Exercises about chapter 1 & 2

1. About variational solutions (Chapter 1)

Exercise 1.1. (Poincaré Wirtinger inequality) Consider $f \in L^1_{\rm loc}(\mathbb{R}^d)$ such that $\nabla f \in L^2(\mathbb{R}^d)$ and $0 \le \rho \in L^1_2(\mathbb{R}^d)$ with unit integral. Prove that

$$\|f - f * \rho\|_{L^2(\mathbb{R}^d)} \le C\Big(\int_{\mathbb{R}^d} \rho(z)|z|^2 dz\Big)^{1/2} \|\nabla f\|_{L^2(\mathbb{R}^d)}.$$

Exercise 1.2. Consider the Fokker-Planck equation

$$\partial_t f = \Delta f + \mathrm{div}(xf) \ \text{ in } \ (0,T)\times\mathbb{R}^d, \quad f(0,\cdot) = f_0 \ \text{ in } \ \mathbb{R}^d. \tag{1.1}$$

Under convenient assumptions on f_0 show the existence and uniqueness of a variational solution to the Fokker-Planck equation (1.1) by using Lions' theorem.

Exercise 1.3. Consider the transport equation

$$\partial_t f = b\cdot\nabla f + cf \ \text{ in } \ (0,T)\times\mathbb{R}^d, \quad f(0,\cdot) = f_0 \in L^2(\mathbb{R}^d), \tag{1.2}$$

and its small viscosity regularized version

$$\partial_t f = \varepsilon\Delta f + b\cdot\nabla f + cf \ \text{ in } \ (0,T)\times\mathbb{R}^d, \quad f(0,\cdot) = f_0 \in L^2(\mathbb{R}^d), \tag{1.3}$$

with $\varepsilon > 0$.

(1) Under convenient assumptions on a, b show the existence of a weak solution to the transport equation (1.2) by using Lions' variant of the Lax-Milgram theorem.

(2) Establish the same result by proving first the existence of a solution to the parabolic equation (1.3) with $\varepsilon > 0$ and next passing to the limit $\varepsilon \to 0$.

Exercise 1.4. Consider the kinetic Fokker-Planck equation

$$\partial_t f = v\cdot\nabla_x f + \Delta_v f \ \text{ in } \ \mathcal{U}, \quad f(0,\cdot) = f_0 \ \text{ in } \ \mathcal{O}, \tag{1.4}$$

and its small viscosity regularized version

$$\partial_t f = \varepsilon\Delta_x f + v\cdot\nabla_x f + \Delta_v f \ \text{ in } \ \mathcal{U}, \quad f(0,\cdot) = f_0 \ \text{ in } \ \mathcal{O}, \tag{1.5}$$

with $\varepsilon > 0$, $\mathcal{O} := \mathbb{R}^d\times\mathbb{R}^d$, $\mathcal{U} := (0,T)\times\mathcal{O}$.

(1) Under convenient assumptions on f_0 show the existence of a weak solution to the kinetic Fokker-Planck equation (1.4) by using Lions' variant of the Lax-Milgram theorem.

(2) Establish the same result by proving first the existence of a solution to the parabolic equation (1.5) with $\varepsilon > 0$ and next passing to the limit $\varepsilon \to 0$.

2. De Giorgi-Nash-Moser and beyond (Chapter 2)

Exercise 2.1. (Interpolation inequality) (1) For any $1 \le p, q \le \infty$, $\theta \in (0,1)$ and $f \in L^p \cap L^q$, prove that $f \in L^r$ and

$$\|f\|_{L^r} \le \|f\|_{L^p}^\theta \|f\|_{L^q}^{1-\theta}, \quad \text{with} \quad \frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}.$$

(2) For any $1 \le p_i, q_i \le \infty$, $\theta \in (0,1)$ and $f \in L^{p_1} L^{p_2} \cap L^{q_1} L^{q_2}$, prove that $f \in L^{r_1} L^{r_2}$ and

$$\|f\|_{L^{r_1} L^{r_2}} \le \|f\|_{L^{p_1} L^{p_2}}^\theta \|f\|_{L^{q_1} L^{q_2}}^{1-\theta}, \quad \text{with} \quad \frac{1}{r_i} = \frac{\theta}{p_i} + \frac{1-\theta}{q_i}.$$

Exercise 2.2. 1. Give another proof of the Nash inequality by using the Sobolev inequality in dimension $d \ge 3$. (Hint. Write the interpolation estimate

$$\|f\|_{L^2} \le \|f\|_{L^1}^\theta \, \|f\|_{L^{2^*}}^{1-\theta}$$

and then use the Sobolev inequality associated to the Lebesgue exponent $p = 2$).

2. Give another proof of the Nash inequality by using the Sobolev inequality in dimension $d = 2$. (Hint. Prove the interpolation estimate

$$\|f\|_{L^2} \le \|f\|_{L^1}^{1/4} \, \|f^{3/2}\|_{L^2}^{1/2},$$

then use the Sobolev inequality associated to the Lebesgue exponent $p = 1$ and $p^* := 2$ and finally the Cauchy-Schwartz inequality in order to bound the second term).

3. Give another proof of the Nash inequality by using the Sobolev inequality in dimension $d = 1$. (Hint. Prove the interpolation estimate

$$\|f\|_{L^2} \le \|f\|_{L^1}^{1/2} \, \|f^{3/2}\|_{L^\infty}^{1/3},$$

then use the Sobolev inequality associated to the Lebesgue exponent $p = 1$ and $p^* := \infty$ and finally the Cauchy-Schwartz inequality in order to bound the second term).

Exercise 2.3. 1) Prove the Poincaré-Wirtinger inequality

$$\|f - f_r\|_{L^2} \le C\, r \|\nabla f\|_{L^2}, \quad f_r(x) := \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\, dy,$$

for any $r > 0$ and some constant $C = C(d) > 0$.

2) Recover the Nash inequality in any dimension $d \ge 1$. *(Hint. Write that $\|f\|_{L^2}^2 = (f, f - f_r) + (f, f_r)$ and deduce that $\|f\|_{L^2}^2 \le C_1\, r\, \|f\|_{L^2}\, \|\nabla f\|_{L^2} + C_2\, r^{-d}\, \|f\|_{L^1}^2$, for any $r > 0$).*