CHAPTER 3 - TRANSPORT EQUATION : CHARACTERISTICS METHOD AND DIPERNA-LIONS RENORMALIZATION THEORY

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This chapter is an introduction to the well-posedness theory for transport equations. We present the classical characteristics method as well as the more modern DiPerna-Lions theory of renormalization of solutions.

Contents

1. Introduction

In this chapter we consider the PDE (transport equation)

$$\partial_t f = \Lambda f = -a \cdot \nabla f \quad \text{in} \quad (0,\infty) \times \mathbb{R}^d, \tag{1.1}$$

for a drift force field $a = a(t,x) : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$, that we complement with an initial condition

$$f(0,x) = f_0(x) \quad \text{in} \quad \mathbb{R}^d,$$

as well as related equations. We assume that a is C^1 and satisfies the globally Lipschitz estimate

$$|a(t,x) - a(t,y)| \le L\,|x-y|, \quad \forall\, t \ge 0,\ x,y \in \mathbb{R}^d, \tag{1.2}$$

for some constant $L \in (0,\infty)$, and that the initial datum satisfies

$$f_0 \in L^p(\mathbb{R}^d),\ 1 \le p \le \infty. \tag{1.3}$$

We prove that there exists a unique solution in the renormalization sense to the transport equation (1.1) associated to the initial datum f_0.

2. Characteristics method and existence of solutions

2.1. **Smooth initial datum.**

As a first step we consider $f_0 \in C^1_c(\mathbb{R}^d;\mathbb{R})$.

Thanks to the Cauchy-Lipschitz theorem on ODE, we know that for any $x \in \mathbb{R}^d$ and $s \ge 0$, the equation

$$\dot x(t) = a(t, x(t)), \quad x(s) = x, \tag{2.1}$$

admits a unique solution $t \mapsto x(t) = \Phi_{t,s}(x) \in C^1(\mathbb{R}_+;\mathbb{R}^d)$. Moreover, for any $s,t \ge 0$, the vectors valued function $\Phi_{t,s} : \mathbb{R}^d \to \mathbb{R}^d$ is a C^1-diffeomorphism which satisfies the semigroup properties $\Phi_{t,t} = \mathrm{Id}$, $\Phi_{t_3,t_2} \circ \Phi_{t_2,t_1} = \Phi_{t_3,t_1}$ for any $t_3,t_2,t_1 \ge 0$, and the mapping $[0,T] \times [0,T] \times B(0,R) \to \mathbb{R}^d$, $(s,t,x) \mapsto \Phi_{s,t}(x)$ is Lipschitz for any $T,R > 0$, and we denote by $L_{T,R}$ this constant.

The characteristics method makes possible to build a solution to the transport equation (1.1) thanks to the solutions (characteristics) of the above ODE problem.

We start with a simple case. Assuming $f_0 \in C^1(\mathbb{R}^d;\mathbb{R})$, we define the function $f \in C^1(\mathbb{R}_+ \times \mathbb{R}^d;\mathbb{R})$

$$\forall\, t \ge 0,\ \forall\, x \in \mathbb{R}^d \quad f(t,x) := f_0(\Phi_t^{-1}(x)), \quad \Phi_t := \Phi_{t,0}. \tag{2.2}$$

From the associated implicit equation $f(t,\Phi_t(x)) = f_0(x)$, we deduce

$$\begin{aligned} 0 &= \frac{d}{dt}[f(t,\Phi_t(x))] = (\partial_t f)(t,\Phi_t(x)) + \dot\Phi_t(x) \cdot (\nabla_x f)(t,\Phi_t(x)) \\ &= (\partial_t f + a \cdot \nabla_x f)(t,\Phi_t(x)). \end{aligned}$$

The above equation holding true for any $t > 0$ and $x \in \mathbb{R}^d$ and the function Φ_t mapping $\mathbb{R}^d$ onto $\mathbb{R}^d$, we deduce that f satisfies the transport equation (1.1) in the sense of the classical differential calculus.

If furtheremore $f_0 \in C^1_c(\mathbb{R}^d)$, we have $f(t) \in C^1_c(\mathbb{R}^d)$ for any $t \ge 0$. Indeed, let take $R > 0$ such that $\operatorname{supp} f_0 \subset B_R$ and denote by R_t a constant such that $\Phi_t(\bar B_R) \subset B_{R_t}$, what is possible because $\Phi_t : \mathbb{R}^d \to \mathbb{R}^d$ is continuous (alternatively, one can observe that $|\Phi_t(x) - \Phi_0(x)| \le L_{t,R} t$ for any $x \in \mathbb{R}^d$ ant $t \ge 0$, so that $R_t := R + tL_{t,R}$ is suitable for any $t \ge 0$). As a consequence, $B_R \cap \Phi_t^{-1}(B^c_{R_t}) = \emptyset$, which implies that $f_0(\Phi_t^{-1}(x)) = 0$ if $x \in B^c_{R_t}$, and therefore $\operatorname{supp} f(t,\cdot) \subset B_{R_t}$. In other words, transport occurs with finite speed: that makes a great difference with the instantaneous positivity of solution (related of a "infinite speed" of propagation of particles) known for the heat equation and more generally for parabolic equations.

Exercise 2.1. *Make explicit the construction and formulas in the three following cases:*
(1) $a(x) = a \in \mathbb{R}^d$ is a constant vector. (Hint. One must find $f(t,x) = f_0(x - at)$).
(2) $a(x) = x$. (Hint. One must find $f(t,x) = f_0(e^{-t}x)$).
(3) $a(x,v) = v$, $f_0 = f_0(x,v) \in C^1(\mathbb{R}^d \times \mathbb{R}^d)$ and look for a solution $f = f(t,x,v) \in C^1((0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d)$. (Hint. One must find $f(t,x,v) = f_0(x - vt, v)$).
(4) Assume that $a = a(x)$ and prove that (S_t) is a group on $C(\mathbb{R}^d)$, where

$$\forall\, f_0 \in C(\mathbb{R}^d),\ \forall\, t \in \mathbb{R},\ \forall\, x \in \mathbb{R}^d \quad (S_t f_0)(x) = f(t,x) := f_0(\Phi_t^{-1}(x)). \tag{2.3}$$

2.2. **L^p initial datum.** As a second step, we want to generalize the construction of solutions to a wider class of initial data as announced in (1.3). We observe that, at least formally, the following computation holds for a given positive solution f of the transport equation (1.1):

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^d} f^p\,dx &= \int_{\mathbb{R}^d} \partial_t f^p\,dx = \int_{\mathbb{R}^d} p f^{p-1}\,\partial_t f\,dx \\
&= -\int_{\mathbb{R}^d} p f^{p-1}\,a\cdot\nabla_x f\,dx = -\int_{\mathbb{R}^d} a\cdot\nabla_x f^p\,dx \\
&= \int_{\mathbb{R}^d} (\mathrm{div}_x a) f^p\,dx \le \|\mathrm{div}_x a\|_{L^\infty}\int_{\mathbb{R}^d} f^p\,dx.
\end{aligned}$$

With the help of the Gronwall lemma, we learn from that differential inequality that the following (still formal) estimate holds

$$\|f(t)\|_{L^p} \le e^{bt/p}\,\|f_0\|_{L^p} \quad \forall\, t \ge 0, \tag{2.4}$$

with $b := \|\mathrm{div}_x a\|_{L^\infty_{tx}}$. As a consequence, we may propose the following natural definition of solution.

Definition 2.2. *We say that $f = f(t,x)$ is a weak solution to the transport equation (1.1) associated to the initial datum $f_0 \in L^p(\mathbb{R}^d)$ if it satisfies the bound*

$$f \in L^\infty(0,T;L^p(\mathbb{R}^d))$$

and it satisfies the equation in the following weak sense:

$$\int_0^T\int_{\mathbb{R}^d} f\,L^*\varphi\,dxdt = \int_{\mathbb{R}^d} f_0\,\varphi(0,.)\,dx, \tag{2.5}$$

for any $\varphi \in C^1_c([0,T) \times \mathbb{R}^d)$. Here, we define the primal operator L by

$$Lg := \partial_t g + a\cdot\nabla_x g$$

and its (formal) dual operator L^* by

$$L^*\varphi := -\partial_t\varphi - div_x(a\varphi).$$

We say that f is a global weak solution if it is a weak solution on $(0,T)$ for any $T > 0$.

Observe that choosing $\varphi \in C^1_c([0,T)) \times C^1_c(\mathbb{R}^d)$, that is $\varphi(t,x) = \chi(t)\,\psi(x)$, $\chi \in C^1_c([0,T))$, $\psi \in C^1_c(\mathbb{R}^d)$, and defining $\Lambda^*\psi := \mathrm{div}_x(a\,\psi)$, any weak solution $f \in L^\infty(0,T;L^p(\mathbb{R}^d))$ satisfies

$$-\int_0^T \Big(\int_{\mathbb{R}^d} f\,\psi\,dx\Big)\partial_t\chi\,dt = \int_0^T \Big(\int_{\mathbb{R}^d} f\,\Lambda^*\psi\,dx\Big)\chi\,dt + \int_{\mathbb{R}^d} f_0\,\psi\,dx\chi(0).$$

That is a weak formulation of the differential equality

$$\frac{d}{dt}\int_{\mathbb{R}^d} f\,\psi\,dx = \int_{\mathbb{R}^d} f\,\Lambda^*\psi\,dx, \tag{2.6}$$

complemented with initial datum

$$\int_{\mathbb{R}^d} f(0,.)\,\psi\,dx = \int_{\mathbb{R}^d} f_0\,\psi\,dx. \tag{2.7}$$

It is worth emphasizing that because of (2.6) there holds

$$\int_{\mathbb{R}^d} f\,\psi\,dx \in C([0,T]), \quad \forall\,\psi \in C^1_c(\mathbb{R}^d),$$

so that (2.7) makes sense. In other words, $f \in C([0,T];(C^1_c(\mathbb{R}^d))')$.

Exercise 2.3. *1. Prove that a smooth function is a classical solution iff it is a weak solution.*
2. Prove that a solution in the sense of (2.6) *is a weak solution in the sense of Definition 2.2. (Hint. Use the fact that the vectorial space generated by* $C^1_c([0,T)) \times C^1_c(\mathbb{R}^d)$ *is dense into* $C^1_c([0,T) \times \mathbb{R}^d)$*).*

Theorem 2.4 (Existence)**.** *For any* $f_0 \in L^p(\mathbb{R}^d)$, $1 \le p \le \infty$, *there exists a global weak solution to the transport equation* (1.1) *which furthermore satisfies*

$$f \in C([0,\infty);L^p(\mathbb{R}^d)) \text{ when } p \in [1,\infty);$$
$$f \in C([0,\infty);L^1_{loc}(\mathbb{R}^d)) \text{ when } p = \infty.$$

If moreover $f_0 \ge 0$ *then* $f(t,.) \ge 0$ *for any* $t \ge 0$.

Exercise 2.5. *Prove that a weak solution* f *is weakly continuous (after modification of* $f(t)$ *on a time set of measure zero) in the following sense:*
(i) $f \in C([0,T]; w*-(C_0(\mathbb{R}^d))')$ *when* $p=1$ *(for the weak topology* $*\sigma(M^1,C_0)$*);*
(ii) $f \in C([0,T]; w-L^p(\mathbb{R}^d))$ *when* $p \in (1,\infty)$ *(for the weak topology* $\sigma(L^p,L^{p'})$*);*
(iii) $f \in C([0,T]; w-L^p_{loc}(\mathbb{R}^d))$ *for any* $p \in [1,\infty)$ *when* $p=\infty$.
(Hint 1. Consider a sequence $\{\psi_m\}$ *in* $C^1_c(\mathbb{R}^d)$ *such that* $\{\psi_m\}$ *is dense in* $C_0(\mathbb{R}^d)$ *(resp.* $L^{p'}$, $1 < p' < \infty$*) and prove that* $t \mapsto \langle f(t),\psi_m\rangle$ *is continuous. Hint 2. See Step 1 in the proof of Corollary 4.4).*

Proof of Theorem 2.4. We split the proof into two steps.
Step 1. Rigorous a priori bounds. Take $f_0 \in C^1_c(\mathbb{R}^d)$ and consider $f(t)$ the solution of (1.1). For any smooth (renormalizing) function $\beta : \mathbb{R} \to \mathbb{R}_+$, $\beta(0)=0$, which is C^1, we clearly have that $\beta(f(t,x))$ is a solution to the same equation associated to the initial datum $\beta(f_0)$ and $\beta(f(t,.)) \in C^1_c(\mathbb{R}^d)$ for any $t \ge 0$. The function

$$(0,T) \to \mathbb{R}_+, \quad t \mapsto \int_{\mathbb{R}^d} \beta(f(t,x))\,dx$$

is clearly C^1 (that is an exercise using the Lebesgue's dominated convergence Theorem) and

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d}\beta(f)\,dx &= \int_{\mathbb{R}^d}\partial_t\beta(f)\,dx = \int_{\mathbb{R}^d}\beta'(f)\partial_t f\,dx\\ &= -\int_{\mathbb{R}^d}\beta'(f)\,a\cdot\nabla_x f\,dx = -\int_{\mathbb{R}^d}a\cdot\nabla_x\beta(f)\,dx\\ &= \int_{\mathbb{R}^d}(\mathrm{div}_x a)\beta(f)\,dx.\end{aligned}$$

Assuming furtheremore that $\beta \ge 0$, we deduce the differential inequality

$$\frac{d}{dt}\int_{\mathbb{R}^d}\beta(f(t,x))\,dx \le b\int_{\mathbb{R}^d}\beta(f(t,x))\,dx,$$

with $b := \|\mathrm{div}_x a\|_{L^\infty_{t,x}}$, and thanks to the Gronwall lemma, we get

$$\int_{\mathbb{R}^d}\beta(f(t,x))\,dx \le e^{bt}\int_{\mathbb{R}^d}\beta(f_0(x))\,dx.$$

Since $f_0 \in L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ by assumption, for any $1 \le p < \infty$, we can define a sequence of renormalized functions (β_n) such that $0 \le \beta_n(s) \nearrow |s|^p$ for any $s \in \mathbb{R}$ and we can pass to the limit in the preceding inequality using the monotonous Lebesgue Theorem at the RHS and the Fatou Lemma at the LHS in order to get

$$\int_{\mathbb{R}^d}|f(t,x)|^p\,dx \le e^{bt}\int_{\mathbb{R}^d}|f_0(x)|^p\,dx,$$

or in other words

$$\|f(t,.)\|_{L^p} \le e^{bt/p}\,\|f_0\|_{L^p}, \quad \forall\, t \ge 0.$$

We can choose $\beta(s) := |s|^p$ when $p \in (1,\infty)$ and $\beta_n(s) := \beta(ns)$ when $p = 1$, with β the vanishing in 0 primitive of the odd function β' defined by $\beta'(s) := s \wedge 1$ for any $s \ge 0$. Passing to the limit $p \to \infty$ in the above equation, we obtain (maximum principle)

$$\|f(t,.)\|_{L^\infty} \le \|f_0\|_{L^\infty}, \quad \forall\, t \ge 0.$$

Moreover, $f \in C([0,T];L^p(\mathbb{R}^d))$ for any $p \in [1,\infty)$.

Step 2. Existence in the case $p \in [1,\infty)$. For any function $f_0 \in L^p(\mathbb{R}^d)$, $1 \le p < \infty$, we may define a sequence of functions $f_{0,n} \in C^1_c(\mathbb{R}^d)$ such that $f_{0,n} \to f_0$ in $L^p(\mathbb{R}^d)$: here comes the restriction $p < \infty$. We may for instance take $f_{0,n} := (\chi_n\, f_0) * \rho_n$, where ρ_n is a sequence of approximations of the identity defined through a mollifier $0 \le \rho \in \mathcal{D}(\mathbb{R}^d)$, $\|\rho\|_{L^1} = 1$, by $\rho_n(x) := n^d\,\rho(nx)$ and χ_n is a sequence of truncation functions defined by $\chi_n(x) := \chi(x/n)$ for some fixed function $\chi \in \mathcal{D}(\mathbb{R}^d)$, $0 \le \chi \le 1$, $\chi(x) = 1$ for any $|x| \le 1$.

Because of the first paragraph we may define $f_n(t)$ as a solution to the transport equation and corresponding to the initial condition $f_{0,n}$. Moreover, thanks to the first step and because the equation is linear we have

$$\sup_{t\in[0,T]}\|f_n(t,.) - f_m(t,.)\|_{L^p} \le e^{bT/p}\,\|f_{0,n} - f_{0,m}\|_{L^p} \to 0, \quad \forall\, T \ge 0,$$

as $n,m \to \infty$. The sequence (f_n) is thus a Cauchy sequence in $C([0,T];L^p(\mathbb{R}^d))$. The space $C([0,T];L^p(\mathbb{R}^d))$ being complete, there exists $f \in C([0,T];L^p(\mathbb{R}^d))$ such that $f_n \to f$ in $C([0,T];L^p(\mathbb{R}^d))$ as $n\to\infty$. Now, writing

$$\begin{aligned} 0 &= -\int_0^T\int_{\mathbb{R}^d}\varphi\left\{\partial_t f_n + a\cdot\nabla f_n\right\}dxdt \\ &= \int_0^T\int_{\mathbb{R}^d} f_n\left\{\partial_t\varphi + \mathrm{div}_x(a\varphi)\right\}dxdt + \int_{\mathbb{R}^d} f_{0,n}\,\varphi(0,.)\,dx, \end{aligned}$$

we may pass to the limit in the above equation and we get that f is a solution in the convenient sense.

If moreover $f_0 \geq 0$ then the same holds for $f_{0,n}$, then for f_n and finally for f. □

Exercise 2.6. *(1) Show that for any characterictics solution f to the transport equation associated to an initial datum $f_0 \in C^1_c(\mathbb{R}^d)$, for any times $T>0$ and radius R, there exists some constants $C_T, R_T \in (0,\infty)$ such that*

$$\sup_{t\in[0,T]}\int_{B_R}|f(t,x)|\,dx \leq C_T\int_{B_{R_T}}|f_0(x)|\,dx.$$

(Hint. Use the property of finite speed propagation of the transport equation).

(2) Adapt the proof of existence to the case $f_0\in L^\infty$.

(3) Prove that for any $f_0\in C_0(\mathbb{R}^d)$ there exists a global weak solution f to the transport equation which furthermore satisfies $f\in C([0,T];C_0(\mathbb{R}^d))$.

3. Weak solutions are renormalized solutions

We now consider the transport equation with an additional source term

$$\partial_t g = -a\cdot\nabla_x g + G. \tag{3.1}$$

We start with a remark. For any $g\in C^1$ a classical solution of (3.1) and any $\beta\in C^1(\mathbb{R};\mathbb{R})$, there holds

$$\partial_t\beta(g) + a\cdot\nabla_x(\beta(g)) = \beta'(g)\,\partial_t g + \beta'(g)\,a\cdot\nabla_x g = \beta'(g)\,G.$$

Definition 3.1. *We say that $g\in L^1_{loc}([0,T]\times\mathbb{R}^d)$ is a renormalized solution to the transport equation* (3.1) *with $G\in L^1_{loc}([0,T]\times\mathbb{R}^d)$, $g_0\in L^1_{loc}(\mathbb{R}^d)$ if g satisfies the equation*

$$\int_0^T\int_{\mathbb{R}^d}\beta(g)\,L^*\varphi = \int_{\mathbb{R}^d}\beta(g_0)\varphi(0,.) + \int_0^T\int_{\mathbb{R}^d}\varphi\,\beta'(g)\,G, \tag{3.2}$$

for any "test function" $\varphi\in C^1_c([0,T)\times\mathbb{R}^d)$ and any "renormalizing function" $\beta\in C^1(\mathbb{R})$ which is globally Lipschitz.

Theorem 3.2. *With the above notations and assumptions, any weak solution $g\in C([0,T];L^1_{loc}(\mathbb{R}^d))$ to the transport equation* (3.1) *is a renormalized solution.*

We start with two elementary but fundamental lemmas.

Lemma 3.3. *Given $G\in L^1_{loc}([0,T]\times\mathbb{R}^d)$, let $g\in L^1_{loc}([0,T]\times\mathbb{R}^d)$ be a weak solution to the PDE*

$$Lg = G \quad \textit{on } (0,T)\times\mathbb{R}^d. \tag{3.3}$$

For a mollifier sequence

$$\rho_\varepsilon(t,x) := \frac{1}{\varepsilon^{d+1}}\rho\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}\right),\quad 0\leq\rho\in\mathcal{D}(\mathbb{R}^{d+1}),\ supp\,\rho\subset(-1,0)\times B(0,1),\ \int_{\mathbb{R}^{d+1}}\rho = 1,$$

and for $\tau \in (0,T)$, $\varepsilon \in (0,\tau)$, we define the function

$$g_\varepsilon := (\rho_\varepsilon *_{t,x} g)(t,x) := \int_0^T \int_{\mathbb{R}^d} g(s,y)\, \rho_\varepsilon(t-s, x-y)\, dsdy.$$

Then $g_\varepsilon \in C^\infty([0,T-\tau)\times\mathbb{R}^d)$ and it satisfies the equation

$$Lg_\varepsilon = G_\varepsilon + r_\varepsilon$$

in the classical differential calculus sense on $[0,T-\tau)\times\mathbb{R}^d$, with

$$G_\varepsilon := \rho_\varepsilon *_{t,x} G, \quad r_\varepsilon := a\cdot\nabla_x g_\varepsilon - (a\cdot\nabla g) * \rho_\varepsilon.$$

Proof of Lemma 3.3 using the theory of distributions. From (3.3), the following equations

$$\begin{aligned} G * \rho_\varepsilon &= (\partial_t g) * \rho_\varepsilon + (a\cdot\nabla g) * \rho_\varepsilon \\ &= \partial_t (g * \rho_\varepsilon) + a\cdot\nabla(g*\rho_\varepsilon) - r_\varepsilon \end{aligned}$$

hold in $\mathcal{D}'((0,T)\times\mathbb{R}^d)$ and then also in the classical sense because all the terms are (at least) continuous functions.

Proof of Lemma 3.3 using the weak formulation. We emphasize that one possible way to define the *"commutator"* r_ε is in a weak sense, namely

$$r_\varepsilon(t,x) := \int_{\mathbb{R}^{d+1}} g(s,y) \Big\{ a(x)\cdot\nabla_x \rho_\varepsilon(t-s,x-y) + \mathrm{div}_y \big[a(y)\, \rho_\varepsilon(t-s,x-y)\big] \Big\}\, dyds.$$

Define $\mathcal{O} := [0,T-\tau)\times\mathbb{R}^d$. For any $(t,x)\in\mathcal{O}$ fixed and any $\varepsilon\in(0,\tau)$, we define

$$(s,y)\mapsto \varphi(s,y) = \varphi_\varepsilon^{t,x}(s,y) := \rho_\varepsilon(t-s,x-y) \in \mathcal{D}((0,T)\times\mathbb{R}^d).$$

We then just write the weak formulation of equation (3.1) for that test function. We get

$$\begin{aligned} 0 &= \int_0^T\int_{\mathbb{R}^d} g\, L^*\varphi - \int_0^T\int_{\mathbb{R}^d} G\,\varphi \\ &= \int_0^T\int_{\mathbb{R}^d} g(s,y)\,\{-\partial_s\varphi^{t,x}(s,y) - \nabla_y(a(y)\,\varphi^{t,x}(s,y))\} - \int_0^T\int_{\mathbb{R}^d} G(s,y)\,\varphi^{t,x}(s,y) \\ &= \int_0^T\int_{\mathbb{R}^d} g(s,y)\,\partial_t\varphi^{t,x}(s,y) - \int_0^T g(s,y)\,\nabla_y(a(y)\,\varphi^{t,x}(s,y)) - \int_0^T\int_{\mathbb{R}^d} G(s,y)\,\varphi^{t,x}(s,y) \\ &= \partial_t g_\varepsilon(t,x) + a\cdot\nabla_x g_\varepsilon(t,x) - r_\varepsilon(t,x) - G_\varepsilon(t,x), \end{aligned}$$

because,

$$-\int_0^T g(s,y)\,\nabla_y(a(y)\,\varphi^{t,x}(s,y)) = \int_0^T\int_{\mathbb{R}^d} a(y)\cdot\nabla_y g(s,y)\,\rho_\varepsilon(t-s,x-y) = (a\cdot\nabla g)*\rho_\varepsilon,$$

by performing one integration by part. □

Lemma 3.4. *Under the assumptions $B\in W^{1,q}_{loc}(\mathbb{R}^d)$ and $g\in L^p_{loc}(\mathbb{R}^d)$ with $1/r = 1/p+1/q \le 1$, then*

$$R_\varepsilon := (B\cdot\nabla g)*\rho_\varepsilon - B\cdot\nabla(g*\rho_\varepsilon) \to 0 \quad L^r_{loc},$$

for any mollifier sequence (ρ_ε).

Remark 3.5. *For a time dependent function $g = g(t,x)$ satisfying the boundedness conditions of Theorem 3.2 the same result (with the same proof) holds, so that the commutator r_ε defined in Lemma 3.3 satisfies $r_\varepsilon \to 0$ in $L^1_{loc}([0,T)\times\mathbb{R}^d)$.*

Proof of Lemma 3.4. We only consider the case $p = 1$, $q = \infty$ and $r = 1$. We start writing

$$\begin{aligned} R_\varepsilon(x) &= -\int_{\mathbb{R}^d} g(y)\Big\{\mathrm{div}_y\Big(B(y)\,\rho_\varepsilon(x-y)\Big) + B(x)\cdot\nabla_x\big(\rho_\varepsilon(x-y)\big)\Big\}\,dy \\ &= \int_{\mathbb{R}^d} g(y)\Big\{\Big(B(y)-B(x)\Big)\cdot\nabla_x(\rho_\varepsilon(x-y))\Big\}\,dy - ((g\,\mathrm{div}\,B)*\rho_\varepsilon)(x) \\ &=: R^1_\varepsilon(x) + R^2_\varepsilon(x). \end{aligned}$$

For the first term, we remark that

$$\begin{aligned} |R^1_\varepsilon(x)| &\le \int |g(y)|\,\Big|\frac{B(y)-B(x)}{\varepsilon}\Big|\,|(\nabla\rho)_\varepsilon(x-y)|\,dy \\ &\le \|\nabla B\|_{L^\infty}\int_{|x-y|\le 1}|g(y)|\,|(\nabla\rho)_\varepsilon(x-y)|\,dy, \end{aligned}$$

so that

$$\int_{B_R}|R^1_\varepsilon(x)|\,dx \le \|\nabla B\|_{L^\infty}\,\|\nabla\rho\|_{L^1}\,\|g\|_{L^1(B_{R+1})}. \tag{3.4}$$

On the other hand, if g is a smooth (say C^1) function

$$\begin{aligned} R^1_\varepsilon(x) &= \nabla_x((gB)*\rho_\varepsilon) - B\cdot\nabla_x(g*\rho_\varepsilon) \\ &\longrightarrow \nabla_x(gB) - B\cdot\nabla_x g = (\mathrm{div}B)\,g. \end{aligned}$$

Since every things make sense at the limit with the sole assumption $\mathrm{div}B \in L^\infty$ and $g \in L^1$, with the help of (3.4) we can use a density argument in order to get the same result without the additional smoothness hypothesis on g. More precisely, for a sequence g_α in C^1 such that $g_\alpha \to g$ in L^1_{loc}, we have

$$R^1_\varepsilon[g_\alpha] \to (\mathrm{div}B)\,g \text{ in } L^1_{loc}, \quad \|R^1_\varepsilon[h]\|_{L^1} \le C\,\|h\|_{L^1}\ \forall\, h,$$

where $R^1_\varepsilon[h]$ stands for the function R^1_ε defined above but for the function h instead of the function g, so that

$$\begin{aligned} R^1_\varepsilon[g] - (\mathrm{div}B)\,g &= \{R^1_\varepsilon[g] - R^1_\varepsilon[g_\alpha]\} + \{R^1_\varepsilon[g_\alpha] - (\mathrm{div}B)\,g_\alpha)\} \\ &\quad + \{(\mathrm{div}B)\,g_\alpha - (\mathrm{div}B)\,g)\} \to 0, \end{aligned}$$

in L^1_{loc} as $\varepsilon \to 0$. For the second term, we clearly have

$$R^2_\varepsilon = (g\,\mathrm{div}\,B)*\rho_\varepsilon \to g\,\mathrm{div}\,B,$$

and we conclude by putting all the terms together. □

Proof of Theorem 3.2. Step 1. We consider a weak solution $g \in L^1_{loc}$ to the PDE

$$Lg = G \quad \text{in} \quad [0,T)\times\mathbb{R}^d.$$

By mollifying the functions with the sequence (ρ_ε) defined in Lemma 3.3 and using Lemma 3.3, we get

$$Lg_\varepsilon = G_\varepsilon + r_\varepsilon \quad \text{in} \quad [0,T)\times\mathbb{R}^d, \quad r_\varepsilon \to 0 \text{ in } L^1_{loc}.$$

Because g_ε is a smooth function, we may perform the following computation (in the sense of the classical differential calculus)

$$L\beta(g_\varepsilon) = \beta'(g_\varepsilon)\,G_\varepsilon + \beta'(g_\varepsilon)\,r_\varepsilon,$$

so that

$$\int_{\mathbb{R}^d} \beta(g_\varepsilon) L^* \varphi = \int_{\mathbb{R}^d} \beta(g_\varepsilon(0,.))\varphi(0,.) + \int_{\mathbb{R}^d} \beta'(g_\varepsilon)\, G_\varepsilon\, \varphi + \int_{\mathbb{R}^d} \beta'(g_\varepsilon)\, r_\varepsilon\, \varphi \tag{3.5}$$

for any $\varphi \in C^1_c([0,T) \times \mathbb{R}^d$. Using that

$$g_\varepsilon \to g, \quad G_\varepsilon \to G, \quad r_\varepsilon \to 0 \quad \text{in } L^1_{loc} \text{ as } \varepsilon \to 0,$$

we may pass to the limit $\varepsilon \to 0$ in the last identity and we obtain (3.2) for any test function $\varphi \in C^1_c((0,T) \times \mathbb{R}^d)$.
Using that $g \in C([0,T); L^1_{loc}(\mathbb{R}^d))$, we additionally have

$$g_\varepsilon \to g \quad \text{in } C([0,T); L^1_{loc}(\mathbb{R}^d)).$$

In particular $g_\varepsilon(0,.) \to g(0,.)$ in $L^1_{loc}(\mathbb{R}^d)$ and we may pass to the limit in equation (3.5) for any test function $\varphi \in C^1_c([0,T) \times \mathbb{R}^d)$. □

4. Consequences of the renormalization result

In this section we present several immediate consequences of the renormalization formula established in Theorem 3.2.

4.1. Uniqueness in $L^p(\mathbb{R}^d)$, $1 \le p < \infty$.

Corollary 4.1. *Assume $p \in [1,\infty)$. For any initial datum $g_0 \in L^p(\mathbb{R}^d)$, the transport equation admits a unique weak solution $g \in C([0,T]; L^p(\mathbb{R}^d))$.*

Proof of Corollary 4.1. Consider two weak solutions g_1 and g_2 to the transport equation (1.1) associated to the same initial datum g_0. The function $g := g_2 - g_1 \in C([0,T]; L^p(\mathbb{R}^d))$ is then a weak solution to the transport equation (1.1) associated to the initial datum $g(0) = 0$. Thanks to Theorem 3.2, it is also a renormalized solution, which means

$$\int_{\mathbb{R}^d} \beta(g(t,.))\, \varphi\, dx = \int_0^t \int_{\mathbb{R}^d} \beta(g)\, \mathrm{div}_x(a\, \varphi)\, dxds,$$

for any renormalizing function $\beta \in W^{1,\infty}(\mathbb{R})$, $\beta(0) = 0$, and any test function $\varphi = \varphi(x) \in C^1_c(\mathbb{R}^d)$. We fix β such that furthermore $0 < \beta(s) \le |s|^p$ for any $s \ne 0$, $\chi \in \mathcal{D}(\mathbb{R}^d)$ such that $\mathbf{1}_{B(0,1)} \le \chi \le \mathbf{1}_{B(0,2)}$, and we take $\varphi(x) = \chi_R(x) = \chi(x/R)$, so that

$$\int_{\mathbb{R}^d} \beta(g(t,.))\, \chi_R\, dx = \int_0^t \int_{\mathbb{R}^d} \beta(g)\, (\mathrm{div}_x a)\, \chi_R\, dxds + \frac{1}{R} \int_0^t \int_{\mathbb{R}^d} \beta(g)\, a \cdot \nabla\chi(x/R)\, dxds.$$

For handling the last term, we observe that $|a(t,x)| \le C\,(1 + |x|)$ for any $x \in \mathbb{R}^d$ and $t \in (0,T)$ from (1.2) and thus

$$\frac{1}{R} |a \cdot \nabla\chi(x/R)| \le 3C \|\nabla\chi\|_{L^\infty} \mathbf{1}_{|x| \ge R}, \quad \forall\, x \in \mathbb{R}^d,\ t \in (0,T),\ R \ge 1.$$

Taking advantage of that $\beta(g) \in C([0,T]; L^1(\mathbb{R}^d))$, we easily pass to the limit as $R \to \infty$ in the above expression, and we get

$$\int_{\mathbb{R}^d} \beta(g(t,.))\, dx = \int_0^t \int_{\mathbb{R}^d} \beta(g)\, (\mathrm{div}_x a)\, dxds. \tag{4.1}$$

By the Gronwall lemma, we conclude that $\beta(g(t,.)) = 0$ and then $g(t,.) = 0$ for any $t \in [0,T]$. □

4.2. **Positivity.** We can recover in a quite elegant way the positivity as an a posteriori property that we deduce from the renormalization formula.

Corollary 4.2. *Consider a solution* $g \in C([0,T]; L^p(\mathbb{R}^d))$, $1 \le p < \infty$, *to the transport equation* (1.1)*. If* $g_0 \ge 0$ *then* $g(t,.) \ge 0$ *for any* $t \ge 0$.

Proof of Corollary 4.2. We argue similarly as in the proof of Corollary 4.1 but fixing a renormalizing function $\beta \in W^{1,\infty}(\mathbb{R})$ such that $\beta(s) = 0$ for any $s \ge 0$, $\beta(s) > 0$ for any $s < 0$. We observe that for $f \in L^p$, we have $f \ge 0$ iff $\|\beta(f)\|_{L^1} = 0$. Since then $\beta(g_0) = 0$, we deduce that (4.1) holds again with that choice of function β and then, thanks to Gronwall lemma, $\beta(g(t,.)) = 0$ for any $t \ge 0$. That means $g(t,.) \ge 0$ for any $t \ge 0$. □

4.3. **A posteriori estimate.**

Corollary 4.3. *Consider a solution* $g \in C([0,T]; L^p(\mathbb{R}^d))$, $1 \le p < \infty$, *to the transport equation* (1.1)*. If* $g_0 \in L^q(\mathbb{R}^d)$, $1 \le q \le \infty$, *then* $g \in L^\infty(0,T; L^q(\mathbb{R}^d))$ *for any* $T > 0$.

Proof of Corollary 4.3. We argue similarily as in the proof of Corollary 4.1 but fixing an arbitrary renormalizing function $\beta \in C^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ such that $|\beta(s)| \le C|s|^p$ for any $s \in \mathbb{R}$, and then $\beta(g) \in C([0,T]; L^1(\mathbb{R}^d))$. For such a choice, we have

$$\int_{\mathbb{R}^d} \beta(g(t,.))\, dx = \int_{\mathbb{R}^d} \beta(g_0)\, dx + \int_0^t \!\! \int_{\mathbb{R}^d} \beta(g)\, (\mathrm{div}_x a)\, dx ds, \qquad \forall\, t \ge 0.$$

From the Gronwall lemma, we obtain with $b = \|\mathrm{div}\, a\|_{L^\infty}$, the estimate

$$\int_{\mathbb{R}^d} \beta(g(t,.))\, dx \le e^{b\, t} \int_{\mathbb{R}^d} \beta(g_0)\, dx \qquad \forall\, t \ge 0. \tag{4.2}$$

Since estimate (4.2) is uniform with respect to β, we may choose a sequence of renormalizing functions (β_n) such that $\beta_n(s) \nearrow |s|^q$ in the case $1 \le q < \infty$ and we get

$$\|g(t,.)\|_{L^q} \le e^{bt/q}\, \|g(t,.)\|_{L^q} \qquad \forall\, t \ge 0.$$

In the case $q = \infty$, we obtain the same conclusion by fixing $\beta \in W^{1,\infty}$ such that $\beta(s) = 0$ for any $|s| \le \|g_0\|_{L^\infty}$, $\beta(s) > 0$ for any $|s| > \|g_0\|_{L^\infty}$ or by passing to the limit $q \to \infty$ in the above inequality. □

4.4. **Continuity.** We can recover the strong L^p continuity property from the renormalization formula for a given solution.

Corollary 4.4. *Let* $g \in L^\infty(0,T; L^p(\mathbb{R}^d))$, $1 < p < \infty$, *be a renormalized solution to the transport equation* (1.1)*. Then* $g \in C([0,T]; L^p(\mathbb{R}^d))$.

Proof of Corollary 4.4. *Step 1.* We claim that $g \in C([0,T]; L^p(\mathbb{R}^d) - w)$ in the sense that

$$t \mapsto \int_{\mathbb{R}^d} g(t,x)\, \psi(x)\, dx \ \text{ is continuous for any } \ \psi \in L^{p'}(\mathbb{R}^d).$$

Taking $\beta(s) = s$ and $\varphi = \chi(t)\psi(x)$, $\chi \in C^1_c([0,T))$, $\psi \in C^1_c(\mathbb{R}^d)$, in the renormalized formulation of Definition 3.1 with vanishing source term, we have

$$\int_0^T u_\psi\, \chi'\, dt = \int_0^T v_\psi\, \chi\, dt + u^0_\psi \chi(0), \tag{4.3}$$

with

$$u_\psi := \int_{\mathbb{R}^d} g\,\psi\,dx, \quad v_\psi := \int_{\mathbb{R}^d} g\,\mathrm{div}(a\psi)\,dx, \quad u^0_\psi := \int_{\mathbb{R}^d} g_0\,\psi\,dx.$$

Because $u_\psi, v_\psi \in L^\infty(0,T)$, equation (4.3) is nothing but a weak formulation of the fact that $u_\psi \in W^{1,\infty}(0,T)$ and $u'_\psi = v_\psi$. From the $W^{1,\infty}(0,T) \subset C([0,T])$ embedding, we deduce that there exists a mesurable set $\mathcal{O}_\psi \subset [0,T]$ and $\tilde u_\psi \in C([0,T])$ such that $\tilde u_\psi \equiv u_\psi$ on $\mathcal{O}_\psi$ and $\mathrm{meas}([0,T]\backslash\mathcal{O}_\psi) = 0$.

We classically know that $L^{p'}(\mathbb{R}^d)$ is separable, and more precisely, there exists a countable family $\{\psi_m\}$ of $C^1_c(\mathbb{R}^d)$ such that for any $\psi \in L^{p'}(\mathbb{R}^d)$ there exists a subsequence (ψ_{n_k}) such that $\psi_{n_k} \to \psi$ in $L^{p'}$ as $k \to \infty$. For any fixed $\psi \in L^{p'}(\mathbb{R}^d)$, we define

$$\tilde u_\psi := \lim_{k\to\infty} \tilde u_{\psi_{n_k}} \in C([0,T])$$

which does exist because $(\tilde u_{\psi_{n_k}})$ is a Cauchy sequence in $C([0,T])$. On the one hand, defining $\mathcal{O} := \cap \mathcal{O}_{\psi_m}$, we have $\tilde u_\psi(t) = u_\psi(t)$ for any $t \in \mathcal{O}$ as well as $\mathrm{meas}([0,T]\backslash\mathcal{O}) = 0$. On the other hand, for any $t \in \mathcal{O}$, we have

$$\begin{aligned} |\tilde u_\psi(t)| &= \lim_{k\to\infty} |\tilde u_{\psi_{n_k}}| = \lim_{k\to\infty} |u_{\psi_{n_k}}| = \lim_{k\to\infty} \Big| \int_{\mathbb{R}^d} g(t)\,\psi_{n_k}\,dx \Big| \\ &\le \|g\|_{L^\infty(0,T;L^p)} \lim_{k\to\infty} \|\psi_{n_k}\|_{L^{p'}} \\ &\le \|g\|_{L^\infty(0,T;L^p)} \|\psi\|_{L^{p'}}, \end{aligned}$$

and then, by density,

$$\forall\, t \in [0,T], \qquad |\tilde u_\psi(t)| \le \|g\|_{L^\infty(0,T;L^p)} \|\psi\|_{L^{p'}}.$$

By construction, the mapping $\psi \mapsto \tilde u_\psi(t)$ is linear, and thus, it is a linear form on $L^{p'}(\mathbb{R}^d)$. In other words, for any $t \in [0,T]$, there exists $\tilde g(t,.) \in L^p(\mathbb{R}^d)$ such that

$$\tilde u_\psi(t) = \int_{\mathbb{R}^d} \tilde g(t,x)\,\psi(x)\,dx, \quad \forall\,\psi \in L^{p'}(\mathbb{R}^d).$$

All together, we have $\tilde g \in C([0,T]; L^p(\mathbb{R}^d) - w)$ and $\tilde g = g$ a.e., which is (the precise statement of) our claim.

Step 2. For $\beta \in C^1(\mathbb{R})$, $|\beta(s)| \le |s|^p$, we can proceed similarly as in the proof of Corollary 4.1 and in Step 1, and we get

$$\int_0^T u\,\chi'\,dt = \int_0^T v\,\chi\,dt, \quad \forall\,\chi \in C^1_c(0,T),$$

with

$$u := \int_{\mathbb{R}^d} \beta(g)\,dx, \quad v := \int_{\mathbb{R}^d} \beta(g)\,(\mathrm{div} a)\,dx,$$

and next, by a approximation argument, with

$$u := \int_{\mathbb{R}^d} |g|^p\,dx, \quad v := \int_{\mathbb{R}^d} |g|^p\,(\mathrm{div} a)\,dx.$$

As a consequence, there exists $\tilde u \in C([0,T])$ such that $\tilde u \equiv u$ on a measurable set $\mathcal{O}$ with $\mathrm{meas}([0,T]\backslash\mathcal{O}) = 0$. On $\mathcal{O}$, we then have $t \mapsto \|g(t)\|_{L^p}$ is uniformly continuous and $t \mapsto g(t)$ is weakly uniformly continuous. Because L^p has a strictlty convex norm, we deduce that the mapping $t \mapsto g(t)$ is strongly uniformly continuous from $\mathcal{O}$ into L^p. Again, we can extend by continuity and density the function g as a function $\tilde g \in C([0,T]; L^p)$ such that $\tilde g = g$ on $\mathcal{O}$. □

Appendix A. Fundamental results on Lebesgue spaces

We refer to [1, Chapter IV] and [3, Chapters 1 & 2], as well as the references therein, for a good introduction to the analysis of Lebesgue spaces. We give hereafter a list of classical results we make use in this chapter and possibly in the next ones.

– **Separability of $L^p(\mathbb{R}^d)$, $1 \le p < \infty$.** For any $f \in L^p(\Omega)$, $\Omega \subset \mathbb{R}^d$ open, there exists a sequence (φ_n) of $C_c^\infty(\Omega)$ such that $\varphi_n \to f$ a.e. and for the L^p norm.

– **Consequence of the strict convexity of $L^p(\mathbb{R}^d)$, $1 < p < \infty$.** For any sequence (f_n) of $L^p(\Omega)$, the two convergences $f_n \rightharpoonup f$ and $\|f_n\|_{L^p} \to \|f\|_{L^p}$ together imply $f_n \to f$ for the L^p norm.

– **de La Vallée Poussin Lemma.** For any $f \in L^1(\mathbb{R}^d)$ there exist $\phi : \mathbb{R} \to \mathbb{R}_+$ and $m : \mathbb{R}^d \to [1, \infty)$ such that

$$\int_{\mathbb{R}^d} [\phi(|f|) + |f|\, m]\, dx < \infty, \quad \phi(s)/s \to \infty \text{ as } s \to \infty, \quad m(x) \to \infty \text{ as } |x| \to \infty.$$

– **Dunford-Pettis Lemma.** Consider a sequence (f_n) of $L^1(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} [\phi(|f_n|) + |f_n|\, m]\, dx \le C,$$

for some constant $C \in \mathbb{R}_+$ and for some functions $\phi : \mathbb{R} \to \mathbb{R}_+$, $\phi(s)/s \to \infty$ as $s \to \infty$ and $m : \mathbb{R}^d \to [1, \infty)$, $m(x) \to \infty$ as $|x| \to \infty$. Then, there exists $f \in L^1(\mathbb{R}^d)$ and a subsequence (f_{n_k}) such that $f_{n_k} \rightharpoonup f$ weakly $\sigma(L^1, L^\infty)$ as $k \to \infty$.

– **Egorov Theorem.** Any sequence (f_n) such that $f_n \to f$ a.e. as $n \to \infty$ converges almost uniformaly on any ball: for any $R, \varepsilon > 0$, there exists $A \subset B_R$ such that $|B_R \backslash A| < \varepsilon$ and $\|f_n - f\|_{L^\infty(A)} \to 0$ as $n \to \infty$.

Appendix B. Complementary results

In this section we state and give a sketch of the proof of two complementary results of existence (for a larger class of equations) and uniqueness (in a L^∞ framework).

Other interesting issues such as the existence problem for the transport equation with a nonlinear RHS term or the wellposedness problem for the transport equation set in a domain $\Omega \subset \mathbb{R}^d$ (and we possibly have to add boundary conditions) will be not considered in the present notes.

B.1. **Semigroup.** In the case when $a = a(x)$ and in the same way as in chapter 2, we can deduce from the existence and uniqueness result on the linear transport equation (1.1) presented in Theorem 2.4 & Corollary 4.1 that the formula

$$(S_t g_0)(x) := g(t, x) \tag{B.1}$$

defines a C_0-semigroup on $L^p(\mathbb{R}^d)$, $1 \le p < \infty$, and on $C_0(\mathbb{R}^d)$, where g is the solution to the transport equation (1.1) associated to the initial datum g_0. We refer to chapter 4 for a precise statement and proof of this claim.

B.2. **Duhamel formula and existence for transport equation with an additional lower term.** We consider the evolution equation with source term

$$\partial_t g = \Lambda\, g + G \quad \text{in} \quad (0, \infty) \times \mathbb{R}^d, \tag{B.2}$$

with

$$(\Lambda g)(x) := -a(x) \cdot \nabla g(x) + c(x)\, g(x) + \int_{\mathbb{R}^d} b(x, y)\, g(y)\, dy$$

Denoting

$$\mathcal{A} g = c\, g + \int_{\mathbb{R}^d} b(\cdot, y)\, g(y)\, dy, \quad \mathcal{B} g = -a \cdot \nabla g,$$

we interpret that equation as a perturbation equation

$$\partial_t g = \mathcal{B} g + \tilde{G}, \quad \tilde{G} = \mathcal{A} g + G.$$

We introduce the semigroup $S_{\mathcal{B}}(t)$ as defined in (B.1), and we claim that the function

$$g(t) = S_{\mathcal{B}}(t) g_0 + \int_0^t S_{\mathcal{B}}(t - s)\, \tilde{G}(s)\, ds \tag{B.3}$$

is a solution to equation (B.2). Indeed, the semigroup $S_{\mathcal{B}}$ satisfies

$$\frac{d}{dt} S_{\mathcal{B}}(t) h = \mathcal{B} S_{\mathcal{B}}(t) h,$$

in a weak sense (that is nothing but (2.6)), and then

$$\begin{aligned} \frac{d}{dt} g(t) &= \frac{d}{dt} S_{\mathcal{B}}(t) g_0 + \int_0^t \frac{d}{dt} S_{\mathcal{B}}(t-s)\, \tilde{G}(s)\, ds + S_{\mathcal{B}}(0)\, \tilde{G}(t) \\ &= \mathcal{B}\Big\{ S_{\mathcal{B}}(t) g_0 + \int_0^t S_{\mathcal{B}}(t-s)\, \tilde{G}(s)\, ds \Big\} + \tilde{G}(t) \\ &= \mathcal{B}\, g(t) + \tilde{G}(t). \end{aligned}$$

All that computations can be justified when written in a weak sense. The method used here is nothing but the well-known variation of the constant method in ODE theory, the expression (B.3) is called the *"Duhamel formula"* and a function $g(t)$ which satisfies (B.3) (in an appropriate and meaningful functional sense) is called a *"mild solution"* to the equation (B.2).

Theorem B.1. *Assume $a \in W^{1,\infty}$, $c \in L^\infty$, $b \in L^{p'}_y(L^p_x)$, $1 \le p < \infty$. For any $g_0 \in L^p$ and $G \in L^1(0,T;L^p)$ there exists a unique mild (weak, renormalized) solution to equation* (B.2).

Proof of Theorem B.1. For any $h \in C([0,T];L^p)$, we define the mapping

$$(\mathcal{U}h)(t) := S_{\mathcal{B}}(t) g_0 + \int_0^t S_{\mathcal{B}}(t-s) \Big\{ \mathcal{A}h(s) + G(s) \Big\}\, ds,$$

and we claim that

$$\mathcal{U} : C([0,T];L^p) \to C([0,T];L^p)$$

with Lipschitz constant bounded by $C\,T$, for some constant $C \in \mathbb{R}_+$. The fact that $\mathcal{U}$ is well defined as a mapping of $C([0,T];L^p)$ is a consequence of the characterics method introduced in the second section. More precisely, for smooth functions h, c, b, g_0, G the above formula makes sense using characteristics, $g = \mathcal{U}h \in C([0,T];L^p)$ and g is a solution to an evolution PDE similar to (B.4), from what we deduce that the same is true with Lebesgue functions as considered in the statement of the theorem. Let us just explain with more details how to get the Lipschitz estimate. We consider two functions $h_1, h_2 \in C([0,T];L^p)$ and we observe that $g := \mathcal{U}h_2 - \mathcal{U}h_1$ is a solution to the transport equation

$$\text{(B.4)} \qquad \partial_t g = -a(x) \cdot \nabla g(x) + c(x)\, h(x) + \int_{\mathbb{R}^d} b(x,y)\, h(y)\, dy, \quad g(0,.) = 0,$$

where $h := h_2 - h_1$. Multiplying that equation by $p\, g\, |g|^{p-2}$, we get

$$\frac{d}{dt} \int_{\mathbb{R}^d} |g|^p \le \int_{\mathbb{R}^d} (\mathrm{div} a)_+ |g|^p + \int_{\mathbb{R}^d} p\, |c|\, |h|\, |g|^{p-1} + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} p\, |b(x,y)|\, |h(y)|\, |g(x)|^{p-1}\, dxdy.$$

We then just point out that using twice the Holder inequality and next the Young inequality

$$\begin{aligned} \int\!\!\int |b(x,y)|\, |h(y)|\, |g(x)|^{p-1}\, dxdy &\le \|g\|_{L^p}^{p/p'} \int \Big(\int |b(x,y)|^p dx \Big)^{1/p} |h(y)| dy \\ &\le \|b\|_{L^{p'}_y(L^p_x)} \|g\|_{L^p}^{p/p'} \|h\|_{L^p} \\ &\le \|b\|_{L^{p'}_y(L^p_x)} \Big(\frac{1}{p'} \|g\|_{L^p}^p + \frac{1}{p} \|h\|_{L^p}^p \Big), \end{aligned}$$

and we conclude thanks to the Gronwall lemma. Choosing T small enough, the mapping $\mathcal{U}$ is a contraction, and we can apply the Banach-Picard contraction theorem. We get the existence of a fixed point $g \in C([0,T];L^p)$, $g = \mathcal{U}g$. Proceding by induction, we obtain in that way a global mild solution to equation (B.2). □

Exercise B.2. *We consider the transport equation with source term*

$$\partial_t g = -a \cdot \nabla_x g + b\, g + G$$

where a, b and G may be time dependent functions.

(1) Write a representation formula for the solution when $G = 0$ but $a = a(t,x)$, $b = b(t,x)$.

(2) Write a representation formula for the solution when $b = 0$ but $a = a(t,x)$, $G = G(t,x)$.

Hint: Prove and try to use the Duhamel formula

$$g(t) = S_{0,t} g_0 + \int_0^t S_{s,t} G(s)\, ds.$$

(3) Write the general representation formula.

B.3. **Explicit formula by the characteristics method.** We consider the transport equation

$$\partial_t f + a \cdot \nabla f + c\, f = G, \quad f(0) = f_0, \tag{B.5}$$

with $a = a(t,x)$, $c = c(t,x)$ and $G = G(t,x)$ smooth functions. With the notation of section 2.1 on the flow associated to the associated ODE (2.1), if a smooth solution f to the above equation does exist, we must have

$$\frac{d}{dt}\Big[f(t,\Phi_t(x))\, e^{\int_0^t c(s,\Phi_s(x))\, ds} \Big] = G(t,\Phi_t(x))\ e^{\int_0^t c(s,\Phi_s(x))\, ds},$$

from which we deduce

$$f(t,\Phi_t(x)) = f_0(x)\, e^{-\int_0^t c(\tau,\Phi_\tau(x))\, d\tau} + \int_0^t G(s,\Phi_s(x))\, e^{-\int_s^t c(\tau,\Phi_\tau(x))\, d\tau}\, ds.$$

Using that $\Phi_t^{-1} = \Phi_{0,t}$ and the semigroup property of $\Phi_{s,t}$, we deduce that the solution to the transport equation (B.5) is given through the explicit formula

$$f(t,x) := f_0(\Phi_{0,t}(x))\, e^{-\int_0^t c(\tau,\Phi_{\tau,t}(x))\, d\tau} + \int_0^t G(s,\Phi_{s,t}(x))\, e^{-\int_s^t c(\tau,\Phi_{\tau,t}(x))\, d\tau}\, ds. \tag{B.6}$$

B.4. **Duality and uniqueness in the case $p = \infty$.**

Theorem B.3. *Assume $a = a(x) \in W^{2,\infty}$. For any $g_0 \in L^\infty$, there exists at most one weak solution $g \in L^\infty((0,T)\times\mathbb{R}^d)$ to the transport equation* (1.1).

Proof of Theorem B.3. Since the equation is linear, we only have to prove that the unique weak solution $g \in L^\infty((0,T)\times\mathbb{R}^d)$ associated to the initial datum $g_0 = 0$ is $g = 0$.
By definition, for any $\psi \in C_c^1([0,T]\times\mathbb{R}^d)$, there holds

$$\int_0^T \int_{\mathbb{R}^d} g\, L^*\psi\, dxdt = -\int_{\mathbb{R}^d} g(T)\, \psi(T)\, dx$$

with $L^*\psi := -\partial_t\psi - \mathrm{div}(a\,\psi)$. We claim that for any $\Psi \in C_c^1((0,T)\times\mathbb{R}^d)$ there exists a function $\psi \in C_c^1([0,T]\times\mathbb{R}^d)$ such that

$$L^*\psi = \Psi, \quad \psi(T) = 0. \tag{B.7}$$

If we accept that fact, we obtain

$$\int_0^T \int_{\mathbb{R}^d} g\, \Psi\, dxdt = 0 \quad \forall\, \Psi \in C_c^1((0,T)\times\mathbb{R}^d),$$

which in turns implies $g = 0$ and that ends the proof.

Here we can solve easily the backward equation (B.7) thanks to the characterics method which leads to an explicit representation formula. In order to make the discussion simpler, we exhibit that formula for the associated forward problem (we do not want to bother with backward time, but one can pass from a formula to another just by changing time $t \to T - t$). We then consider the equation

$$\partial_t\psi + a\cdot\nabla\psi + c\,\psi = \Psi, \quad \psi(0) = 0,$$

with $c := \mathrm{div}\, a$. As in the preceding section and observing that $\Phi_t^{-1} = \Phi_{-t}$ because the associated ODE $\dot{x} = a(x)$ is time autonomous, we introduce the function

$$\psi(t,x) := \int_0^t \Psi(s,\Phi_{s-t}(x))\, e^{-\int_s^t c(\Phi_{\tau-t}(x))\, d\tau}\, ds.$$

It is clear that ψ defined by the above formula is the solution to our dual problem from which we get (reversing time) the solution to (B.7) we were trying to find. $\square$

Appendix C. Transport equation in conservative form

In this section we extend the existence and uniqueness theory to a bounded Radon measures framework, and then a probability measures framework, for the important class of transport equations which may be writen in a *"conservative form"*.

More precisely, we consider a time depend vectors field $a = a(t, y) : (0, T) \times \mathbb{R}^d \to \mathbb{R}^d$ of class $C^1 \cap \mathrm{Lip}$ and we note L the Lipschitz constant of a in the second variable:

$$\forall\, t \in [0, T],\ \forall\, x, y \in \mathbb{R}^d \qquad |a(t, x) - a(t, y)| \le L\, |x - y|.$$

We are interested in the transport equation in conservative form

$$\frac{\partial f}{\partial t} + \nabla\,(a\, f) = 0 \quad \text{in} \quad \mathcal{D}'((0, T) \times \mathbb{R}^d), \tag{C.8}$$

where $f = f(t, dx) = df_t(x)$ is a mapping from $(0, T)$ into the space of bounded Radon measures $M^1(\mathbb{R}^d)$ or the space of probability measures $\mathscr{P}(\mathbb{R}^d)$. We recall that

$$M^1(\mathbb{R}^d) := \{f \in (C_c(\mathbb{R}^d))';\ \|f\|_{TV} := \sup_{\|\varphi\|_\infty \le 1} |\langle f, \varphi\rangle| < \infty\}$$

and

$$\mathscr{P}(\mathbb{R}^d) := \{f \in (C_c(\mathbb{R}^d))';\ f \ge 0,\ \langle f, 1\rangle = 1\} \subset M^1(\mathbb{R}^d).$$

We also define

$$\mathscr{P}_1(\mathbb{R}^d) := \{f \in \mathscr{P}(\mathbb{R}^d);\ \langle f, |x|\rangle < 1\}$$

and the Monge-Kantorovich-Wasserstein distance on $\mathscr{P}_1(\mathbb{R}^d)$ by

$$\forall\, f, g \in \mathscr{P}_1(\mathbb{R}^d), \quad W_1(f, g) = \|f - g\|_{(Lip)'} := \sup_{\varphi \in C^1; \|\nabla\varphi\|_\infty \le 1} \langle f - g, \varphi\rangle.$$

We point out that $W_1(f, g)$ is well-defined and finite for any $f, g \in \mathscr{P}_1(\mathbb{R}^d)$ because

$$|\langle f - g, \varphi\rangle| = |\langle f - g, \varphi - \varphi(0)\rangle| \le \langle |f| + |g|, |x|\rangle < \infty,$$

for any $\varphi \in C^1(\mathbb{R}^d)$, $\|\nabla\varphi\|_\infty \le 1$, and that $W_1(f, g) = 0$ implies $f = g$ because $C^1_c(\mathbb{R}^d) \subset C_c(\mathbb{R}^d)$ with continuous and dense embedding.

We finally denote as $M^1(\mathbb{R}^d) - w$ the weak topology on $M^1(\mathbb{R}^d)$ defined through the weak convergence: we say that a sequence (f_n) in $M^1(\mathbb{R}^d)$ weakly converges to $f \in M^1(\mathbb{R}^d)$, we write $f_n \rightharpoonup f$ in $M^1(\mathbb{R}^d) - w$, if

$$\int \varphi\, f_n \to \int \varphi\, f \quad \forall\, \varphi \in C_c(\mathbb{R}^d).$$

We emphasize that for $f_n, f \in \mathscr{P}_1(\mathbb{R}^d)$ the following holds: the convergence $W_1(f_n, f) \to f$ implies that $f_n \rightharpoonup f$; the convergence $f_n \rightharpoonup f$ implies the *"tightness"* of sequence (f_n) and then the (stronger) convergence

$$\int \varphi\, f_n \to \int \varphi\, f \quad \forall\, \varphi \in C_b(\mathbb{R}^d).$$

Definition C.4. (Image Measure). *Let $(E, \mathcal{E}, \mu)$ be a measure space, F be a set and $\Phi : E \to F$ a mapping. We define the σ-algebra $\mathcal{F}$ on F by $\mathcal{F} := \{A \subset F;\ \Phi^{-1}(A) \in \mathcal{E}\}$ (it is the smallest σ-algebra on F for which Φ is measurable) and we define the measure ν on $\mathcal{F}$ by $\forall\, A \in \mathcal{F}$, $\nu[A] := \mu[\Phi^{-1}(A)]$. We denote $\nu = \Phi \sharp \mu$ and we say that ν is the image measure of μ by Φ. By definition, for any measurable function $\varphi : (F, \mathcal{F}) \to \mathbb{R}_+$, we have*

$$\int_F \varphi\, d(\Phi \sharp \mu) = \int_E \varphi \circ \Phi\, d\mu.$$

Theorem C.5. (Characterictics). *For any $f_0 \in M^1(\mathbb{R}^d)$, the unique solution $f \in C([0, T]; M^1(\mathbb{R}^d) - w)$ to the transport equation* (C.8) *associated to the initial datum f_0 is given by*

$$f(t, .) = \Phi_t \sharp f_0 \quad \forall\, t \in [0, T], \tag{C.9}$$

where Φ_t denotes the flow associated to the ODE of characterictics defined in section 2.1. Moreover, given two initial data f_0, $g_0 \in \mathscr{P}_1(\mathbb{R}^d)$, the corresponding solutins $f, g \in C([0, \infty); M^1(\mathbb{R}^d) - w)$ to the transport equation (C.8) *satisfy*

$$\forall\, t \in [0, T] \qquad W_1(f_t, g_t) \le e^{L\, t}\, W_1(f_0, g_0). \tag{C.10}$$

Remark C.6. *For a deterministic system associated to a vectors field a, we say that* (2.1) *is a Lagragian description of the dynamics while* (C.8) *is an Eulerian description. The formula* (C.9) *shows the equivalence between these two points of view.*

Proof of Theorem C.7. Step 1. We prove that $f(t) := \Phi_t \sharp f_0$ is a solution to (C.8). Fix $\varphi \in \mathcal{D}(\mathbb{R}^d)$ and just compute

$$\begin{aligned}\Big\langle \frac{\partial f}{\partial t}, \varphi \Big\rangle = \frac{d}{dt} \int_{\mathbb{R}^d} \varphi\, f(t) &= \frac{d}{dt} \int_{\mathbb{R}^d} \varphi(\Phi_t(y_0))\, f_0(dy_0) \\ &= \int_{\mathbb{R}^d} (\nabla \varphi)(\Phi_t(y_0)) \cdot \frac{d}{dt}(\Phi_t(y_0))\, f_0(dy_0) \\ &= \int_{\mathbb{R}^d} (\nabla \varphi)(\Phi_t(y_0)) \cdot a(t, \Phi_t(y_0))\, f_0(dy_0) \\ &= \int_{\mathbb{R}^d} (\nabla \varphi)(y) \cdot a(t,y)\, f(t,dy) \\ &= -\Big\langle \nabla(a\, f), \varphi \Big\rangle,\end{aligned}$$

in the sense of duality in $\mathcal{D}'((0,T))$. That means that (C.8) holds in the sense of duality in $(\mathcal{D}(0,T) \times \mathcal{D}(\mathbb{R}^d))'$, and thanks to a density argument, in the sense of duality in $\mathcal{D}'((0,T) \times \mathbb{R}^d)$.

Step 2. We establish the uniqueness of the solution. Because the equation is linear, we just have to prove that $f_T = 0$ if $f_0 = 0$. We argue by duality. We define the backward flow Ψ_t by setting $\Psi_t(z) = z(t)$, where $z(t)$ is the solution to the ODE

$$z'(t) = a(t, z(t)), \quad z(T) = z.$$

For a given function $\varphi_T \in C^1_c(\mathbb{R}^d)$, we define $\varphi(t,y) := \varphi_T(\Psi_t^{-1}(y)) \in C^1_b([0,T] \times \mathbb{R}^d)$. From the implicit equation $\varphi(t, z(t)) = \varphi_T(z)$, we obtain

$$\begin{aligned} 0 &= \frac{d}{dt}\left[\varphi(t,z(t))\right] = (\partial_t \varphi)(t, z(t)) + (\nabla \varphi)(t, z(t))\, z'(t) \\ &= \left[\partial_t \varphi + a \cdot \nabla \varphi\right](t, z(t)),\end{aligned}$$

and the following transport equation holds (in the sense of classical differential calculus)

$$\partial_t \varphi + a \cdot \nabla \varphi = 0 \quad \text{in} \quad [0,T] \times \mathbb{R}^d.$$

We then compute

$$\begin{aligned}\frac{d}{dt}\langle f_t, \varphi_t \rangle &= \int_{\mathbb{R}^d} [\partial_t \varphi(x)]\, f_t(dx) + \langle \partial_t f_t, \varphi_t \rangle \\ &= \int_{\mathbb{R}^d} \left[a \cdot \nabla \varphi_t(x)\right]\, f_t(dx) + \int_{\mathbb{R}^d} \left[-a \cdot \nabla \varphi_t(x)\right]\, f_t(dx) = 0.\end{aligned}$$

It implies

$$\int_{\mathbb{R}^d} \varphi_T(x)\, f_T(dx) = \int_{\mathbb{R}^d} \varphi_0(x)\, f_0(dx) = 0,$$

for any $\varphi_T \in C^1_b(\mathbb{R}^d)$, which means $f_T \equiv 0$.

Step 3. We start recalling that the flow Φ_t satisfies

$$\forall\, t \in [0,T] \quad \|\nabla_y \Phi_t\|_\infty \le e^{tL}. \tag{C.11}$$

Indeed, for $x_0, y_0 \in \mathbb{R}^d$, the two solutions solutions $x_t = \Phi_t(x_0)$, $y_t = \Phi_t(y_0)$ satisfy

$$\frac{d}{dt}|x_t - y_t| \le |\dot{x}_t - \dot{y}_t| \le |a(t,x_t) - a(t,y_t)| \le L\, |x_t - y_t|,$$

and we conclude thanks to the Gronwall lemma.

Now, thanks to Theorem C.7 and by definition of W_1 and $\sharp$, we have

$$\begin{aligned} W_1(f_t,g_t) &= W_1(\Phi_t\sharp f_0,\Phi_t\sharp g_0) \\ &= \sup_{\|\nabla\varphi\|\le 1}\int_{\mathbb{R}^d}\varphi\, d(\Phi_t\sharp f_0-\Phi_t\sharp g_0) \\ &= \sup_{\|\nabla\varphi\|\le 1}\int_{\mathbb{R}^d}\varphi\circ\Phi_t\, d(f_0-g_0) \\ &\le \sup_{\|\nabla\varphi\|\le 1}\|\nabla(\varphi\circ\Phi_t)\| \sup_{\|\nabla\psi\|\le 1}\int_{\mathbb{R}^d}\psi\, d(f_0-g_0) \\ &\le \|\nabla\Phi_t\|\, W_1(f_0,g_0) \end{aligned}$$

and we conclude thanks to (C.11). □

Remark C.7. *1. When the solution has a density with respect to the Lebesgue measure $f_t(dy) = g_t(y)\,dy$ with $g_t\in L^1(\mathbb{R}^d)$, the change of variables theorem in the definition of the image measure implies*

$$g_0(x) = g_t(\Phi_t(x))\, det(D\Phi_t(x)).$$

In particular, $g_0(y)$ is not equal to $g_t(\Phi_t(y))$ in general.

2. However, one can classically show that $J(t,y) := det(D\Phi_t(y))$ satisfies the Liouville equation

$$\frac{d}{dt}J(t,y) = [(div\, a)(t,\Phi_t(y))]\, J(t,y), \quad J(0,y) = det(Id) = 1.$$

In the case of a free-divergence verctors field a, namely $div\, a = 0$, we deduce of it the incompressibility of the flow $J(t,y)\equiv 1$. In that case, $g_t(\Phi_t(x)) = g_0(x)$.

3. When $div\, a = 0$, we can obtain $g_t(\Phi_t(y)) = g_0(y)$ (and thus recover the incompressibility of the flow) in a maybe much simpler way. We come back to the uniqueness argument in the proof of Theorem C.7. We define $h(t,z) := g_0(\Phi_t^{-1}(z))$ for $g_0\in C^1_b(\mathbb{R}^d)$, and we compute

$$0 = \frac{d}{dt}\left[h(t,y(t))\right] = \left[\partial_t h + a\cdot\nabla h\right](t,y(t)).$$

We deduce

$$0 = \partial_t h + a\cdot\nabla h = \partial_t h + \nabla(a\, h), \quad h(0,.) = g_0.$$

From the uniqueness of the solution, there holds $h = g$, and then

$$g(t,\Phi_t(x)) = h(t,\Phi_t(x)) = g_0(x) = g(t,\Phi_t(x))\, J(t,x).$$

Choosing $g_0\to 1$, we get $J\equiv 1$.

4. For $f_0 = \delta_{x_0}$, we have

$$\Phi_t\,\sharp\delta_{x_0} = \delta_{\Phi_t(x_0)}.$$

Indeed, for any test function $\varphi\in C_b(\mathbb{R}^d)$, we write

$$\begin{aligned} \int_{\mathbb{R}^d}\varphi(x)(\Phi_t\,\sharp\delta_{x_0})(dx) &= \int_{\mathbb{R}^d}\varphi(\Phi_t(x))\,\delta_{x_0}(dx) \\ &= \varphi(\Phi_t(x_0)) = \int_{\mathbb{R}^d}\varphi(x)\,\delta_{\Phi_t(x_0)}(dx). \end{aligned}$$

Lemma C.8. *For an initial $f_0\in L^1(\mathbb{R}^d)$, the solution $f\in C([0,T];M^1(\mathbb{R}^d)-w)$ to the conservative transport equation* (C.8) *satisfies*

(1) the mass conservation property:

$$\int_{\mathbb{R}^d} f(t,x)\,dx = \int_{\mathbb{R}^d} f_0\,dx \quad \forall\, t\in[0,T];$$

(2) the L^1 stability property:

$$\int_{\mathbb{R}^d}|f(t,x)|\,dx = \int_{\mathbb{R}^d}|f_0|\,dx \quad \forall\, t\in[0,T].$$

Proof of Lemma C.8. We only prove (2), point (1) can be proved similarly. We write

$$\partial_t f = -a\cdot\nabla f - (\mathrm{div} a)\, f,$$

which has an unique solution $f \in C([0,\infty); L^1_{loc}(\mathbb{R}^d))$ thanks to Theorem 2.4 or its variant Theorem B.1. For any renormalizing function $\beta \in C^1 \cap W^{1,\infty}$, we have

$$\partial_t \beta(f) = -a \cdot \nabla \beta(f) - (\mathrm{div} a)\, f\, \beta'(f) \quad \text{in } \mathcal{D}'((0,T))$$

and then

$$\frac{d}{dt} \int_{\mathbb{R}^d} \beta(f)\, \chi = \int_{\mathbb{R}^d} \big\{ \chi (\mathrm{div} a)\, [\beta(f) - f\, \beta'(f)]\, dx + (a \cdot \nabla \chi)\, \beta(f) \big\}$$

for any $\chi \in \mathcal{D}(\mathbb{R}^d)$. We first take $\beta_\varepsilon(s) = s^2/2$ for $|s| \le \varepsilon$, $\beta_\varepsilon(s) = |s| - \varepsilon/2$ for $|s| \ge \varepsilon$, and observing that $|\beta_\varepsilon(s) - s\, \beta_\varepsilon'(s)| \le \varepsilon \ \forall\, s \in \mathbb{R}$, $\forall\, \varepsilon \in (0,1)$ as well as $\beta_\varepsilon(s) \to |s|$ as $\varepsilon \to 0$, we may pass to the limit $\varepsilon \to 0$ and we get

$$\Big| \int_{\mathbb{R}^d} (|f(t)| - |f_0|)\, \chi - \int_0^t \int_{\mathbb{R}^d} (a \cdot \nabla \chi)\, |f(s)|\, dx ds \Big|$$
$$= \lim_{\varepsilon \to 0} \int_0^t \int_{\mathbb{R}^d} \big\{ \chi (\mathrm{div} a)\, [\beta_\varepsilon(f) - f\, \beta_\varepsilon'(f)]\, dx ds \big\} = 0.$$

Taking $\chi(x) = \psi(x/R)$ with $\psi \in \mathcal{D}(\mathbb{R}^d)$, $\psi \equiv 1$ on $B(0,1)$, $0 \le \psi \le 1$, $\mathrm{supp}\, \psi \subset B(0,2)$, we may pass to the limit $R \to \infty$, and we get

$$\int_{\mathbb{R}^d} |f(t)|\, dx = \int_{\mathbb{R}^d} |f_0| + \lim_{R \to \infty} \frac{1}{R} \int_0^t \int_{\mathbb{R}^d} a(s,x) \cdot \nabla \psi(x/R))\, |f(s)|\, dx ds = \int_{\mathbb{R}^d} |f_0|,$$

so that the L^1 stability property is proved. □

Appendix D. Exercices

Exercise D.1. *We define the weak distance on $L^1(\mathbb{R}^d)$ by*

$$D(f,g) := \sup_{\|\varphi\|_{W^{1,\infty}} \le 1} \int_{\mathbb{R}^d} (f-g)\, \varphi\, dx,$$

for any $f, g \in L^1(\mathbb{R}^d)$. The topology induced by the distance D on any fixed ball of $L^1(\mathbb{R}^d)$ is the weak $*$ topology $\sigma(M^1(\mathbb{R}^d), C_0(\mathbb{R}^d))$. Prove that any solution $f \in C([0,T]; L^1(\mathbb{R}^d))$ to the transport equation (1.1) *is Lipschitz continuous for the distance D.*

Exercise D.2. *Consider a weak solution $g \in L^1_{loc}([0,T] \times \mathbb{R}^d)$ to the transport equation* (3.1) *associated to an initial datum $g_0 \in L^1_{loc}(\mathbb{R}^d)$ and a source term $G \in L^1_{loc}([0,T] \times \mathbb{R}^d)$ in the following sense:*

$$\int_0^T \int_{\mathbb{R}^d} g\, L^* \varphi = \int_{\mathbb{R}^d} g_0 \varphi(0,.) + \int_0^T \int_{\mathbb{R}^d} \varphi\, G,$$

for any $\varphi \in C^1_c([0,T) \times \mathbb{R}^d)$. Prove that the approximation sequence (g_ε) introduced in Theorem 3.2 is a Cauchy sequence in $C([0,T]; L^1(B_R))$, for any $R > 0$, and deduce that $g \in C([0,T]; L^1_{loc}(\mathbb{R}^d))$. (Hint. Write the renormalized formulation of the equation satisfied by $g_\varepsilon - g_{\varepsilon'}$ for a renormalizing function $\beta(s) \sim |s|$, a test function $\mathbf{1}_{B(0,R)} \le \chi \in \mathcal{D}(\mathbb{R}^d)$, Corollary 4.3).

Exercise D.3. *Consider a renormalized solution $g \in L^1_{loc}([0,T] \times \mathbb{R}^d)$ in the sense of Definition 3.1 to the transport equation* (1.1) *associated to the initial datum $g_0 \in L^p(\mathbb{R}^d)$, $1 < p < \infty$, and such that $\beta(g) \in L^1((0,T) \times \mathbb{R}^d)$ for any $\beta \in \mathcal{A}$, the class of renormalizing functions $\beta \in C^1(\mathbb{R})$ such that $\beta(0) = 0$ and $\beta' \in C_c(\mathbb{R}\backslash\{0\})$.*

(1) Prove that for any given $\beta \in \mathcal{A}$, the function

$$t \mapsto \int_{\mathbb{R}^d} \beta(g)\, dx \quad \textit{is continuous},$$

up to the modification of g on a negligible set of times. (Hint. Repeat Step 1 of the proof of Corollary 4.4).

(2) Deduce that $g \in L^\infty(0,T; L^p(\mathbb{R}^d))$. (Hint. Repeat the proof of Corollary 4.3).

(3) Deduce that $g \in C([0,T]; L^p(\mathbb{R}^d))$.

(4) We define $L^0((0,T) \times \mathbb{R}^d)$ as the set of measurable functions g on $(0,T) \times \mathbb{R}^d$ such that $meas(|g| \ge \varepsilon) < \infty$ for any $\varepsilon > 0$. Consider a weak solution $g \in L^1_{loc}([0,T] \times \mathbb{R}^d) \cap L^0((0,T) \times \mathbb{R}^d)$ to the transport equation (1.1) *associated to an initial datum $g_0 \in L^p(\mathbb{R}^d)$, $1 < p < \infty$, in the sense of equation* (2.5)*. Prove that $g \in C([0,T]; L^p(\mathbb{R}^d))$.*

Exercise D.4. *Consider a renormalized solution $g \in L^\infty(0,T;L^1(\mathbb{R}^d))$ in the sense of Definition 3.1 to the transport equation* (1.1) *associated to the initial datum $g_0 \in L^1(\mathbb{R}^d)$.*

(1) Prove that $g_0\,\omega \in L^1(\mathbb{R}^d)$ for a smooth and positive function ω such that $\langle x\rangle|\nabla\omega| \le \omega$ and $\omega(x) \to \infty$ when $|x| \to \infty$. (Hint. That is the de La Vallée Poussin Lemma).

(2) Prove that $g\,\omega \in L^\infty(0,T;L^1(\mathbb{R}^d))$.

(3) Using the continuity property $g \in C([0,T];L^1_{loc}(\mathbb{R}^d))$ established in Exercise D.2, prove that $g \in C([0,T];L^1(\mathbb{R}^d))$.

We propose an alternative proof by following a line of arguments developed in Exercise D.3.

(4) Prove that $\Phi(g_0) \in L^1(\mathbb{R}^d)$ for a smooth and positive function Φ such that $\Phi(s)/|s| \to \infty$ when $s \to \infty$. (Hint. That is again the de La Vallée Poussin Lemma).

(5) Prove that $\Phi(g) \in L^\infty(0,T;L^1(\mathbb{R}^d))$ and deduce that $g \in C([0,T];L^1(\mathbb{R}^d) - w)$. (Hint. Use the Dunford-Pettis Lemma).

(6) Prove that there exists a one-to-one function $\beta \in C^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ such that $t \mapsto \beta(g(t,.))$ and $t \mapsto \beta(g(t,.))^2$ are weakly continuous functions (for instance in $L^2_{loc}(\mathbb{R}^2)$). (Hint. Repeat some arguments developed during the proof of Corollary 4.4, see also Exercise D.3).

(7) Deduce that $t \mapsto g(t,.)$ is continuous in the a.e. sense in $\mathbb{R}^d$ and next that $g \in C([0,T];L^1(\mathbb{R}^d))$. (Hint. Use the Egorov Theorem in the last step).

Exercise D.5. *We define*

$$J(t,x) := \exp\Big(\int_0^t (div\,a)(s,\Phi_s(x))\,ds\Big).$$

(1) Show that for $f_0 \in C^1_c(\mathbb{R}^d)$, the function f defined implicitely by

$$f(t,\Phi_t(x))J(t,x) = f_0(x) \quad \forall\, t \in [0,T],\ \forall\, x \in \mathbb{R}^d,$$

is the (unique) solution to the transport equation in divergenece form (C.8) *associated to the inital datum f_0.*

(2) Show or use Liouville Theorem $J(t,.) = det\,D\Phi_t$, in order to get an alternative proof of Lemma C.8.

Exercise D.6. *(1) Prove the existence of a (weak in the sense of distributions) solution $f \in L^\infty(0,T;L^2(\mathbb{R}^d))$ to the first order equation*

$$\partial_t f = a(x)\cdot\nabla f(x) + c(x)\,f(x) + \int_{\mathbb{R}^d} b(y,x)\,f(y)\,dy,$$

with the usual assumptions on a, c, b by the vanishing viscosity method: that is by passing to the limit in the familly of equation

$$\partial_t f_\varepsilon = \varepsilon\Delta f_\varepsilon + a\cdot\nabla f_\varepsilon + c\,f_\varepsilon + \int_{\mathbb{R}^d} b(y,x)\,f_\varepsilon(y)\,dy,$$

as $\varepsilon \to 0$.

(2) Prove that the above solution is a renomalized solution.

(3) Prove that $f \in C([0,T];L^2(\mathbb{R}^d))$.

Exercise D.7. *Consider the transport equation*

$$Lf = \partial_t f + a\cdot\nabla f = 0$$

with a Lipschitz vector field $a : \mathbb{R}^d \to \mathbb{R}^d$.

(1) Prove the existence of a weak solution $f \in L^\infty(0,T;L^p)$ for any $T > 0$ and any initial datum $f_0 \in L^p(\mathbb{R}^d)$, $p \in (1,\infty]$. Why $f \in C([0,\infty);L^p)$ if $p \in (1,\infty)$?

(2) Prove the uniqueness of the solution in the case $p \in (1,\infty)$.

(3) Prove the existence and uniqueness of the solution in the case $p = 1$. (Hint. Use the Dunford-Pettis Theorem).

(4) Prove the uniqueness of the solution in the case $p = \infty$. (Hint. Use a duality argument).

(4a) Consider $f \in L^\infty((0,T)\times\mathbb{R}^d) \cap C([0,T];L^1_{loc}(\mathbb{R}^d)$ a weak solution to the transport equation with vanishing initial datum and prove that there exits (f_ε) a sequence in $C^1([0,T)\times\mathbb{R}^d)$ such that $f_\varepsilon \to f$ a.e., (f_ε) is bounded in $L^\infty((0,T)\times\mathbb{R}^d)$ and

$$Lf_\varepsilon = r_\varepsilon \to 0 \quad in \quad L^1_{loc}([0,T]\times\mathbb{R}^d).$$

(4b) Consider $\Psi \in L^1((0,T)\times\mathbb{R}^d)$ *and prove that there exists a sequence of functions* (ψ_ε) *in* $C^1([0,T]\times\mathbb{R}^d)$ *and a function* $\psi \in C([0,T];L^1(\mathbb{R}^d))$ *such that* $\psi_\varepsilon \to \psi$ *a.e. and in* $C([0,T];L^1(\mathbb{R}^d))$, $\psi_\varepsilon(T) = 0$ *and*

$$L^*\psi_\varepsilon - \Psi = R_\varepsilon \to 0 \quad in \quad L^1_{loc}([0,T]\times\mathbb{R}^d),$$

(4c) For a function $\chi \in \mathcal{D}(\mathbb{R}^d)$, $0 \le \chi \le 1$, $\chi(0) = 1$, *and a real number* $R > 0$, *define* $\chi_R(x) := \chi(x/R)$. *For any* $\varepsilon, R > 0$, *establish that*

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^d} f_\varepsilon L^*(\psi_\varepsilon\chi_R) &= \int_0^T\int_{\mathbb{R}^d} r_\varepsilon\psi_\varepsilon\chi_R \\ &= \int_0^T\int_{\mathbb{R}^d} f_\varepsilon(\Psi + R_\varepsilon)\chi_R - \int_0^T\int_{\mathbb{R}^d} f_\varepsilon\psi_\varepsilon\, div(a\chi_R).\end{aligned}$$

Passing to the limit $\varepsilon \to 0$ *and next* $R \to \infty$ *in the last identity, prove that*

$$\int_0^T\int_{\mathbb{R}^d} f\Psi = 0$$

and conclude.

Appendix E. References

The main result of the chapter, namely Theorem 3.2, is due to R. DiPerna and P.-L. Lions and has been established in [2].

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