CHAPTER 6 - SEMIGROUP AND LONGTIME BEHAVIOUR

Contents

In this chapter we introduce two techniques which are useful for analyzing the longtime asymptotic of evolution PDEs. On the one hand, it is the use of splitting tool though the itarated Duhamel formula. On the other hand, it is an introduction of the analysis of stochastic semigroup following Harris-Meyn-Tweedie type approach.

1. Weighted L^1 decay through semigroups factorization technique

In this section, we establish the following weighted L^1 decay through a semigroups factorization technique and the already known weighted L^2 decay (consequence and equivalent to the Poincaré inequality) prsented in chapter 3. We consider the Fokker-Planck equation

$$\frac{\partial}{\partial t} f = \mathcal{L}\, f = \Delta f + \nabla \cdot (f\, \nabla V) \quad \text{in } (0,\infty)\times\mathbb{R}^d \tag{1.1}$$

$$f(0,x) = f_0(x) \quad \text{on } \mathbb{R}^d, \tag{1.2}$$

and we assume that the *"confinement potential"* V is the harmonic potential

$$V(x) := \frac{|x|^2}{2} + V_0, \quad V_0 := \frac{d}{2}\log 2\pi.$$

We recall

$$\|f(t,.) - \langle f_0\rangle\, G\|_E \le e^{-\lambda_P t}\, \|f_0 - \langle f_0\rangle\, G\|_E \quad \text{as} \quad t\to\infty, \tag{1.3}$$

where $\|\cdot\|_E$ stands for the norm of the Hilbert space $E := L^2(G^{-1})$ defined by

$$\|f\|_E^2 := \int_{\mathbb{R}^d} f^2\, G^{-1}\, dx$$

and λ_P is the best (larger) constant in the Poincaré inequality.

Theorem 1.1. *For any $a \in (-\lambda_P, 0)$ and for any $k > k^* := \lambda_P$ there exists $C_{k,a}$ such that for any $\varphi \in L^1_k$, the associated solution f to the Fokker-Planck equation* (1.1)-(1.2) *satisfies*

$$\|f - \langle\varphi\rangle\, G\|_{L^1_k} \le C_{k,a}\, e^{a\,t}\, \|\varphi - \langle\varphi\rangle\, G\|_{L^1_k}. \tag{1.4}$$

A refined version of the proof below shows that the same estimate holds with $a := -\lambda_P$.

Proof of Theorem 1.1. In order to simplify a bit the presentation, we only present the proof in the case of the dimension $d \le 3$, but the same arguments can be generalized to any dimension $d \ge 1$.
Step 1. The splitting. We introduce the splitting $\mathcal{L} = \mathcal{A} + \mathcal{B}$ with

$$\mathcal{B} f := \Delta f + \nabla \cdot (f\, x) - M\, f\, \chi_R, \quad \mathcal{A} f := M\, f\, \chi_R,$$

where $\chi_R(x) = \chi(x/R)$, $\chi \in \mathcal{D}(\mathbb{R}^d)$, $\mathbf{1}_{B_1} \le \chi \le \mathbf{1}_{B_2}$, and where $R, M > 0$ are two real constants to be chosen later. We define, in any Banach space $\mathcal{X}$ such that $G \in \mathcal{X} \subset L^1$, the projection operator

$$\Pi f := \langle f \rangle\, G,$$

which thus satisfies $\Pi^2 = \Pi$ and $\Pi \in \mathcal{B}(\mathcal{X})$. When $S_\mathcal{L}$ is well defined as a semigroup in $\mathcal{X}$, we have

$$S_\mathcal{L}(I - \Pi) = (I - \Pi)\, S_\mathcal{L} = (I - \Pi)\, S_\mathcal{L}(I - \Pi) \tag{1.5}$$

as a consequence of the projection property $(I - \Pi)^2 = (I - \Pi)^2$, of the facts that G is a stationary solution to the Fokker-Planck equation and that the mass is preserved by the associated flow. Now, iteraration the Duhamel formula

$$S_\mathcal{L} = S_\mathcal{B} + S_\mathcal{L} * \mathcal{A} S_\mathcal{B},$$

we have

$$S_\mathcal{L} = S_\mathcal{B} + S_\mathcal{B} * (\mathcal{A} S_\mathcal{B}) + S_\mathcal{L} * (\mathcal{A} S_\mathcal{B}) * (\mathcal{A} S_\mathcal{B}). \tag{1.6}$$

The two identities (1.5) and (1.6) together and using the shorthand $\Pi^\perp = I - \Pi$, we have

$$S_\mathcal{L} \Pi^\perp \;=\; \Pi^\perp S_\mathcal{B} \Pi^\perp + \Pi^\perp S_\mathcal{B} * (\mathcal{A} S_\mathcal{B}) \Pi^\perp + S_\mathcal{L} \Pi^\perp * (\mathcal{A} S_\mathcal{B}) * (\mathcal{A} S_\mathcal{B} \Pi^\perp) =: \sum_{i=1}^{3} \mathcal{T}_i(t).$$

In order to get (1.4), we will establish that

$$S_\mathcal{B}(t) : L^1_k \to L^1_k, \text{ with bound } \mathcal{O}(e^{a' t}), \ \forall\, t \ge 0, \ \forall\, a' > a^*, \ \forall\, k > k^*, \tag{1.7}$$

$$S_\mathcal{B}(t) : L^1_K \to L^2_K, \text{ with bound } \mathcal{O}\big(\frac{e^{a' t}}{t^{3/4}}\big), \ \forall\, t > 0, \ \forall\, a' > a^*, \ \forall\, K > K^*, \tag{1.8}$$

$$\mathcal{A} : L^1_k \to L^1_K, \ \mathcal{A} : L^2_K \to L^2(G^{-1}), \ \forall\, K > k^*, \ \forall\, k > k^*, \tag{1.9}$$

with $K^* := \lambda_P + d/2$. We also recall that

$$S_\mathcal{L}(t) \Pi^\perp : L^2(G^{-1}) \to L^2(G^{-1}), \text{ with bound } \mathcal{O}(e^{-\lambda_P t}), \ \forall\, t \ge 0, \tag{1.10}$$

which is nothing but (1.3). We finally observe that

$$u * w(t) = \mathcal{O}(e^{a t}) \text{ and } u * v * w(t) = \mathcal{O}(e^{a t}), \quad \forall\, t \ge 0, \ \forall\, a > a^*, \tag{1.11}$$

if

$$u(t) = \mathcal{O}(e^{a' t}), \quad v(t) = \mathcal{O}\big(\frac{e^{a' t}}{t^{3/4}}\big), \quad w(t) = \mathcal{O}(e^{a' t}), \quad \forall\, t > 0, \ \forall\, a' > a^*. \tag{1.12}$$

The first estimate in (1.11) is obtained by writing

$$u * w(t) = \int_0^t u(s) w(t-s)\, ds \lesssim \int_0^t e^{a' s} e^{a'(t-s)}\, ds \lesssim t e^{a' t} \lesssim e^{a t},$$

for any $t \ge 0$ and any $a > a' > a^*$. For the second estimate in (1.11), we first write

$$v * w(t) = \int_0^t v(s) w(t-s)\, ds \lesssim \int_0^t \frac{e^{a'' s}}{s^{3/4}} e^{a''(t-s)}\, ds \lesssim t^{1/4} e^{a'' t} \lesssim e^{a' t},$$

for any $t \ge 0$ and any $a' > a'' > a^*$, and we conclude by combining that estimate with the first estimate in (1.11).
Step 2. The conclusion. With the help of the estimates stated in step 1, we are in position to prove (1.4) or equivalently that

$$\| \mathcal{T}_i(t) \|_{L^1_k \to L^1_k} \lesssim e^{a t}, \ \forall\, t \ge 0, \ \forall\, a > a^*, \ \forall\, k > k^*, \tag{1.13}$$

for any $i = 1, 2, 3$. For $i = 1$, (1.13) is nothing but (1.7) together with $\Pi^\perp \in \mathcal{B}(L^1_k)$. For proving (1.13) when $i = 2$, we use the first estimate in (1.11) with

$$u(t) := \| \Pi^\perp S_\mathcal{B}(t) \|_{L^1_k \to L^1_k}, \quad w(t) := \| \mathcal{A} S_\mathcal{B}(t) \Pi^\perp \|_{L^1_k \to L^1_k},$$

where both functions satisfy the hypothesizes of (1.12) because of $\Pi^\perp \in \mathcal{B}(L^1_k)$, of the first estimate on $\mathcal{A}$ with $K = k$ in (1.9) and of the estimate (1.8) on $S_\mathcal{B}(t)$ in L^1_k. For proving (1.13) when $i = 3$, we use the second estimate in (1.11) with

$$u(t) := \|S_\mathcal{L}(t)\Pi^\perp\|_{L^2(G^{-1})\to L^1_k}, \quad v(t) := \|\mathcal{A}S_\mathcal{B}(t)\|_{L^1_K\to L^2(G^{-1})}, \quad w(t) := \|\mathcal{A}S_\mathcal{B}(t)\Pi^\perp\|_{L^1_k\to L^1_K},$$

where the three functions satisfy the hypothesizes of (1.12). To check the estimate on u, we use (1.10) and $L^2(G^{-1}) \subset L^1_k$. For the estimate on v, we use (1.8) and the second estimate on $\mathcal{A}$ in (1.9). Finally, to check the estimate on w, we use the first estimate on $\mathcal{A}$ in (1.9), the estimate (1.7) on $S_\mathcal{B}(t)$ in L^1_k and $\Pi^\perp \in \mathcal{B}(L^1_k)$.

In order to conclude the proof of Theorem 1.1, we thus need to establish (1.7), (1.8) and (1.9). That is done in the three following steps.

Step 3. Proof of (1.9). The operator $\mathcal{A}$ is clearly bounded in any Lebesgue space and more precisely

$$\|\mathcal{A}\, f\|_{L^p(m)} \le C_{R,M}\, \|f\|_{L^p_\ell}, \quad \forall\, f \in L^p_\ell,\ \forall\, p = 1, 2,$$

for $m := \langle x\rangle^K$ or $m := G^{-1}$ and with

$$C_{R,M} := M\|\frac{m}{\langle\cdot\rangle^{p\ell}}\|^{1/p}_{L^\infty(B_{2R})}.$$

Step 4. Proof of (1.7). For any $k, \varepsilon > 0$ and for any $M, R > 0$ large enough (which may depend on k and ε) the operator $\mathcal{B}$ is dissipative in L^1_k in the sense that

$$\forall\, f \in \mathcal{D}(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} (\mathcal{B}f)\,(\text{sign} f)\,\langle x\rangle^k \le (\varepsilon - k)\,\|f\|_{L^1_k}. \tag{1.14}$$

We immediately deduce (1.7) from (1.14) and the Gronwall lemma. In oder to establish (1.14), we set $\beta(s) = |s|$ (and more rigorously we must take a smooth version of that function) and $m = \langle x\rangle^k$, and we compute

$$\begin{aligned}
\int (\mathcal{L}\, f)\,\beta'(f)\, m &= \int (\Delta f + d\, f + x\cdot\nabla f)\,\beta'(f)\, m \\
&= \int \{-\nabla f\,\nabla(\beta'(f)m) + d\,|f|\, m + m\, x\cdot\nabla|f|\} \\
&= -\int |\nabla f|^2\,\beta''(f)\, m + \int |f|\,\{\Delta m + d - \nabla(x\, m)\} \\
&\le \int |f|\,\{\Delta m - x\cdot\nabla m\},
\end{aligned}$$

where we have used that β is a convex function. Defining

$$\begin{aligned}
\psi &:= \Delta m - x\cdot\nabla m - M\chi_R\, m \\
&= (k^2\,|x|^2\,\langle x\rangle^{-4} - k\,|x|^2\,\langle x\rangle^{-2} - M\,\chi_R)\, m
\end{aligned}$$

we easily see that we can choose $M, R > 0$ large enough such that $\psi \le (\varepsilon - k)\, m$ and then (1.14) follows.

Step 5. Proof of (1.8). Fix now $K > K^*$ and $a > -\lambda_P$. There holds

$$\|S_\mathcal{B}(t)\varphi\|_{L^2_K} \le \frac{C_{a,K}}{t^{d/4}}\, e^{a\,t}\,\|\varphi\|_{L^1_K}, \quad \forall\,\varphi \in L^1_K, \tag{1.15}$$

which immediately implies (1.8) since we are restricted to the case of a dimension $d \le 3$. We set $m = \langle x\rangle^K$. A similar computation as in step 4 gives

$$\begin{aligned}
\int (\mathcal{B}\, f)\, f\, m^2 &= -\int |\nabla(f\, m)|^2 + \int |f|^2 \Big\{\frac{|\nabla m|^2}{m^2} + \frac{d}{2} - x\cdot\nabla m - M\,\chi_R\Big\} m^2 \\
&= -\int |\nabla(f\, m)|^2 + (\frac{d}{2} + \varepsilon - K)\int |f|^2\, m^2,
\end{aligned}$$

for $M, R > 0$ chosen large enough. Denoting by $f(t) = S_\mathcal{B}(t)\varphi$ the solution to the evolution PDE

$$\partial_t f = \mathcal{B}f, \qquad f(0) = \varphi,$$

we (formally) have

$$\frac{1}{2}\frac{d}{dt}\int f^2\, m^2 = \int (\mathcal{B} f)\, f\, m^2 \le -\int |\nabla (f\, m)|^2 + a \int |f|^2\, m^2.$$

On the one hand, throwing away the last (negative) term at the RHS of the above differential inequality and using Nash trick, we get

$$\|f(t)\, m\|_{L^2} \le \frac{C}{t^{d/4}}\, \|f(0)\, m\|_{L^1}, \quad \forall\, t > 0. \tag{1.16}$$

On the other hand, throwing away the first (negative) term at the RHS of the above differential inequality and using the Gronwall lemma exactly as in step 4, we get

$$\|f(t)\, m\|_{L^2} \le C e^{a(t-t_0)}\, \|f(t_0)\, m\|_{L^2}, \quad \forall\, t \ge t_0 \ge 0. \tag{1.17}$$

Using (1.16) for $t \in (0,1]$ and (1.17) for $t \ge 1$, we deduce (1.15). □

2. Asymptotic of Stochastic semigroups

2.1. **Generalities.** From now on, we will be interested in Stochastic semigroups which is a class of semigroups which enjoy both a positivity and a "conservativity" property. The importance of Stochastic semigroups comes from its deep relation with Markov processes in stochastic theory as well as from the fact that a quite satisfactory description of the longtime behaviour of such a semigroups can be performed.

We start with the notion of positivity. It can be formulated in the abstract framework of Banach lattices $(X, \|\cdot\|, \ge)$ which are Banach spaces endowed with compatible order relation or equivalently with an appropriate positive cone X_+. To be more concrete, we just observe that the following three examples are Banach lattices when endowed with their usual order relation:

• $X := C_0(E)$, the space of continuous functions which tend to 0 at infinity (when E is not a compact set) endowed with the uniform norm $\|\cdot\|$;

• $X := L^p(E) = L^p(E, \mathcal{E}, \mu)$, the Lebesgue space of functions associated to the Borel σ-algebra $\mathcal{E}$, a positive σ-finite measure μ and an exponent $p \in [1, \infty]$;

• $X := M^1(E) = (C_0(E))'$, the space of Radon measures defined as the dual space of $C_0(E)$.

Here E denotes a σ-locally compact metric space (typically $E \subset \mathbb{R}^d$) and in the last example the positivity can be defined by duality: $\mu \ge 0$ if $\langle \mu, \varphi\rangle \ge 0$ for any $0 \le \varphi \in C_0(E)$.

Lemma 2.1. *Consider X a Banach lattice (one of the above examples), a bounded linear operator A on X and its dual operator A^* on X'. The following equivalence holds:*
(1) A is positive, namely $Af \ge 0$ for any $f \in X$, $f \ge 0$;
(2) A^* is positive, namely $A^*\varphi \ge 0$ for any $\varphi \in X'$, $\varphi \ge 0$.

The (elementary) proof is left as an exercise. We emphasize that $\langle f, \varphi\rangle \ge 0$ for any $\varphi \in X'_+$ (resp. for any $f \in X_+$) implies $f \in X_+$ (resp. $\varphi \in X'_+$).

There are two "equivalent" (or "dual") ways to formulate the notion of Stochastic and Markov semigroup.

Definition 2.2. *On a Banach lattice $Y \supset C_0(E)$ we say that (P_t) is a Markov semigroup if*
(1) (P_t) is a continuous semigroup in Y;
(2) (P_t) is positive, namely $P_t \ge 0$ for any $t \ge 0$;
(3) (P_t) is conservative, namely $\mathbf{1} \in Y$ and $P_t\mathbf{1} = \mathbf{1}$ for any $t \ge 0$.

Definition 2.3. *On a Banach lattice $X \subset M^1(E)$ we say that (S_t) is a stochastic semigroup if*
(1) (S_t) is a (strongly or weakly $*$ continuous) continuous semigroup in X;
(2) (S_t) is positive, namely $S_t \ge 0$ for any $t \ge 0$;
(3) (S_t) is conservative, namely $\langle S_t f\rangle = \langle f\rangle$, $\forall\, t \ge 0$, $\forall\, f \in X$, where $\langle g\rangle := \langle g, \mathbf{1}\rangle$.

The two notions are dual. In particular, if (P_t) is a Markov semigroup on $Y \supset C_0(E)$, the dual semigroup (S_t) defined by $S_t := P_t^*$ on $X := Y'$ is a stochastic semigroup. In the sequel we will only consider stochastic semigroups defined on $X \subset L^1(E)$.

Stochastic semigroup and semigroup of contractions for the L^1 are closely linked.

Proposition 2.4. *A Stochastic semigroup is a semigroup of contractions for the L^1 norm. In the other way round, a mass conservative semigroup of contractions for the L^1 norm is positive, and thus it is a Stochastic semigroup.*

Proof of Proposition 2.4. We fix $f \in X$ and $t \ge 0$. We write

$$\begin{aligned} |S_t f| &= |S_t f_+ - S_t f_-| \\ &\le |S_t f_+| + |S_t f_-| \\ &= S_t f_+ + S_t f_- \\ &= S_t |f|, \end{aligned}$$

where we have used the positivity property in the third line. We deduce

$$\int |S_t f| \le \int S_t |f| = \int |f|,$$

because of the mass conservation. For the reciprocal part, we consider $f \ge 0$. From both the contraction property and the mass conservation, we have

$$\|S_t f\|_1 \le \|f\|_1 = \int f = \int S_t f.$$

As a consequence,

$$\|(S_t f)_-\|_{L^1} = \frac{1}{2} \int (|S_t f| - S_t f) \le 0$$

so that $(S_t f)_- = 0$ and thus $S_t f \ge 0$. That proves the positivity property.

We may also characterize a Stochastic semigroup in terms of its generator.

Theorem 2.5. *Let $S = S_{\mathcal{L}}$ be a strongly continuous semigroup on a Banach space $X \subset L^1$. There is equivalence between*
(a) $S_{\mathcal{L}}$ is a Stochastic semigroup;
(b) $\mathcal{L}^* 1 = 0$ and $\mathcal{L}$ satisfies Kato's inequality

$$(\operatorname{sign} f) \mathcal{L} f \le \mathcal{L}|f|, \quad \forall f \in D(\mathcal{L}).$$

Partial proof of Theorem 2.5. Step 1. We prove (a) $\Rightarrow$ (b). On the one hand, for any $f \in D(\mathcal{L})$ and any $0 \le \psi \in D(\mathcal{L}^*)$, we have

$$\begin{aligned} \langle \psi, (\operatorname{sign} f) \mathcal{L} f \rangle &= \lim_{t \to 0} \frac{1}{t} \langle \psi, (\operatorname{sign} f)(S(t) f - f) \rangle \\ &\le \lim_{t \to 0} \frac{1}{t} \langle \psi, |S(t) f| - |f| \rangle \\ &\le \lim_{t \to 0} \frac{1}{t} \langle \psi, S(t)|f| - |f| \rangle \\ &= \lim_{t \to 0} \frac{1}{t} \langle S^*(t)\psi - \psi, |f| \rangle \\ &= \langle \mathcal{L}^* \psi, |f| \rangle, \end{aligned}$$

where we have used the inequality $(\operatorname{sign} f) g \le |g|$ in the second line and the positivity assumption in the third line. That inequality is the weak formulation of Kato's inequality. On the other hand and similarly, for any $f \in D(\mathcal{L})$, we have

$$\begin{aligned} \langle \mathcal{L}^* \mathbf{1}, f \rangle &= \langle \mathbf{1}, \mathcal{L} f \rangle \\ &= \lim_{t \to 0} \frac{1}{t} \langle \mathbf{1}, S(t) f - f \rangle = 0, \end{aligned}$$

by just using the mass conservation property.

Step 2. We prove (b) $\Rightarrow$ (a). On the one hand, for any $f \in D(\mathcal{L})$ and $t \ge 0$, we denote $f_t := S_t f$ and we write

$$\langle S_t f - f \rangle = \Big\langle \int_0^t \mathcal{L} f_s \, ds, \mathbf{1} \Big\rangle = \int_0^t \langle f_s, \mathcal{L}^* \mathbf{1} \rangle \, ds = 0.$$

On the other hand, in order to conclude it is enough to prove that (S_t) is a semigroup of contractions. We consider $f \in D(\mathcal{L}^2)$, $t \ge 0$, $n \in \mathbb{N}^*$, we introduce the notation $f_t := S_t f$, $t_k := kt/n$, and we write

$$\begin{aligned} |S_t f| - |f| &= \sum_{k=0}^{n-1} (|f_{t_{k+1}}| - |f_{t_k}|) \\ &\le \sum_{k=0}^{n-1} \mathrm{sign} f_{t_{k+1}} (f_{t_{k+1}} - f_{t_k}) \\ &= \sum_{k=0}^{n-1} \mathrm{sign} f_{t_{k+1}} \int_{t_k}^{t_{k+1}} \mathcal{L} f_s \, ds \\ &= \sum_{k=0}^{n-1} \mathrm{sign} f_{t_{k+1}} \Big\{ \frac{1}{n} \mathcal{L} f_{t_{k+1}} + \int_{t_k}^{t_{k+1}} \mathcal{L}(f_s - f_{t_{k+1}}) \, ds \Big\} \\ &\le \sum_{k=0}^{n-1} \Big\{ \frac{1}{n} \mathcal{L} |f_{t_{k+1}}| + \mathrm{sign} f_{t_{k+1}} \int_{t_k}^{t_{k+1}} \int_{t_{k+1}}^{s} (S_u \mathcal{L}^2 f) \, du ds \Big\}, \end{aligned}$$

where we have used the inequality $(\mathrm{sign} f) g \le |g|$ in the second line and Kato's inequality in the last line. Taking the mean and using the mass conservation, we have

$$\begin{aligned} \|S_t f\| - \|f\| &\le \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \int_{s}^{t_{k+1}} \|S_u \mathcal{L}^2 f\| \, du ds \\ &\le \frac{1}{n} \int_0^t \|S_u \mathcal{L}^2 f\| \, du \to 0, \end{aligned}$$

as $n \to \infty$. □

Exercise 2.6. *Consider $S_{\mathcal{L}^*}$ a (constant preserving) Markov semigroup and $\Phi : \mathbb{R} \to \mathbb{R}$ a concave function. Prove that $\mathcal{L}^* \Phi(m) \le \Phi'(m) \mathcal{L}^* m$. (Hint. Use that $\Phi(a) = \inf\{\ell(a);\ \ell$ affine such that $\ell \ge \Phi\}$ in order to prove $S_t^*(\Phi(m)) \le \Phi(S_t^* m)$ and $\Phi(b) - \Phi(a) \ge \Phi'(a)(b-a)$).*

2.2. **Strong positivity condition and Doblin Theorem.** We consider the case of a strong positivity condition.

Theorem 2.7 (Doeblin)**.** *Consider a Stochastic semigroup S_t such that*

$$S_T f \ge \alpha \, \nu \, \langle f \rangle, \quad \forall \, f \in X_+,$$

for some constants $T > 0$ and $\alpha \in (0,1)$ and some probability measure ν. There holds

$$\|S_t f\|_{L^1} \le C \, e^{at} \|f\|_{L^1}, \quad \forall \, t \ge 0, \ \forall \, f \in X, \ \langle f \rangle = 0,$$

for some constants $C \ge 1$ and $a < 0$.

Proof of Theorem 2.7. We fix $f \in X$ such that $\langle f \rangle = 0$ and we define $\eta := \alpha \nu \langle f_+ \rangle = \alpha \nu \langle f_- \rangle$. We write

$$\begin{aligned} |S_T f| &= |S_T f_+ - \eta - S_T f_- + \eta| \\ &\le |S_T f_+ - \eta| + |S_T f_- - \eta| \\ &= S_T f_+ - \eta + S_T f_- - \eta, \end{aligned}$$

where in the last equality we have used the Doeblin condition. Integrating, we deduce

$$\begin{aligned} \int |S_T f| &\le \int S_T f_+ - \alpha \langle \nu \rangle \langle f_+ \rangle + \int S_T f_- - \alpha \langle \nu \rangle \langle f_- \rangle \\ &\le \int f_+ - \alpha \, \langle f_+ \rangle + \int f_- - \alpha \, \langle f_- \rangle \\ &\le (1-\alpha) \int |f|. \end{aligned}$$

By induction, we obtain $a := [\log(1-\alpha)]/T$ and $C := \exp[|a| T]$. □

2.3. **Geometric stability under Harris and Lyapunov conditions.** We consider now a semigroup S with generator $\mathcal{L}$ and we assume that

(H1) there exists some weight function $m : \mathbb{R}^d \to [1,\infty)$ satisfying $m(x) \to \infty$ as $x \to \infty$ and there exist some constants $\alpha > 0, b > 0$ such that

$$\mathcal{L}^* m \le -\alpha\, m + b;$$

(H2) for any $R > 0$, there exists a constant $T \ge T_0 > 0$ and a positive and not zero measure $\nu = \nu_R$ such that

$$S_T f \ge \nu \int_{B_R} f, \quad \forall\, f \in X_+.$$

Theorem 2.8 (Doeblin)**.** *Consider a Stochastic semigroup S on $X := L^1(m)$ which satisfies (H1) and (H2). There holds*

$$\|S_t f\|_{L^1(m)} \le C\, e^{at} \|f\|_{L^1(m)}, \quad \forall\, t \ge 0,\ \forall\, f \in X,\ \langle f \rangle = 0,$$

for some constants $C \ge 1$ and $a < 0$.

We start with a variant of the key argument in the above Doeblin's Theorem.

Lemma 2.9 (Doeblin's variant)**.** *Under assumption (H2), if $f \in L^1(m)$, with $m(x) \to \infty$ as $|x| \to \infty$, satisfies*

$$\|f\|_{L^1} \ge \frac{4}{m(R)} \|f\|_{L^1(m)} \quad \text{and} \quad \langle f \rangle = 0, \tag{2.1}$$

we then have

$$\|S_T f\|_{L^1} \le \big(1 - \frac{\langle \nu \rangle}{2}\big) \|f\|_{L^1}.$$

Proof of Lemma 2.9. From the hypothesis (2.1), we have

$$\begin{aligned} \int_{B_R} f_\pm &= \int f_\pm - \int_{B_R^c} f_\pm \\ &\ge \frac{1}{2} \int |f| - \frac{1}{m(R)} \int |f| m \ge \frac{1}{4} \int |f|. \end{aligned}$$

Together with (H2), we get

$$S_T f_\pm \ge \frac{\nu}{4} \int |f| =: \eta.$$

We deduce

$$|S_T f| \le |S_T f_+ - \eta| + |S_T f_- - \eta| = S_T f_+ - \eta + S_T f_- - \eta = S_T |f| - 2\eta,$$

and next

$$\int |S_T f| \le \int S_T |f| - 2 \int \eta = \int |f| - \langle \nu \rangle \frac{1}{2} \int |f|,$$

which is nothing but the announced estimate. □

Proof of Theorem 2.8. We split the proof in several steps. We fix $f_0 \in L^1(m)$, $\langle f_0 \rangle = 0$ and we denote $f_t := S_t f_0$.

Step 1. From (H1), we have

$$\frac{d}{dt} \|f_t\|_{L^1(m)} \le -\alpha \|f_t\|_{L^1(m)} + b \|f_t\|_{L^1},$$

from what we deduce

$$\|f_t\|_{L^1(m)} \le e^{-\alpha t} \|f_0\|_{L^1(m)} + \big(1 - e^{-\alpha t}\big) \frac{b}{\alpha} \|f_0\|_{L^1} \quad \forall\, t \ge 0.$$

In other words, for any $T \ge T_0 > 0$, we have

$$\|S_T f_0\|_{L^1(m)} \le \gamma_L \|f_0\|_{L^1(m)} + K \|f_0\|_{L^1}, \tag{2.2}$$

with $\gamma_L \in (0,1)$ and $K > 0$, both constants depending only of T_0. We fix $R > 0$ large enough such that $K/A < 1 - \gamma_L$ with $A := m(R)/4$.

On the other hand, we recall that

$$\|S_T f_0\|_{L^1} \le \|f_0\|_{L^1}, \quad \forall\, T \ge 0, \tag{2.3}$$

and because of Lemma 2.9, there exists $\gamma_H \in (0,1)$ and $T \ge T_0$ only depending on R defined above such that

$$\|S_T f_0\|_{L^1} \le \gamma_H \|f_0\|_{L^1} \quad \text{when} \quad A\,\|f_0\|_{L^1} \ge \|f_0\|_{L^1(m)}. \tag{2.4}$$

Step 2. We introduce the modified norm

$$|||g||| := \|g\|_{L^1} + \beta \|g\|_{L^1(m)}$$

and we observe that we have the alternative

$$A\,\|f_0\|_{L^1} \ge \|f_0\|_{L^1(m)} \quad \text{or} \quad A\,\|f_0\|_{L^1} < \|f_0\|_{L^1(m)}.$$

In the first case of the alternative, using the Lyapunov estimate (2.2) and the coupling estimate (2.4), we have

$$\begin{aligned} |||S_T f_0||| &= \|S_T f_0\|_{L^1} + \beta \|S_T f_0\|_{L^1(m)} \\ &\le (\gamma_H + \beta K)\|f_0\|_{L^1} + \beta \gamma_L \|f_0\|_{L^1(m)} \\ &\le \gamma_1 |||f_0|||, \end{aligned}$$

with $\gamma_1 := \max(\gamma_H + \beta K, \gamma_L) < 1$, by fixing from now on $\beta > 0$ small enough. In the second case of the alternative, using the Lyapunov estimate (2.2) and the non expansion estimate (2.4), we have

$$\begin{aligned} |||S_T f_0||| &= \|S_T f_0\|_{L^1} + \beta \|S_T f_0\|_{L^1(m)} \\ &\le (1 + \beta K - \beta\delta)\|f_0\|_{L^1} + \beta(\gamma_L + \delta/A) \|S_T f_0\|_{L^1(m)} \\ &\le \gamma_2 |||f_0|||, \end{aligned}$$

with $\gamma_2 := \max(1 + \beta K - \beta\delta, \gamma_L + \delta/A)$ for any $0 < \beta\delta < 1 + \beta K$. We take $\delta := K + \varepsilon$, $\varepsilon > 0$, so that we get

$$\gamma_2 = \max(1 - \beta\varepsilon, (\gamma_L + K/A) + \varepsilon/A) < 1,$$

by choosing $\varepsilon > 0$ small enough and by recalling from the very definition of A that $\gamma_L + K/A < 1$. In any cases, we have thus established that

$$|||S_T f_0||| \le \gamma |||f_0|||, \quad \text{with} \quad \gamma := \max(\gamma_1, \gamma_2) < 1.$$

We then conclude as in the proof of Theorem 2.7.

Step 3 (Alternative argument). Alternatively, the two estimates (2.3) and (2.4) together give

$$\|S f_0\|_{L^1} \le \gamma_H \|f_0\|_{L^1} + \frac{1-\gamma_H}{A} \|f_0\|_{L^1(m)}. \tag{2.5}$$

Together with step 1, we deduce that

$$U^{n+1} = M U^n$$

with

$$U^n := \begin{pmatrix} \|S_T^n f_0\|_{L^1(m)} \\ \|S_T^n f_0\|_{L^1} \end{pmatrix} \quad \text{and} \quad M := \begin{pmatrix} \gamma_L & K \\ \frac{1-\gamma_H}{A} & \gamma_H \end{pmatrix}.$$

The eigenvalues of M are

$$\mu_\pm := \frac{1}{2}\big(T \pm \sqrt{T^2 - 4D}\big),$$

with

$$T := \mathrm{tr} M = \gamma_L + \gamma_H, \quad D := \det M = \gamma_L \gamma_H - (1-\gamma_H)\frac{K}{A}.$$

We observe that

$$\gamma_L \gamma_H > D > \gamma_L \gamma_H - (1-\gamma_H)(1-\gamma_L) = T - 1,$$

so that

$$(\gamma_H - \gamma_L)^2 = T^2 - 4\gamma_L\gamma_H < T^2 - 4D < T^2 - 4(T-1) = (T-2)^2$$

and finally

$$\theta := \max(|\mu_+|, |\mu_-|) < \max(\gamma_H, \gamma_L, |T-1|, 1) = 1.$$

We have established that $\|M^n\| \le C\,\theta^n \to 0$ for some constant $C \ge 1$, and we then conclude as in the proof of Theorem 2.7. $\square$

3. An example: the renewal equation

We will discuss now the renewal equation for which we apply some of the results of the preceding sections in order to get some insight about its qualitative behavior in the large time asymptotic. We are thus interesting by the renewal equation

$$\begin{cases} \partial_t f + \partial_x f + a f = 0 \\ f(t,0) = \rho_{f(t)}, \quad f(0,x) = f_0(x), \end{cases} \tag{3.1}$$

where $f = f(t,x)$, $t \geq 0$, $x \geq 0$, and

$$\rho_g := \int_0^\infty g(y)\, a(y)\, dy.$$

Here f typically represents a population of cells (particles) which are aging (getting holder), die (disappear) with rate $a \geq 0$, born again (reappear) with age $x = 0$ and has distribution f_0 at initial time. At least at a formal level, any solution of (3.1) satisfies

$$\frac{d}{dt}\int_0^\infty f\, dx = \int_0^\infty (-\partial_x f - af)\, dx = \big[-f\big]_0^\infty - \int_0^\infty af\, dx = 0,$$

so that the mass is conserved. Similarly, we have

$$\frac{d}{dt}\int_0^\infty |f|\, dx = \int_0^\infty (-\partial_x |f| - a|f|)\, dx = \big[-|f|\big]_0^\infty - \int_0^\infty a|f|\, dx \leq 0,$$

so that the sign of the solution is preserved by observing that $g_- = (|g| + g)/2$ and using the above two informations. That seems to indicate that if (3.1) defines a semigroup, this one is a L^1 Stochastic semigroup.

Preliminarily, we consider the (simpler) transport equation with boundary condition

$$\begin{cases} \partial_t f + \partial_x f + a f = 0 \\ f(t,0) = \rho(t), \quad f(0,x) = f_0(x), \end{cases} \tag{3.2}$$

with f_0 and ρ are given data. We observe that when f is smooth (C^1) and satisfies (3.2), we have

$$\frac{d}{ds}[f(t+s,x+s)e^{A(x+s)}] = 0, \quad A(x) := \int_0^x a(y)\, dy,$$

from what we deduce

$$f(t,x)e^{A(x)} = f(t-s,x-s)e^{A(x-s)},$$

when both terms are well defined. Choosing either $s = t$ or $s = x$, we get

$$f(t,x) = f_0(x-t)\, e^{A(x-t)-A(x)}\, \mathbf{1}_{x>t} + \rho(t-x)\, e^{-A(x)}\, \mathbf{1}_{x<t}. \tag{3.3}$$

In the other way round, we may check that for any smooth functions a, f_0, ρ, the above formula gives a classical solution to (3.2) at least in the region $\{(t,x) \in \mathbb{R}^2_+,\ x \neq t\}$, and thus a weak solution to (3.2) in the sense

$$\int_0^\infty\!\!\int_0^\infty f\,(-\partial_t\varphi - \partial_x\varphi + a\varphi)\, dxdt - \int_0^\infty f_0(x)\varphi(0,x)\, dx - \int_0^\infty \rho(t)\varphi(t,0)\, dt = 0, \tag{3.4}$$

for any $\varphi \in C^1_c(\mathbb{R}^2_+)$. It is worth noticing that this last equation is also the weak formulation of the evolution equation with source term

$$\partial_t f + \partial_x f + af = \rho(t)\delta_0, \quad f(0,x) = f_0(x),$$

defined on the all line (that is for any $x \in \mathbb{R}$).

At least at a formal level, for any solution f to (3.2), we may compute

$$\frac{d}{dt}\int_0^\infty |f|\, dx = \big[-|f|\big]_0^\infty - \int_0^\infty a|f|\, dx \leq |\rho(t)|,$$

so that

$$\sup_{[0,T]} \|f(t)\|_{L^1} \leq \|f_0\|_{L^1} + \int_0^T |\rho(t)|\, dt. \tag{3.5}$$

Lemma 3.1. *Assume $a \in L^\infty$. For any $f_0 \in L^1(\mathbb{R}_+)$ and $\alpha \in L^1(0,T)$ there exists a unique weak solution $f \in C([0,T]; L^1(\mathbb{R}_+))$ associated to equation* (3.2).

Proof Lemma 3.1. Step 1. Existence. When $a \in C_b(\mathbb{R}_+)$ and $f_0, \rho \in C^1_c(\mathbb{R}_+)$ the solution is explicitly given thanks to the characteristics formula (3.3). In the general case, we consider three sequences (a_ε), $(f_{0,\varepsilon})$ and (ρ_ε) of $C_b(\mathbb{R}_+)$ and $C^1_c(\mathbb{R}_+)$ which converge appropriately, namely $a_\varepsilon \to a$ a.e. and (a_ε) bounded in L^∞, $f_{0,\varepsilon} \to f_0$ in $L^1(\mathbb{R}_+)$ and $\rho_\varepsilon \to \rho$ in $L^1(0,T)$, and we see immediately from (3.5) that the functions (f_ε) and f defined thanks to the characteristics formula (3.3) satisfy $f_\varepsilon \to f$ in $C([0,T];L^1)$. As a consequence, we may pass to the limit in (3.2) and we deduce that f is a weak solution to equation (3.2).

Step 2. Uniqueness. Consider two weak solutions f_1 and f_2 to equation (3.2). The difference $f := f_2 - f_1$ satisfies

$$\int_0^\infty \int_0^\infty f\,(-\partial_t\varphi - \partial_x\varphi + a\varphi)\,dxdt = 0, \tag{3.6}$$

for any $\varphi \in C^1_c(\mathbb{R}^2_+)$ and thus also for any $\varphi \in C_c(\mathbb{R}^2_+) \cap W^{1,\infty}(\mathbb{R}^2_+)$. Introducing the semigroup

$$(S_t g)(x) := g(x-t)\,e^{A(x-t)-A(x)}\,\mathbf{1}_{x>t},$$

associated to equation (3.2) with no boundary term, its dual is

$$(S^*_t \psi)(x) := \psi(x+t)\,e^{A(x)-A(x+t)}, \quad \forall\,\psi \in L^\infty(\mathbb{R}_+),$$

and (S^*_t) is well-defined as a semigroup in $C_c \cap W^{1,\infty}(\mathbb{R}_+)$. Now, for $\psi \in C^1_c(\mathbb{R}^2_+)$, we define

$$\begin{aligned}\varphi(t,x) &:= \int_t^T (S^*_{s-t}\psi(s,\cdot))(x)\,ds \\ &= \int_t^T \psi(s,x+s-t)\,e^{A(x)-A(x+s-t)}\,ds \in C_c(\mathbb{R}^2_+) \cap W^{1,\infty}(\mathbb{R}^2_+),\end{aligned}$$

and we compute

$$\partial_x\varphi(t,x) = \int_t^T [\partial_x\psi(s,x+s-t) + \psi(s,x+s-t)(a(x)-a(x+s-t))]\,e^{A(x)-A(x+s-t)}\,ds,$$

from what we deduce

$$\begin{aligned}\partial_t\varphi(t,x) &= -\psi(t,x) + \int_t^T [-\partial_x\psi(s,x+s-t) + \psi(s,x+s-t)a(x+s-t)]\,e^{A(x)-A(x+s-t)}\,ds \\ &= -\psi(t,x) - \partial_x\varphi(t,x) + a(x)\varphi(t,x).\end{aligned}$$

Using then this test function φ in (3.6), we get

$$\int_0^\infty \int_0^\infty f\,\psi\,dxdt = 0, \quad \forall\,\psi \in C^1_c(\mathbb{R}^2_+),$$

and finally $f_1 = f_2$. □

We are now in position to come back to the renewal equation (3.1).

Lemma 3.2. *Assume $a \in L^\infty$. For any $f_0 \in L^1(\mathbb{R}_+)$, there exists a unique global weak solution $f \in C(\mathbb{R}_+;L^1(\mathbb{R}_+))$ associated to equation* (3.1). *We may then associate to the renewal evolution a Stochastic semigroup.*

Proof Lemma 3.2. We define $\mathcal{E}_T := C([0,T];L^1(\mathbb{R}_+))$ and for any $g \in \mathcal{E}_T$, we define $f := \Phi(g) \in \mathcal{E}_T$ the unique solution to equation (3.2) associated to f_0 and $\rho(t) := \rho_{g(t)} \in C([0,T])$. For two given functions $g_1, g_2 \in \mathcal{E}_T$ and the two associated images $f_i := \Phi(g_i)$, we observe that $f := f_2 - f_1$ is a weak solution to equation (3.2) associated to $f(0) = 0$ and $\rho(t) := \rho_{g_2(t)-g_1(t)}$. The estimate (3.5) reads here

$$\begin{aligned}\sup_{[0,T]} \|(f_2-f_1)(t)\|_{L^1} &\le \int_0^T |\rho_{g_2(t)-g_1(t)})|\,dt \le \int_0^T \int_0^\infty a(y)|(g_2-g_1)(t,y)\,dydt \\ &\le T\,\|a\|_{L^\infty} \sup_{[0,T]} \|(g_2-g_1)(t)\|_{L^1}.\end{aligned}$$

Taking first T small enough such that $T\,\|a\|_{L^\infty} < 1$, we get the existence and uniqueness of a fixed point $f = \Phi(f) \in \mathcal{E}_T$, which is nothing but a weak solution to the renewal equation (3.1). Iterating the argument, we get the desired global weak solution $f \in C(\mathbb{R}_+;L^1(\mathbb{R}_+))$.

We may apply the results of the first section in the semigroup chapter 3 in order to get the existence of a semigroup S_t associated to the evolution problem (3.1). This semigroup is clearly positive. That can be seen by construction for instance. Indeed, if $g \in \mathcal{E}_{T,+} := \{g \in \mathcal{E}_T,\, g \geq 0\}$, then $f = \Phi(g) \in \mathcal{E}_{T,+}$ from the representation formula (3.3), and the fixed point argument can be made in that set. Next, from (3.4), we classical deduce (see chapter 2) that

$$\int_0^\infty f\,\varphi_R\,dx = \int_0^\infty f_0\,\varphi_R\,dx + \int_0^t \int_0^\infty (\partial_x\varphi_R + a\varphi_R)\,dxds + \int_0^t \rho(s)\,ds$$

for $\varphi_R(x) := \varphi(x/R)$, $\varphi \in C^1_c(\mathbb{R}_+)$, $\mathbf{1}_{[0,1]} \leq \varphi \leq \mathbf{1}_{[0,2]}$. We get the mass conservation by passing to the limit as $R \to \infty$. □

Lemma 3.3. *Assume furthermore* $\liminf a \geq a_0 > 0$. *There exists a unique stationary solution* $F \in W^{1,\infty}(\mathbb{R}_+)$ *to the stationary problem*

$$\partial_x F + aF = 0, \quad F(0) = \rho_F, \quad F \geq 0, \quad \langle F \rangle = 1.$$

Proof Lemma 3.3. From the first equation we have $F(x) = Ce^{-A(x)}$, so that the boundary condition is immediately fulfilled and the normalized condition is fulfilled by choosing $C := \langle e^{-A(x)} \rangle^{-1}$. It is worth noticing that the additional assumption implies $\langle e^{-A(x)} \rangle < \infty$ so that $C > 0$ and the same is true for F. □

Lemma 3.4. *We still assume* $a \in L^\infty$ *and* $\liminf a \geq a_0 > 0$. *There exist* $C \geq 1$ *and* $\alpha < 0$ *such that for any* $f_0 \in L^1(\mathbb{R}_+)$ *the associated global solutionf to the renewal equation* (3.1) *satisfies*

$$\|f(t) - \langle f_0 \rangle F\|_{L^1} \leq C\,e^{\alpha t}\,\|f_0 - \langle f_0 \rangle F\|_{L^1}, \quad \forall\, t \geq 0.$$

Proof Lemma 3.4. We check Harris condition. We observe that $a \geq a_0/2\,\mathbf{1}_{x \geq x_0}$ for some $x_0 > 0$. We then set $T := 2x_0 > 0$ and we take $0 \leq f_0 \in L^1(\mathbb{R}_+)$. From (3.3), we have

$$f(T,x) \geq \rho_{f(T-x,\cdot)}\,e^{-A(x)}\,\mathbf{1}_{x<T/2}. \tag{3.7}$$

with

$$\begin{aligned} \rho_{f(T-x,\cdot)} &= \int_0^\infty a(y) f(T-x,y)\,dy \\ &\geq \frac{a_0}{2}\int_{x_0}^\infty f(T-x,y)\,dy, \end{aligned}$$

Using the representation formula (3.3) again, we have

$$\begin{aligned} f(T-x,y) &\geq f_0(y+x-T)\,e^{-(A(y)-A(y-(x-T)))}\,\mathbf{1}_{y>T-x} \\ &\geq f_0(y+x-T)\,e^{-(x-T)\|a\|_\infty}\,\mathbf{1}_{y>T-x}, \end{aligned}$$

so that

$$\begin{aligned} \rho_{f(T-x,\cdot)} &\geq \frac{a_0}{2}\int_{x_0}^\infty f_0(y+x-T)\,\mathbf{1}_{y>T-x}\,dy\,e^{-(x-T)\|a\|_\infty} \\ &\geq \frac{a_0}{2}\int_0^\infty f_0(z)\,\mathbf{1}_{z>x_0+x-T}\,dz\,e^{-(x-T)\|a\|_\infty}. \end{aligned}$$

Together with (3.7), we obtain

$$\begin{aligned} f(T,x) &\geq \frac{a_0}{2}\int_0^\infty f_0(z)\,\mathbf{1}_{z>x_0+x-T}\,dz\,e^{-(x-T)\|a\|_\infty}\,e^{-A(x)}\,\mathbf{1}_{x<T/2} \\ &= \nu(x)\int_0^\infty f_0(z)\,dz, \quad \nu(x) := \frac{a_0}{2}\,e^{-(x-T)\|a\|_\infty}\,e^{-A(x)}\,\mathbf{1}_{x<T/2}, \end{aligned}$$

which is precisely a Harris type lower bound. We conclude thanks to Theorem 2.7. □