h|^2G\,dx = -\int|D^2h|^2G\,dx - \int|\nabla h|^2G\,dx \le -\int|\nabla h|^2G\,dx.$$

Similarly, as above, we have

$$\|\nabla h_0\|^2 - \|\nabla h_T\|^2 = -\int_0^T \frac{d}{dt}\|\nabla h_t\|^2\,dt \ge \int_0^T \|\nabla h_t\|^2\,dt,$$

and therefore

$$\|\nabla h_0\|^2 \ge \int_0^\infty 2\|\nabla h_t\|^2\,dt. \tag{2.15}$$

Gathering (2.14) and (2.15), we conclude with the following Poincaré inequality with optimal constant.

Proposition 2.5 (Poincaré inequality with optimal constant)**.** *For any $h\in\mathcal{D}(\mathbb{R}^d)$ with $\langle hG\rangle = 0$,*

$$\|\nabla h\|_{L^2(G)} \ge \|h\|_{L^2(G)}.$$

We deduce from the above Poincaré inequality with optimal constant, the identity (2.13) and the Gronwall lemma, the following optimal decay estimate

$$\|h_t\|_{L^2(G)} \le e^{-t}\|h_0\|_{L^2(G)}, \quad \forall\, t\ge 0,$$

for any $h_0\in L^2(G)$ such that $\langle h_0G\rangle = 0$.

3. Exercises and Complements

Exercise 3.1. *Establish* (2.10) *in the following situations:*

(i) $V(x) := \langle x\rangle^\alpha$ *with* $\alpha\ge 1$*;*

(ii) *there exist* $\alpha>0$ *and* $R\ge 0$ *such that*

$$x\cdot\nabla V(x)\ge\alpha \qquad \forall\, x\notin B_R;$$

(iii) *there exist* $a\in(0,1)$*,* $c>0$ *and* $R\ge 0$ *such that*

$$a\,|\nabla V(x)|^2 - \Delta V(x)\ge c \qquad \forall\, x\notin B_R;$$

(iv) V *is convex (or it is a compact supported perturbation of a convex function) and satisfies* $e^{-V}\in L^1(\mathbb{R}^d)$.

Exercise 3.2. *Generalize the Poincaré inequality to a general superlinear potential* $V(x) = \langle x\rangle^\alpha/\alpha + V_0$*,* $\alpha\ge 1$*, in the following strong (weighted) formulation*

$$\int|\nabla g|^2\,\mathcal{G} \ge \kappa\int|g-\langle g\rangle_{\mathcal{G}}|^2\,(1+|\nabla V|^2)\,\mathcal{G} \qquad \forall\, g\in\mathcal{D}(\mathbb{R}^d),$$

where we have defined $\mathcal{G} := e^{-V}\in\mathbf{P}(\mathbb{R}^d)$ *(for an appropriate choice of* $V_0\in\mathbb{R}$*).*