Université Paris-Dauphine **2019-2020**
An introduction to evolution PDEs December 2019

On the Landau equation

We aim to establish the existence of solutions to the Landau equation

$$\partial_t f(t,v) = Q(f,f)(t,v), \quad f(0,v) = f_0(v), \tag{0.1}$$

on the density function $f = f(t,v) \ge 0$, $t \ge 0$, $v \in \mathbb{R}^d$, $d \ge 2$, where the Landau kernel is defined by the formula

$$Q(f,f)(v) := \frac{\partial}{\partial v_i}\Big\{ \int_{\mathbb{R}^d} a_{ij}(v-v_*) \Big(f(v_*) \frac{\partial f}{\partial v_j}(v) - f(v) \frac{\partial f}{\partial v_j}(v_*) \Big)\, dv_* \Big\}.$$

Here and the sequel we use Einstein's convention of sommation of repeated indices. The matrix $a = (a_{ij})$ is defined by

$$a(z) = |z|^2 \Pi(z), \quad \Pi_{ij}(z) := \delta_{ij} - \hat z_i \hat z_j, \quad \hat z_k := \frac{z_k}{|z|},$$

so that Π is the is the orthogonal projection on the hyperplan $z^\perp := \{y \in \mathbb{R}^d;\ y \cdot z = 0\}$.

1 Part 0 - Some functional estimates.

(01) Prove that for any $0 \le f \in L^1_2$, there holds

$$\int_{\mathbb{R}^d} f(\log f)_- \le \frac{1}{2} M_2(f) + C(d), \quad M_2(f) := \int_{\mathbb{R}^d} f |v|^2\, dv,$$

and deduce that

$$\int_{\mathbb{R}^d} f(\log f)_+ dv \le \int_{\mathbb{R}^d} f \log f dv + \frac{1}{2} M_2(f) + C(d).$$

(Hint. One may prove and use the estimate

$$s\,(\log s)_- \le \sqrt{s}\,\mathbf{1}_{0 \le s \le e^{-a|v|^k}} + s\,a|v|^k\,\mathbf{1}_{e^{-a|v|^k} \le s \le 1}, \quad \forall\, s \ge 0.)$$

Elements of correction. We may indeed observe that

$$\begin{aligned} s\,(\log s)_- &= s\,(\log s)_-\,\mathbf{1}_{0 \le s \le e^{-a|v|^k}} + s\,(\log s)_-\,\mathbf{1}_{e^{-a|v|^k} \le s \le 1} \\ &\le \sqrt{s}\,\mathbf{1}_{0 \le s \le e^{-a|v|^k}} + s\,a|v|^k\,\mathbf{1}_{e^{-a|v|^k} \le s \le 1} \\ &\le e^{-a|v|^k/2} + s\,a|v|^k, \end{aligned}$$

by using the inequality $s(\log s)_- \le \sqrt{s}$ on $[0,1]$ (coming from $\phi(s) := 1 + \sqrt{s}\log s \ge \phi(e^{-2}) > 0$ on $[0,1]$) and the fact that $s \mapsto (\log s)_-$ is a decreasing function. Using the above inequality with $a = 1/2$ and $k = 2$, we have

$$\int f(\log f)_- \le \int e^{-|v|^2/4} dv + \frac{1}{2} \int f|v|^2,$$

and we conclude thanks to

$$\int_{\mathbb{R}^d} f(\log f)_+ = \int_{\mathbb{R}^d} f \log f + \int_{\mathbb{R}^d} f(\log f)_-.$$

(02) Prove that for for any $0 \le f \in L^1$ such that $\int f = 1$, there holds

$$\int_{\mathbb{R}^d} |\nabla f| dv \le \int_{\mathbb{R}^d} \frac{|\nabla f|^2}{f} dv.$$

(Hint. Use the Cauchy-Schwarz inequality).

Elements of correction. Write $|\nabla f| = (|\nabla f| f^{-1/2}) f^{1/2}$ and use the Cauchy-Schwarz inequality.

2 Part I - Physical properties and a priori estimates.

(1) Observe that $a(z)z = 0$ for any $z \in \mathbb{R}^d$ and $a(z)\xi\xi \ge 0$ for any $z, \xi \in \mathbb{R}^d$. Here and below, we use the bilinear form notation $auv = {}^t v a u = v \cdot au$. In particular, the symmetric matrix a is positive but not strictly positive.

Elements of correction. We observe that

$$\Pi(z)\xi\xi := \Pi_{ij}(z)\xi_i\xi_j = (\delta_{ij} - \hat{z}_i\hat{z}_j)\xi_i\xi_j = |\xi|^2 - (\hat{z}\cdot\xi)^2 = |\xi|^2(1 - (\hat{z}\cdot\hat{\xi})^2) \ge 0$$

with equality if $\xi = z$.

(2) For any nice functions $f, \varphi : \mathbb{R}^d \to \mathbb{R}$, $f \ge 0$, prove that

$$\int Q(f,f)\varphi\, dv = \frac{1}{2} \iint a(v - v_*)\big(f\nabla_* f_* - f_*\nabla f\big)\big(\nabla\varphi - \nabla_*\varphi_*\big)\, dv dv_*,$$

where $f_* = f(v_*)$, $\nabla_*\psi_* = (\nabla\psi)(v_*)$. Deduce that

$$\int Q(f,f)\varphi\, dv = 0, \quad \text{for } \varphi = 1, v_i, |v|^2,$$

and

$$-D(f) := \int Q(f,f)\log f\, dv \le 0.$$

Establish then

$$\Big|\int Q(f,f)\varphi\, dv\Big| \le D(f)^{1/2}\Big(\frac{1}{2}\iint f f_* a(v - v_*)(\nabla\varphi - \nabla_*\varphi_*)(\nabla\varphi - \nabla_*\varphi_*)\, dv dv_*\Big)^{1/2}.$$

Elements of correction. By definition

$$\int Q(f,f)\varphi\, dv = \iint a(v - v_*)\Big(f\nabla_* f_* - f_*\nabla f\Big)\nabla\varphi\, dv dv_*.$$

Changing the name of variables and using that $a(-z) = a(z)$, we also have

$$\int Q(f,f)\varphi\, dv = -\iint a(v - v_*)\Big(f\nabla_* f_* - f_*\nabla f\Big)\nabla\varphi_*\, dv dv_*.$$

The first identity follows by taking the arithmetic mean of the two expressions. The RHS of the identity vanishes when particularizing $\varphi = 1, v$ so that $\nabla\varphi - \nabla_*\varphi_* = 0$. Choosing $\varphi = |v|^2$, we also have $a(v - v_*)(\nabla\varphi - \nabla_*\varphi_*) = a(v - v_*)(2v - 2v_*) = 0$ from (1). By definition and (1)

$$-D(f) = \frac{1}{2}\iint f f_* a(v - v_*) X X\, dv dv_* \ge 0, \quad X := \frac{\nabla_* f_*}{f^*} - \frac{\nabla f}{f}.$$

The bilinear form

$$(A, B) \mapsto \mathscr{E}(A, B) := \frac{1}{2}\iint f f_* a(v - v_*) A B\, dv dv_*$$

being positive and symmetric, it verifies the Cauchy-Schwarz inequality which writes

$$\Big|\int Q(f,f)\varphi\,dv\Big| = \Big|\mathscr{E}(X,Y)\Big| \le \mathscr{E}(X,X)^{1/2}\mathscr{E}(Y,Y)^{1/2},$$

with $Y := \nabla\varphi - \nabla_*\varphi_*$.

(3) For $H_0 \in \mathbb{R}$, we define $\mathcal{E}_{H_0}$ the set of functions

$$\begin{aligned}\mathcal{E}_{H_0} \quad := \quad & \Big\{ f \in L^1_2(\mathbb{R}^d);\ f \ge 0,\ \int f\,dv = 1,\ \int f\,v\,dv = 0, \\ & \int f\,|v|^2\,dv \le d,\ H(f) := \int f\,\log f\,dv \le H_0 \Big\}.\end{aligned}$$

Prove that there exists a constant C_0 such that

$$H_-(f) := \int f(\log f)_- dv \le C_0, \quad \forall\, f \in \mathcal{E}_{H_0},$$

and define $D_0 := H_0 + C_0$. Deduce that for any nice positive solution f to the Landau equation such that $f_0 \in \mathcal{E}_{H_0}$, there holds

$$f \in \mathcal{F}_T := \Big\{ g \in C([0,T]; L^1_2);\ g(t) \in \mathcal{E}_{H_0}, \forall\, t \in (0,T),\ \int_0^T D(g(t))\,dt \le D_0 \Big\}.$$

We say that $f \in C([0,T); L^1)$ is a weak solution to the Landau equation if $f \in \mathcal{F}_T$ and (0.1) holds in the distributional sense. Why the definition is meaningful?

Elements of correction. The first estimate on $H_-(f_0)$ follows from (01). From (2), a nice solution to the Landau equation starting from $f_0 \in \mathcal{E}_{H_0}$ satisfies

$$\int f_t = \int f_0 = 1, \quad \int f_t v = \int f_0 v = 0, \quad \int f_t |v|^2 = \int f_0 |v|^2 \le d$$

and

$$\int f_t \log f_t \le \int f_t \log f_t + \int_0^t D(f_s) ds = \int f_0 \log f_0 \le H_0,$$

so that $f_t \in \mathcal{E}_{H_0}$ for any $t > 0$. Because

$$\int_0^T D(f_s) ds \le \int f_0 \log f_0 + \int f_t (\log f_T)_- \le H_0 + C_0,$$

we deduce that $f \in \mathcal{F}_T$ (in fact, we have to assume $f \in C([0,T); L^1_2)$ or modify the definition of $\mathcal{F}_T$ accordingly). With the definition of Y made in (2), we have

$$\begin{aligned}\mathcal{E}(Y,Y) \quad &\le \quad \frac{1}{2}\iint f f |v - v_*|^2 \|2\nabla\varphi\|^2_{L^\infty} dv dv_* \\ &= \quad 4\int f|v|^2 \int f\,\|\nabla\varphi\|^2_{L^\infty} \le d\|\nabla\varphi\|^2_{L^\infty},\end{aligned}$$

so that $\langle Q(f,f),\varphi\rangle$ is well defined as a distribution (of order 1) thanks to the last estimate in (2).

(4) Prove that

$$Q(f,f) = \partial_i(\bar a_{ij}\partial_j f - \bar b_i f) = \partial^2_{ij}(\bar a_{ij} f) - 2\partial_i(\bar b_i f) = \bar a_{ij}\partial^2_{ij} f - \bar c f,$$

with

$$\bar a_{ij} = \bar a^f_{ij} := a_{ij} * f, \quad \bar b_i = \bar b^f_i := b_i * f, \quad \bar c = \bar c^f := c * f, \tag{2.2}$$

and

$$b_i := \sum_{j=1}^d \partial_j a_{ij} = -(d-1) z_i, \quad c := \sum_{i=1}^d \partial_i b_i = -(d-1)d.$$

For $f \in \mathscr{E}_{H_0}$, prove that there existe $C \in (0,\infty)$ such that

$$|\bar a_{ij}| \le C(1+|v|^2), \quad |\bar b_i| \le C(1+|v|),$$

Elements of correction. We compute for instance

$$|\bar a_{ij}| \le \int |a_{ij}| f_* dv_* \int |v-v_*|^2 f_* dv_* = |v|^2 \int f_* dv_* + \int f_* |v_*|^2 dv_* \le |v|^2 + d,$$

for $f \in \mathscr{E}_{H_0}$. The other points are similar or elementary computations.

3 Part II - On the ellipticity of $\bar a$.

We fix $H_0 \in \mathbb{R}$ and $f \in \mathcal{E}_{H_0}$.

(5a) Show that there exists a function $\eta \ge 0$ (only depending of D_0) such that

$$\forall\, A \subset \mathbb{R}^d, \quad \int_A f\, dv \le \eta(|A|)$$

and $\eta(r) \to 0$ when $r \to 0$. Here $|A|$ denotes the Lebesgue measure of A. Deduce that

$$\forall\, R, \varepsilon > 0, \quad \int f\, \mathbf{1}_{|v|\le R} \mathbf{1}_{|v_i|\le\varepsilon}\, dv \le \eta_R(\varepsilon)$$

and $\eta_R(r) \to 0$ when $r \to 0$.

Elements of correction. For any $M > 1$, we may write

$$\begin{aligned} \int_A f\, dv &= \int_A f \mathbf{1}_{f\le M}\, dv + \int_A f \mathbf{1}_{f>M}\, dv \\ &\le M|A| + \int_A f \frac{(\log f)_+}{\log M} \mathbf{1}_{f>M}\, dv \le M|A| + \frac{D_0}{\log M} =: \Xi(|A|, M). \end{aligned}$$

We then define $\eta(r) := \Xi(r, \sqrt{2})$ if $r > 1/2$ and $\eta(r) := \Xi(r, r^{-1/2})$ if $r \in (0, 1/2)$. For the second estimate, we write

$$\int f\, \mathbf{1}_{|v|\le R} \mathbf{1}_{|v_i|\le\varepsilon}\, dv \le \eta(|B_R \cap \{|v_i| \le \varepsilon\}) \le \eta(2^d R^{d-1} \varepsilon) =: \eta_R(\varepsilon).$$

(5b) Show that

$$\int f \mathbf{1}_{|v|\le R} \ge 1 - \frac{d}{R^2}.$$

Elements of correction. We write

$$\begin{aligned} \int f \mathbf{1}_{|v|\le R} &= 1 - \int f \mathbf{1}_{|v|>R} \\ &\ge 1 - \frac{1}{R^2} \int f |v|^2 \mathbf{1}_{|v|>R} \ge 1 - \frac{d}{R^2}. \end{aligned}$$

(5c) Deduce from the two previous questions that

$$\forall\, i = 1, \dots, d, \quad T_i := \int f v_i^2 dv \ge \lambda,$$

for some constant $\lambda > 0$ which only depends on D_0. Generalize the last estimate into

$$\forall \xi \in \mathbb{R}^d, \quad T(\xi) := \int f|v\cdot\xi|^2 dv \geq \lambda|\xi|^2.$$

Elements of correction. For $\varepsilon, R > 0$, we write

$$\begin{aligned} T_i &\geq \varepsilon^2 \int f\mathbf{1}_{|v_i|>\varepsilon}\mathbf{1}_{|v|\leq R} dv \\ &\geq \varepsilon^2\Big(\int f\mathbf{1}_{|v|\leq R} dv - \int f\mathbf{1}_{|v_i|\leq\varepsilon}\mathbf{1}_{|v|\leq R} dv\Big) \\ &\geq \varepsilon^2\Big(1 - \frac{d}{R^2} - \eta_R(\varepsilon)\Big). \end{aligned}$$

We fix first R such that $d/R^2 \leq 1/4$ and next $\varepsilon > 0$ such that $\eta_R(\varepsilon) \leq 1/4$. The estimate holds with $\lambda := \varepsilon^2/2$. The same result holds by replacing v_i by $v\cdot\hat\xi$ for any fixed $\xi \in \mathbb{R}^d\backslash\{0\}$.

(6) Deduce that

$$\forall v, \xi \in \mathbb{R}^d, \quad \bar a(v)\xi\xi := \sum_{i,j=1}^d \bar a_{ij}(v)\xi_i\xi_j \geq (d-1)\lambda\,|\xi|^2.$$

Prove that any weak solution formally satisfies

$$\frac{d}{dt}H(f) = -\int \bar a_{ij}\frac{\partial_i f\partial_j f}{f} - \int \bar c f,$$

and thus the following bound on the Fisher information

$$I(f) := \int \frac{|\nabla f|^2}{f} \in L^1(0,T)$$

Elements of correction. We write

$$\begin{aligned} \bar a(v)\xi\xi &= \int (|v-v_*|^2\delta_{ij} - (v_i - v_{*i})(v_j - v_{*j}))f_* dv_* \xi_i\xi_j \\ &= |\xi|^2 \int [(|v|^2 + |v_*|^2)\delta_{ij} - (v_iv_j + v_{*i}v_{*j})] f_* dv_* \hat\xi_i\hat\xi_j \\ &= a(v)\xi\xi + |\xi|^2\int [|v_*|^2 - (v_*\cdot\hat\xi)^2] f_* dv_*. \end{aligned}$$

The first term is nonnegative from (1). For the second term, we introduce an euclidian basis $(e_1, \dots, e_d)$ such that $e_1 = \hat\xi$ and we observe that

$$|v_*|^2 - (v_*\cdot\hat\xi)^2 = \sum_{i=2}^d (v_*\cdot e_i)^2.$$

Together with (5c), we deduce

$$\int [|v_*|^2 - (v_*\cdot\hat\xi)^2] f_* dv_* = \sum_{i=2}^d \int (v_*\cdot e_i)^2 f_* dv_* \geq (d-1)\lambda.$$

Writing the Landau equation as

$$\partial_t f = \partial_i(\bar a_{ij}\partial_j f - \bar b_i f)$$

from (4), we have

$$\begin{aligned} \frac{d}{dt}H(f) &= \int (\partial_t f)(1 + \log f) \\ &= -\int (\bar a_{ij}\partial_j f - \bar b_i f)\partial_i(\log f) \\ &= -\int \bar a_{ij}\frac{\partial_i f\partial_j f}{f} + \int \bar b_i\partial_i f = -\int \bar a_{ij}\frac{\partial_i f\partial_j f}{f} - \int \bar c f. \end{aligned}$$

From (4) and the above estimate, we deduce

$$\frac{d}{dt}H(f) \quad \le \quad -\lambda(d-1)I(f) + (d-1)d,$$

so that

$$H(f_T) + \lambda(d-1)\int_0^T I(f_s)ds \le H(f_0) + (d-1)dT.$$

We get $I(f) \in L^1(0,T)$ as in (3).

4 Part III - Weak stability.

We consider here a sequence of weak solutions (f_n) to the Landau equation such that $f_n \in \mathcal{F}_T$ for any $n \ge 1$.
(7) Prove that

$$\int_0^T \int |\nabla_v f_n| \, dvdt \le C_T$$

and that

$$\frac{d}{dt}\int f_n \varphi \, dv \text{ is bounded in } L^\infty(0,T), \quad \forall \varphi \in C^2_c(\mathbb{R}^d).$$

Deduce that (f_n) belongs to a compact set of $L^1((0,T)\times\mathbb{R}^d)$. Up to the extraction of a subsequence, we then have

$$f_n \to f \text{ strongly in } L^1((0,T)\times\mathbb{R}^d).$$

Deduce that

$$Q(f_n,f_n) \rightharpoonup Q(f,f) \text{ weakly in } \mathcal{D}'((0,T)\times\mathbb{R}^d)$$

and that f is a weak solution to the Landau equation.

Elements of correction. From (02) and (6), we have

$$\int_0^T \int |\nabla f_n| \le \int_0^T I(f_n)dt \le C_T. \tag{4.3}$$

From (2) (or (4)), we have

$$\left| \frac{d}{dt}\int f_n \varphi \, dv \right| = |\langle Q(f_n,f_n),\varphi\rangle| \le C\|\nabla\varphi\|_{L^\infty}. \tag{4.4}$$

We then argue as in Aubin-Lions Lemma. For a mollifier (ρ_ε) in $\mathcal{D}(\mathbb{R}^d)$, we write

$$f_n = f_n * \rho_\varepsilon + [f_n - f_n * \rho_\varepsilon],$$

with $(f_n * \rho_\varepsilon)$ bounded in $W^{1,\infty}_{tx}$ by (4.4) and

$$\|f_n - f_n * \rho_\varepsilon\|_{L^1} \le \varepsilon \|\nabla_v f_n\|_{L^1} \le \varepsilon C_T$$

uniformly in $n \ge 1$ by a L^1 version of the Poincaré-Wirtinger inequality (same proof as for the L^2 version). We deduce $f_n \to f$ strongly in $L^1((0,T)\times B_R)$ and thus strongly in $L^1((0,T)\times \mathbb{R}^d)$ thanks to the moment estimate coming from $f_n \in \mathcal{F}_T$. We write

$$\int Q(f_n,f_n)\varphi dv = \int \bar a^n_{ij} f_n \partial^2_{ij}\varphi + 2\bar b^n_i f_n \partial_i \varphi$$

where we have used the second expression of $Q(f,f)$ in (4), several integrations by part and the notations $\bar\sigma^n := \sigma * f_n$. Here we need probably an additional estimate. The simplest way consists in assuming additionally $M_4(f_0) < \infty$ and to prove

$$\sup_{[0,T]} M_4(f_t) \le C(M_4(f_0)), \quad M_4(h) := \int h|v|^4 dv.$$

Writing the Landau equation as

$$\partial_t f = \partial^2_{ij}(\bar a_{ij} f) - 2\partial_i(\bar b_i f)$$

and observing that $\partial_i |v|^4 = 4|v|^2 v_i$, $\partial^2_{ij}|v|^4 = 4|v|^2\delta_{ij} + 8 v_i v_j$, we may indeed compute

$$\frac{d}{dt} M_4(f) \quad = \quad \int f \bar a_{ij}(4|v|^2\delta_{ij} + 8 v_i v_j) + 8 \int f \bar b_i |v|^2 v_i.$$

Observing next that

$$a_{ij}\delta_{ij} = (d-1)|v-v_*|^2, \quad a_{ij} v_i v_j = |v|^2|v_*|^2 - (v\cdot v_*)^2, \quad b_i |v|^2 v_i = -(d-1)|v|^4 + (d-1)|v|^2 v\cdot v_*,$$

and using that the first moment vanishes, we get

$$\begin{aligned}\frac{d}{dt} M_4(f) \quad &= \quad \iint f f_* [4(d+1)|v|^2|v_*|^2 - 8(v\cdot v_*)^2 - 2(d-1)|v|^4] \\ &\le \quad 4(d+3) M_2(f)^2 - 2(d-1) M_4(f) \le 4(d+3)d^2 - M_4(f),\end{aligned}$$

and finally

$$M_4(f_t) \le e^{-t} M_4(f_0) + 4(d+3)d^2(1-e^{-t}), \quad \forall t \ge 0.$$

We next assume that $f_n \in \mathcal{F}^\sharp_T$, where $\mathcal{F}^\sharp_T$ is defined accordingly with this additional estimate. We now write

$$\begin{aligned}\bar a^n_{ij} \quad &= \quad \int_{B_R} f_n(t,v_*)(|v-v_*|^2\delta_{ij} - (v-v_*)_i(v-v_*)_j) dv_* \\ &\quad + \int_{B_R^c} f_n(t,v_*)(|v-v_*|^2\delta_{ij} - (v-v_*)_i(v-v_*)_j) dv_* =: A_{n,R} + B_{n,R}.\end{aligned}$$

Because of the strong convergence $f_n \to f$ in L^1, we have

$$A_{n,R} \to A_R := \int_{B_R} f(t,v_*)(|v-v_*|^2\delta_{ij} - (v-v_*)_i(v-v_*)_j) dv_*$$

a.e. and locally bounded in $L^\infty([0,T]\times\mathbb{R}^d)$. Because of the additional moment estimate, we have

$$B_{n,R} \le \int_{B_R^c} f_n(t,v_*)(|v|^2 + |v_*|^2) dv_* \le \frac{1}{R^2}(1+|v|^2) \sup_{[0,T]} M_4(f_n) \to 0,$$

as $R\to\infty$ uniformly in $n \ge 1$. These two pieces of information together imply

$$\bar a^n_{ij} \to \bar a_{ij} := a_{ij} * f \quad \text{a.e. and locally bounded in} \quad L^\infty([0,T]\times\mathbb{R}^d),$$

and then

$$\int \bar a^n_{ij} f_n \partial^2_{ij}\varphi \to \int \bar a_{ij} f \partial^2_{ij}\varphi.$$

The other contribution in $\langle Q(f_n,f_n),\varphi\rangle$ can be handled in a similar (and even simpler) way.

(8) (Difficult, here $d=3$) Take $f \in \mathcal{E}_{H_0}$ with energy equals to d. Establish that $D(f) = 0$ if, and only if,

$$\frac{\nabla f}{f} - \frac{\nabla f_*}{f_*} = \lambda(v,v_*)(v-v_*), \quad \forall v, v_* \in \mathbb{R}^d,$$

for some scalar function $(v,v_*) \mapsto \lambda(v,v_*)$. Establish then that the last equation is equivalent to

$$\log f = \lambda_1 |v|^2/2 + \lambda_2 v + \lambda_3, \quad \forall v \in \mathbb{R}^d,$$

for some constants $\lambda_1 \in \mathbb{R}$, $\lambda_2 \in \mathbb{R}^d$, $\lambda_3 \in \mathbb{R}$. Conclude that

$$D(f) = 0 \text{ if, and only if, } f = M(v) := (2\pi)^{-3/2}\exp(-|v|^2/2).$$

(9) (very difficult, because needs many steps) Prove that for any global weak solution f associated to $f_0 \in L^1_3 \cap \mathcal{E}_{H_0}$ with energy equals d, there holds $f(t) \rightharpoonup M$ when $t\to\infty$. (Hint. Accept that the energy $M_2(f(t)) = d$ and prove that the third moment $M_3(f(t))$ is uniformly bounded).

5 Part IV - Existence.

In this part, we may accept the following abstract version of the J.-L. Lions theorem:

Consider two Hilbert spaces H and V such that $V \subset H = H' \subset V'$, with dense embeddings. Assume that $a(t) : V \times V \to \mathbb{R}$ is defined for any $t \in [0,T]$ as a bounded bilinear form (and thus as an element of $\mathcal{L}(V,V')$) and is such that
(i) $[0,T] \to \mathcal{L}(V,V')$, $t \mapsto a(t)$ is continuous;
(ii) $\exists \alpha > 0$ and $\kappa \in \mathbb{R}$ such that $a(t,u,u) \le -\alpha \|u\|_V^2 + \kappa \|u\|_H^2$, $\forall\, t \in [0,T]$, $\forall\, u \in V$.
Then, for any $u_0 \in H$, there exists a unique function $u \in X_T := C([0,T];H) \cap L^2(0,T;V) \cap L^2(0,T;V')$ such that

$$(u(t), \varphi(t)) = (u_0, \varphi(0)) + \int_0^t a(s, u(s), \varphi(s))\, ds,$$

for any $\varphi \in X_T$.

(10) We fix $k = d+4$. Show that $\mathcal{H} := L^2_k \subset L^1_3$ and that $H_0 := H(f_0) \in \mathbb{R}$ if $0 \le f_0 \in L^2_k$. In the sequel, we first assume that $f_0 \in \mathcal{E}_{H_0} \cap \mathcal{H}$.

Elements of correction. The inclusion $L^2_k \subset L^1_3$ comes from the Cauchy-Schwarz inequality while $H(f_0) < \infty$ comes from the inequality $f(\log f)_+ \le f^2$ and the material of (01).

(11) For $f \in C([0,T]; \mathcal{E}_{H_0})$, we define $\bar a$, $\bar b$ and $\bar c$ thanks to (2.2) and then

$$\tilde a_{ij} := \bar a_{ij} + \varepsilon |v|^2 \delta_{ij}, \quad \tilde b_i := \bar b_i - \varepsilon \frac{d+4}{2} v_i, \quad \varepsilon \in (0,\lambda).$$

We define $\mathcal{V} := H^1_{k+2}$ and then

$$\forall\, g \in \mathcal{V}, \quad Lg := \partial_i \big(\tilde a_{ij} \partial_j g - \tilde b_i g \big) \in \mathcal{V}'.$$

Show that for some constant $C_i \in (0,\infty)$, there hold

$$(Lg,g)_{\mathcal{H}} \le -\varepsilon \|g\|^2_{\mathcal{V}} + C_1 \|g\|^2_{\mathcal{H}}, \quad |(Lg,h)_{\mathcal{H}}| \le C_2 \|g\|_{\mathcal{V}} \|h\|_{\mathcal{V}}, \quad \forall\, g,h \in \mathcal{V}.$$

Deduce that there exists a unique variational solution

$$g \in \mathcal{X}_T := C([0,T];\mathcal{H}) \cap L^2(0,T;\mathcal{V}) \cap H^1(0,T;\mathcal{V}')$$

to the parabolic equation

$$\partial_t g = Lg, \quad g(0) = f_0.$$

Prove furthermore that $g \in \mathcal{F}_T$.

Elements of correction. We observe that

$$(Lg,g)_{\mathcal{H}} \;=\; -\int \tilde a_{ij} \partial_i g \partial_j g \langle v \rangle^{2k} + \frac{1}{2} \int g^2 \partial_j (\tilde a_{ij} \partial_i \langle v \rangle^{2k}) + \frac{1}{2} \int g^2 [(\partial_i \tilde b_i) \langle v \rangle^{2k} - \tilde b_i \partial_i \langle v \rangle^{2k}],$$

with from (6)

$$\tilde a \xi \xi = \bar a \xi \xi + \varepsilon |v|^2 |\xi|^2 \ge \varepsilon (1 + |v|^2) |\xi|^2,$$

and thus

$$\int \tilde a_{ij} \partial_i g \partial_j g \langle v \rangle^{2k} \ge \varepsilon \|\nabla g\|^2_{L^2_{k+1}} = \varepsilon \|g\|^2_{\mathcal{V}} - \varepsilon \|g\|^2_{L^2_k},$$

if we define the space $\mathcal{V}$ in that way ! We conclude to the coercivity estimate thanks to the bounds

$$\partial_j (\tilde a_{ij} \partial_i \langle v \rangle^{2k}), \;\; (\partial_i \tilde b_i) \langle v \rangle^{2k}, \;\; \tilde b_i \partial_i \langle v \rangle^{2k} = \mathcal{O}(\langle v \rangle^{2k}).$$

Existence and uniqueness of a solution in X_T comes from J.-L. Lions' theorem. We next have to repeat the estimates established in (3) in order to get $g \in \mathcal{F}_T$. We also have to repeat the estimates established in (7) in order to get $g \in \mathcal{F}_T^\sharp$. We may for instance compute

$$\begin{aligned}
\langle Lg, \varphi \rangle &= \int g[\partial_j(\tilde{a}_{ij}\partial_i\varphi) + \tilde{b}_i\partial_i\varphi] \\
&= \int g[\tilde{a}_{ij}\partial^2_{ij}\varphi + 2\bar{b}_i\partial_i\varphi - \varepsilon\frac{d}{2}v_i\partial_i\varphi]
\end{aligned}$$

We deduce first $\langle Lg, 1\rangle = 0$ and the mass conservation. We deduce next $\langle Lg, v\rangle = C\langle g, v\rangle$ and the first moment conservation. We also deduce

$$\begin{aligned}
\langle Lg, |v|^2\rangle &= \int\int g f_* \{(a_{ij} + \varepsilon|v|^2\delta_{ij})2\delta_{ij} - 4(d-1)(v-v_*)_i v_i - \varepsilon d|v|^2)\} \\
&= \int\int g f_* \{2(d-1)|v-v_*|^2 - 4(d-1)|v|^2\} \\
&= 2(d-1)\int f_*|v_*|^2 - 2(d-1)\int g|v|^2
\end{aligned}$$

and the energy conservation when $M_2(f_t) = M_2(f_0) = d$ or uniform estimate when $M_2(f_t) \le d$, $M_2(f_0) \le d$. For the entropy, we compute

$$\langle Lg, \log g\rangle = -\int \tilde{a}_{ij}\frac{\partial_i g \partial_j g}{g} - \int \tilde{c}g,$$

with $\tilde{c} := \partial_i \tilde{b}_i$, and we get a similar estimate as in (6) and (7). For the fourth moment, we compute

$$\begin{aligned}
\frac{d}{dt}M_4(g) &= \int g(\bar{a}_{ij} + \varepsilon|v|^2\delta_{ij})(4|v|^2\delta_{ij} + 8v_iv_j) + 8\int g(\bar{b}_i - \varepsilon\frac{d+4}{2}v_i)|v|^2v_i. \\
&= \ldots - 14\varepsilon M_4(g)
\end{aligned}$$

and a similar estimate as in (7) holds.

(12) Prove that there exists a unique fonction

$$f_\varepsilon \in C([0,T]; L^2_k) \cap L^2(0,T; H^1_k) \cap \mathcal{F}_T^\sharp$$

solution to the nonlinear parabolic equation

$$\partial_t f_\varepsilon = \partial_i(\tilde{a}^{f_\varepsilon}_{ij}\partial_j f_\varepsilon + \tilde{b}^{f_\varepsilon}_i f_\varepsilon), \quad f_\varepsilon(0) = f_0,$$

where $\tilde{a}^{f_\varepsilon}_{ij}$ denotes the

Elements of correction. We have to proceed similarly as for the nonlinear McKean-Vlasov equation in Chapter 1.

(13) For $f_0 \in \mathcal{E}_{H_0}$ and $T > 0$, prove that there exists at least one weak solution $f \in \mathcal{F}_T$ to the Landau equation.

Elements of correction. We have to use the a priori estimates as deduced in (11) and to use the stability result established in (7).