On the Landau equation

1 Part I - Equations and physical properties.

We recall the nonlinear Landau equation

$$\partial_t F(t,v) = Q(F,F)(t,v), \quad F(0,v) = F_0(v), \tag{1.1}$$

on the density function $F = F(t,v) \ge 0$, $t \ge 0$, $v \in \mathbb{R}^d$, $d \ge 2$, where the Landau kernel is defined by the formula

$$Q(f,g)(v) := \frac{\partial}{\partial v_i}\Big\{\int_{\mathbb{R}^d} a_{ij}(v-v^*)\Big(f(v^*)\frac{\partial g}{\partial v_j}(v) - f(v)\frac{\partial g}{\partial v_j}(v^*)\Big)\,dv^*\Big\}.$$

Here and the sequel we use Einstein's convention of sommation of repeated indices. The matrix $a = (a_{ij})$ is defined by

$$a(z) = |z|^{2+\gamma}\Pi(z), \quad \Pi_{ij}(z) := \delta_{ij} - \hat z_i \hat z_j, \quad \hat z_k := \frac{z_k}{|z|}, \quad \gamma \in [-3,2], \tag{1.2}$$

so that Π is the is the orthogonal projection on the hyperplan $z^\perp := \{y \in \mathbb{R}^d;\ y \cdot z = 0\}$.

(1) We recall that $a(z)z = 0$ for any $z \in \mathbb{R}^d$ and $a(z)\xi\xi \ge 0$ for any $z, \xi \in \mathbb{R}^d$ (with the use of the bilinear form notation $auv = v^T a u = v \cdot au$, v^T denoting the line vector transpose of the column vector v). Deduce that

$$Q(M,M) = 0, \quad \text{where} \quad M(v) := \frac{1}{(2\pi)^{d/2}} e^{-|v|^2/2}.$$

Introducing the change of unknown $F = M + Mh$, show (formally) that F is a solution to the Landau equation (1.1) if, and only if, h is a solution to the rescaled Landau equation

$$\partial_t h = Lh + \mathcal{C}(h,h), \quad h(0,v) = h_0(v), \tag{1.3}$$

where

$$\begin{aligned} \mathcal{C}(f,g) &:= M^{-1}\partial_i\Big\{\int_{\mathbb{R}^d} MM^* a_{ij}\Big(f^*\partial_j g - f\partial_{*j} g^*\Big)\,dv^*\Big\}, \\ Lh &:= \mathcal{C}(1,h), \end{aligned}$$

and we use the shorthands $\partial_i = \frac{\partial}{\partial v_i}$, $h^* = h(v^*)$, $\partial_{*j} h^* = (\partial_j h)(v^*)$.

We will also consider the linearized Landau equation

$$\partial_t h = Lh, \quad h(0,v) = h_0(v), \tag{1.4}$$

(2) For any nice functions $f, g, \varphi : \mathbb{R}^d \to \mathbb{R}$, prove that

$$\int \mathcal{C}(f,g)\varphi\, M dv = \frac{1}{2}\iint MM^* a(f\nabla_* g^* - f^*\nabla g)(\nabla\varphi - \nabla_*\varphi^*)\, dv dv^*,$$

where $\nabla_* h^* = (\nabla h)(v^*)$. Deduce that

$$\int \mathcal{C}(f,g)\varphi\, M dv = 0, \quad \text{for}\, \varphi = 1, v_i, |v|^2,$$

and

$$\begin{aligned} D_\gamma(h) &:= -\int (Lh)h\, M dv \\ &= \frac{1}{2}\iint MM^* a\,(\nabla_* h^* - \nabla h)(\nabla_* h^* - \nabla h)\, dv dv^*, \end{aligned}$$

where $\gamma \in [-3, 2]$ is defined in (1.2).

(3) We define the scalar product

$$(g,h) := \int ghM\, dv$$

and the associated Hilbert space

$$L^2(M) := \{h : \mathbb{R}^d \to \mathbb{R} \text{ measurable};\ \|h\|^2 = (h,h) < \infty\}.$$

With the notation

$$\langle g\rangle := \int g(v)M(v)\, dv,$$

we define the subspace

$$L^2_0(M) := \{h \in L^2(M);\ \langle h\rangle = \langle hv_i\rangle = \langle h|v|^2\rangle = 0,\ \forall\, i\}.$$

Prove that, at least formally, any solution h to the linearized Landau equation (1.4) satisfies

$$h(t,\cdot) \in L^2_0(M), \quad \forall\, t \geq 0, \quad \text{if} \quad h_0 \in L^2_0(M),$$

and

$$\frac{1}{2}\frac{d}{dt}\|h(t,\cdot)\|^2 = -D_\gamma(h(t,\cdot)), \quad \forall\, t \geq 0.$$

2 A Poincaré like inequality (when $\gamma = 0$)

We assume $\gamma = 0$. In this section, we fix $h \in L^2_0(M)$. The following algebraic computations are not really difficult but probably a bit heavy. Do not hesitate in accepting (2.6) and carrying on.

(4) Prove that

$$D_0(h) = \frac{1}{2} \int_{\mathbb{R}^{2d}} Y^T [|u|^2 I - u \otimes u] Y \, M M^* \, dv dv^*,$$

with the notations $Y := \nabla h - \nabla_* h^*$, $u = v - v^*$, $h^* = h(v^*)$, $\nabla_* h^* = (\nabla h)(v^*)$. Using the symmetries and the notation $h_i := \partial_i h$, prove next that

$$D_0(h) = \sum_{i,j} (B_{ij} + C_{ij}),$$

with

$$\begin{aligned} B_{ij} &:= \int (v_i - v_i^*)^2 (h_j^2 - h_j h_j^*) M M^* dv dv^* \\ C_{ij} &:= \int (v_i - v_i^*)(v_j - v_j^*)(h_i h_j^* - h_i h_j) M M^* dv dv^*. \end{aligned}$$

(5) For any $i, j = 1, \dots, d$, with the notation

$$T_{ij} = T_{ij}(h) := \langle v_i v_j h \rangle, \tag{2.5}$$

prove that

$$\langle 1 \rangle = 1, \quad \langle v_j \rangle = \langle h_j \rangle = 0, \quad \langle v_i v_j \rangle = \delta_{ij}, \quad \langle v_i h_j \rangle = T_{ij}.$$

(6) Expanding and using symetries, deduce that

$$B_{ij} = \langle (v_i^2 + 1) h_j^2 \rangle + 2 T_{ij}^2.$$

(7) With the same type of arguments, prove that

$$C_{ij} = -\langle v_j v_i h_i h_j + \delta_{ij} h_i^2 \rangle - T_{ij}^2 - T_{ii} T_{jj}.$$

(8) Observing that

$$\sum_i T_{ii} = 0$$

and

$$\sum_{ij} (v_i^2 h_j^2 - v_j v_i h_i h_j) \;\; = \;\; |v|^2 |\Pi(v) \nabla h|^2,$$

deduce that

$$D_0(h) = (d-1) \int |\nabla h|^2 M + \int |v|^2 |\Pi(v) \nabla h|^2 M + \sum_{ij} T_{ij}^2. \tag{2.6}$$

(9) We introduce the anisotropic gradient $\widetilde{\nabla}_v h$ of a function h by

$$\widetilde{\nabla}_v h = \Pi^\perp(v)\nabla_v h + [v]\Pi(v)\nabla_v h, \quad [v] := (1+|v|^2)^{1/2}, \tag{2.7}$$

with $\Pi^\perp(v) := I - \Pi(v)$ and the related Sobolev norm

$$\|h\|^2_{*,\gamma} := \|[v]^{\gamma/2}\widetilde{\nabla} h\|^2 + \|[v]^{(2+\gamma)/2}\, h\|^2.$$

Deduce from (2.6) that

$$D_0(h) \geq \|\nabla h\|^2 \geq \|h\|^2. \tag{2.8}$$

Also prove that there exists a constant $\lambda > 0$ such that

$$D_0(h) \geq \|\widetilde{\nabla} h\|^2 \geq \lambda \|h\|^2_{*,0}. \tag{2.9}$$

3 Nonlinear a priori estimate on the Landau equation and long-time behavior (when $\gamma = 0$)

We still assume $\gamma = 0$. In this section, we fix $h_0 \in L^2_0(M)$ and we come back to the rescaled Landau equation (1.3).
(10) For any nice functions $f, g, h : \mathbb{R}^d \to \mathbb{R}$, prove that

$$(\mathcal{C}(f,g),h) = -\int_{\mathbb{R}^d} \Big(\bar{a}^f_{ij}\partial_j(gM) - fM\bar{b}^g_i\Big)\partial_i h\, dv, \tag{3.10}$$

with

$$\begin{aligned} \bar{a}^f_{ij} &= a_{ij} * (Mf) = \int a_{ij}(u) M^* f^* dv^* \\ \bar{b}^g_i &:= b_i * (Mg) = \int b_i(u) M^* f^* dv^*, \quad b_i := \sum_{j=1}^d \partial_j a_{ij} = -(d-1)u_i. \end{aligned}$$

Observe then that for $g, f \in L^2_0(M)$ and $\gamma = 0$, the coefficients simplify into

$$\bar{a}^f_{ij} = T_{ij}(f), \quad \bar{b}^g_i = 0,$$

with the notation of (2.5). Deduce that there exists a constant $K >$ such that

$$|T_{ij}(f)| \leq \frac{K}{2}\|f\|_{L^2(M)}, \quad \forall\, f \in L^2_0(M), \tag{3.11}$$

and

$$|(\mathcal{C}(f,g),h)| \leq K\|f\|_{L^2(M)}\|\nabla g\|_{L^2(M)}\|\nabla h\|_{L^2(M)}, \quad \forall\, f,g,h \in L^2_0(M). \tag{3.12}$$

(11) Prove that, at least formally, any solution h to the rescaled Landau equation (1.3) satisfies

$$\frac{1}{2}\frac{d}{dt}\|h\|^2 \leq -\|\nabla h\|^2 + K\|\nabla h\|^2\|h\|. \tag{3.13}$$

(12) Consider $h \in C^1(\mathbb{R}_+; L^2_0(M)) \cap C(\mathbb{R}_+; H^1(M))$ which satisfies (3.13) and

$$\|h(0,\cdot)\| \le \eta < 1/K.$$

Establish first that

$$\|h(t,\cdot)\| \le \eta, \quad \forall\, t \ge 0,$$

next that

$$\|h(t,\cdot)\| \le e^{(K\eta-1)\lambda t}\|h(0,\cdot)\|, \quad \forall\, t \ge 0,$$

and finally that for any $\alpha \in (0,\lambda)$ there exists $C_\alpha > 0$ such that

$$\|h(t,\cdot)\| \le C_\alpha\, e^{-\alpha t}, \quad \forall\, t \ge 0.$$

4 Existence of solutions (when $\gamma = 0$)

(13) We assume $h_0 \in L^2_0(M)$ and $\|h_0\| \le \eta < 1/K$. For $g \in C([0,T]; L^2_0(M))$ such that $\sup_{t\ge 0} \|g(t,\cdot)\| \le \eta < 1/K$, we consider the linear equation

$$\partial_t h = Lh + \mathcal{C}(g,h), \quad h(0,\cdot) = h_0. \tag{4.14}$$

Establish that there exists a variational solution $h \in C([0,T]; L^2_0(M)) \cap L^2(0,T; H^1(M))$ to equation (4.14) and this one satisfies

$$\|h(t,\cdot)\| \le \eta, \quad \forall\, t \ge 0,$$

and

$$(1 - K\eta) \int_0^T \|\nabla h(t,\cdot)\|^2\, dt \le \frac{1}{2}\eta^2, \quad \forall\, T > 0.$$

(14) For g_i as in question (12) and h_i the associated solution to equation (4.14), prove that $h^\sharp := h_2 - h_1$ satisfies

$$\frac{1}{2}\frac{d}{dt}\|h^\sharp\|^2 \le (K\eta - 1)\|\nabla h^\sharp\|^2 + K\|g^\sharp\|\|\nabla h_1\|\|\nabla h^\sharp\|,$$

with $g := g_2 - g_1$. Deduce that there exists $K' > 0$ such that

$$\sup_{[0,T]} \|h^\sharp(t,.)\|^2 \le K' \int_0^T \|g^\sharp(t,.)\|^2 \|\nabla h_1(t,.)\|^2 dt.$$

Deduce next that the mapping $g \mapsto h$ defined by (4.14) is a contraction for $\eta > 0$ small enough.

(15) Conclude to the existence and uniqueness of a solution $h \in C([0,T]; L^2_0(M)) \cap L^2(0,T; H^1(M))$, $\forall\, T > 0$, which satisfies (3.13).

5 The case $\gamma \in (0, 2]$

In this part, we assume $\gamma \in (0, 2]$.
(16) We fix $h \in L^2_0(M)$. Prove that for any $R \in (0, 1)$,

$$D_\gamma(h) \geq R^\gamma D_0(h) - \varepsilon_R(h),$$

with

$$\varepsilon_R(h) \;\; := \;\; \frac{R^\gamma}{2} \int_{\mathbb{R}^{2d}} \mathbf{1}_{|u| \leq R} Y^T [|u|^2 I - u \otimes u] Y \, M M_* \, dv dv_*$$

Deduce

$$D_\gamma(h) \geq 2 \|\nabla h\|^2_{L^2(M^{1/2})} ((d-1) \, R^\gamma - R^{\gamma+2})$$

for any $R \in (0, 1)$, and next that there exists $K_1 > 0$ such that

$$D_\gamma(h) \geq K_1 \|h\|^2_{L^2(M)}.$$

Prove (or accept!) that

$$D_\gamma(h) \geq C_1 \, \|h\|^2_{*,\gamma} - C_2 \, \|h\|^2_{L^2(M)}.$$

The two last inequalities together, deduce that there exists $\lambda > 0$ such that

$$D_\gamma(h) \geq \lambda \, \|h\|^2_{*,\gamma}.$$

(17) Recalling that $a(u)u = 0$, establish that

$$\begin{aligned} |a_{ij} * (Mf) v_i v_j| \;\; &= \;\; \left| \int a_{ij} v^*_i v^*_j f^* M^* dv^* \right| \leq C_1 [v]^{\gamma+2} \|f\| \\ |a_{ij} * (Mf) v_i| \;\; &= \;\; \left| \int a_{ij} v^*_i f^* M^* dv^* \right| \leq C_2 [v]^{\gamma+2} \|f\|. \end{aligned}$$

Introducing the splitting $\nabla g = \nabla^\| g + \nabla^\perp g$ and $\nabla h = \nabla^\| h + \nabla^\perp h$ in formula (3.10) with $\nabla^\| f = \Pi(v) \nabla f$, $\nabla^\perp f = \Pi^\perp(v) \nabla f$, so that

$$\partial^\perp_i f := (\nabla^\perp f)_i = \frac{v_i}{|v|} \Big(\frac{v}{|v|} \cdot \nabla f \Big),$$

prove (or accept!) that

$$|(\mathcal{C}(f, g), h)| \leq K \|f\|_{L^2(M)} \|g\|_{*,\gamma} \|h\|_{*,\gamma}, \quad \forall \, f, g, h \in L^2_0(M). \tag{5.15}$$

(18) For $\eta > 0$ small enough and any $h_0 \in L^2_0(M)$ such that $\|h\|_{L^2(M)} \leq \eta$, prove the existence and uniqueness of a solution $h \in C^1(\mathbb{R}_+; L^2_0(M))$ which satisfies

$$\sup_{t \geq 0} \|h(t, .)\|^2 + \int_0^\infty \|h(t, .)\|^2_{*,\gamma} \, dt \leq \eta$$

and for $\alpha, C > 0$

$$\|h(t, \cdot)\| \leq C \, e^{-\alpha t}, \quad \forall \, t \geq 0.$$