1 Problem I (2016-2017)

We consider the evolution PDE

$$\partial_t f = \Delta f + \mathrm{div}(E f), \tag{1.1}$$

on the unknown $f = f(t,x)$, $t \ge 0$, $x \in \mathbb{R}^d$, with $E = E(x)$ a given smooth force field which satisfies for some $\gamma \ge 1$

$$\forall\, |x| \ge 1, \quad |E(x)| \le C\,|x|^{\gamma-1}, \quad \mathrm{div}E(x) \le C|x|^{\gamma-2}, \quad x \cdot E \ge |x|^\gamma.$$

We complement the equation with an initial condition

$$f(0,x) = f_0(x).$$

Question 1. Which strategy can be used in order to exhibit a semigroup $S(t)$ in $L^p(\mathbb{R}^d)$, which provides solutions to (1.1) for initial data in $L^p(\mathbb{R}^d)$? Is the semigroup positive? mass conservative? Explain briefly why there exists a function $G = G(x)$ such that

$$0 \le G \in L^2(m), \quad \langle G \rangle := \int G = 1, \quad \mathcal{L}G = 0.$$

We rather accept this fact ! We accept that $G > 0$. For any nice function $f : \mathbb{R}^d \to \mathbb{R}$ we denote $h := f/G$ and, reciprocally, for any nice function $h : \mathbb{R}^d \to \mathbb{R}$ we denote $f := Gh$.

In the sequel we will not try to justify rigorously the a priori estimates we will establish, but we will carry on the proofs just as if there do exist nice (smooth and fast decaying) solutions. We denote by C or C_i some constants which may differ from line to line.

Question 2. Prove that for any weight function $m : \mathbb{R}^d \to [1,\infty)$ and any nice function $f : \mathbb{R}^d \to \mathbb{R}$, there holds

$$\int (\mathcal{L}f) f m = -\int |\nabla f|^2 m + \frac{1}{2}\int f^2 \mathcal{L}^* m,$$

where we will make explicit the expression of $\mathcal{L}^*$.

Question 3. Prove that there exist $w : \mathbb{R}^d \to [1,\infty)$, $\alpha > 0$ and $b, R_0 \ge 0$ such that

$$\mathcal{L}^* w \le -\alpha\, w + b \mathbf{1}_{B_{R_0}}.$$

Question 4. For some constant $\lambda \geq 0$ to be specified later, we define $W := w + \lambda$. Deduce from the previous question that

$$\int h^2 w \, G \leq \frac{1}{\alpha} \int h^2 \, (b \, \mathbf{1}_{B_{R_0}} - \mathcal{L}^* W) \, G,$$

for any nice function $h : \mathbb{R}^d \to \mathbb{R}$.

Question 5. Take a nice function $h : \mathbb{R}^d \to \mathbb{R}$ such that $\langle hG \rangle = 0$ and denote $G(\Omega) := \langle G \, \mathbf{1}_\Omega \rangle$. Prove that for any $R \geq R_0$ there exists $\kappa_R \in (0, \infty)$ such that

$$\int h^2 \, \mathbf{1}_{B_R} \, G \leq \kappa_R \int_{B_R} |\nabla h|^2 \, G + \frac{1}{G(B_R)} \Big(\int_{B_R^c} h \, G \Big)^2$$

and deduce that

$$\int h^2 \, \mathbf{1}_{B_R} \, G \leq \frac{\kappa_R}{1 + \lambda} \int |\nabla h|^2 \, W \, G + \frac{G(B_R^c)}{G(B_R)} \int h^2 \, w \, G.$$

Question 6. Establish finally that there exist some constants $\lambda, K_1 \in (0, \infty)$ such that

$$\frac{2}{K_1} \int h^2 w G \leq \int \Big(W |\nabla h|^2 - \frac{1}{2} h^2 \mathcal{L}^* W \Big) G = \int (-\mathcal{L} f) f G^{-1} W,$$

for all nice function h such that $\langle hG \rangle = 0$.

Question 7. Consider a nice solution f to (1.1) associated to an initial datum f_0 such that $\langle f_0 \rangle = 0$. Establish that f satisfies

$$\frac{1}{2} \frac{d}{dt} \int h_t^2 \, W G = - \int |\nabla h|^2 \, W G + \frac{1}{2} \int h^2 G \mathcal{L}^* W.$$

Deduce that there exists $K_2 \in (0, \infty)$ such that f satisfies the decay estimate

$$\int f_t^2 \, W G^{-1} \, dx \leq e^{-K_2 t} \int f_0^2 \, W G^{-1} \, dx, \quad \forall \, t \geq 0.$$