Exercises about chapter 1

1. About variational solutions (Chapter 1)

Exercise 1.1. Consider $f \in L^1(\mathbb{R}^d)$ such that $\mathrm{div} f \in L^1(\mathbb{R}^d)$. Show that

$$\int_{\mathbb{R}^d} \mathrm{div} f \, dx = 0.$$

Solution or Hint for Exercise 1.1. That is true for $f \in C^1_c(\mathbb{R}^d)$. For $f \in L^1(\mathbb{R}^d)$ we introduce a mollifier (ρ_ε), a truncation fiunction χ_M and $\rho_\varepsilon * (f\chi_M) \in C^1_c(\mathbb{R}^d)$.

Exercise 1.2. Consider the parabolic type equation

$$\partial_t f = \partial_i (A_{ij}\, \partial_j f) + b_i\, \partial_i f + cf \tag{1.1}$$

with time dependent coefficients

$$A,\, b,\, c \in L^\infty((0,T) \times \mathbb{R}^d),$$

and under uniformly elliptic condition

$$\forall\, t \in (0,T),\ \forall\, x \in \mathbb{R}^d,\ \forall\, \xi \in \mathbb{R}^d \quad A_{ij}(t,x)\, \xi_i \xi_j \ge \alpha\, |\xi|^2, \quad \alpha > 0. \tag{1.2}$$

1) For any initial datum $f_0 \in L^2(\mathbb{R}^d)$, recover the second and generalized version of J.-L. Lions theorem about existence of variational solutions $f \in X_T$ by using the first version.

(Hint. Define

$$A_k := \frac{n}{T} \int_{t_{k-1}}^{t_k} A(t,\cdot)\, dt, \quad i = 1, \dots, n, \quad t_k := kT/n,$$

and a similar way b_k, c_k, and prove that there exists a unique variational solution $g_k \in X_{T/n}$ associated to the A_k, b_k, c_k and the initial condition g_0 when $k = 1$, $g_{k-1}(T/n)$ when $k \ge 2$. Build next a solution $g^n \in X_T$ to the equation (1.1) *associated to the piecewise constant functions $A^n(t) = A_k$ if $t \in [t_k, t_{k+1})$, $k = 0, \dots, n-1$, and b^n, c^n defined similarly. Conclude by passing to the limit $n \to \infty$).*

2) For the above problem, show that $f \ge 0$ if $f_0 \ge 0$ and $G \ge 0$. (Hint. Show that the sequence (g_k) defined in step 2 of the proof of the existence part is such that $g_k \ge 0$ for any $k \in \mathbb{N}$).

Exercise 1.3. Consider a parabolic equation where the operator $\mathcal{L}$ incloses a kernel term

$$\mathcal{L} f := \Delta f + b \cdot \nabla f + cf + \mathcal{K} f, \quad (\mathcal{K} f)(x) := \int_{\mathbb{R}^d} k(x,y) f(y) dy$$

with coefficients satisfying

$$b, c \in L^\infty(\mathbb{R}^d), \quad k \in L^2(\mathbb{R}^d \times \mathbb{R}^d).$$

and establish the existence of a variational solution in the usual X_T space.

Exercise 1.4. Consider the parabolic equation with coefficients $b \in L^\infty + L^d$ and $c \in L^1_{\mathrm{loc}}$, $c_+ \in L^\infty + L^{d/2}$ with $d \ge 3$. Establish the existence of a variational solution in the space X_T associated to $H := L^2$ and $V := \{g \in H^1;\ \sqrt{c_-} g \in L^2\}$. *(Hint. Observe that $f(|b| \mathbf{1}_{|b| > M} + \sqrt{c_+} \mathbf{1}_{c_+ > M}) \to 0$ in L^2 when $M \to \infty$).*

Exercise 1.5. For $b, c \in L^\infty(\mathbb{R}^d)$, $f_0 \in L^p(\mathbb{R}^d)$, $1 \le p \le \infty$, we consider the linear parabolic equation

$$\partial_t f = \Lambda f := \Delta f + b \cdot \nabla f + cf, \quad f(0) = f_0. \tag{1.3}$$

We introduce the usual notations $H := L^2$, $V := H^1$ and X_T the associated space for some given $T > 0$.

1) Consider a convex function $\beta \in C^2(\mathbb{R})$ such that $\beta(0) = \beta'(0) = 0$ and $\beta'' \in L^\infty$. Prove that any variational solution $f \in X_T$ to the above linear parabolic equation satisfies

$$\int_{\mathbb{R}^d} \beta(f_t)\,dx \le \int_{\mathbb{R}^d} \beta(f_0)\,dx + \int_0^t \int_{\mathbb{R}^d} \{c\,f\,\beta'(f) - (\operatorname{div} b)\,\beta(f)\}\,dxds,$$

for any $t \ge 0$.

2) Assuming moreover that $\beta \ge 0$ and there exists a constant $K \in (0,\infty)$ such that $0 \le s\,\beta'(s) \le K\beta(s)$ for any $s \in \mathbb{R}$, deduce that for some constant $C := C(b,c,K)$, there holds

$$\int_{\mathbb{R}^d} \beta(f_t)\,dx \le e^{Ct} \int_{\mathbb{R}^d} \beta(f_0)\,dx, \quad \forall\, t \ge 0.$$

3) Prove that for any $p \in [1,2]$, for some constant $C := C(b,c)$ and for any $f_0 \in L^2 \cap L^p$, there holds

$$\|f(t)\|_{L^p} \le e^{Ct}\|f_0\|_{L^p}, \quad \forall\, t \ge 0.$$

(Hint. Define β on $\mathbb{R}_+$ and extend it to $\mathbb{R}$ by symmetry. More precisely, define $\beta''_\alpha(s) = 2\theta \mathbf{1}_{s\le\alpha} + p(p-1)s^{p-2}\mathbf{1}_{s>\alpha}$, with $2\theta = p(p-1)\alpha^{p-2}$ and then the primitives which vanish at the origin, which are thus defined by $\beta'_\alpha(s) = 2\theta s\mathbf{1}_{s\le\alpha} + (ps^{p-1} + p(p-2)\alpha^{p-1})\mathbf{1}_{s>\alpha}$, $\beta_\alpha(s) = \theta s^2\mathbf{1}_{s\le\alpha} + (s^p + p(p-2)\alpha^{p-1}s + A\alpha^p)\mathbf{1}_{s>\alpha}$, $A := p(p-1)/2 - 1 - p(p-2)$. Observe that $s\beta'_\alpha(s) \le 2\beta_\alpha(s)$ because $s\beta''_\alpha(s) \le \beta'_\alpha(s)$ and $\beta_\alpha(s) \le \beta(s)$ because $\beta''_\alpha(s) \le \beta''(s)$).

4) Prove that for any $p \in [2,\infty]$ and for some constant $C := C(a,c,p)$ there holds

$$\|f(t)\|_{L^p} \le e^{Ct}\|f_0\|_{L^p}, \quad \forall\, t \ge 0.$$

(Hint. Define $\beta''_R(s) = p(p-1)s^{p-2}\mathbf{1}_{s\le R} + 2\theta\mathbf{1}_{s>R}$, with $2\theta = p(p-1)R^{p-2}$, and then the primitives which vanish in the origin and which are thus defined by $\beta'_R(s) = ps^{p-1}\mathbf{1}_{s\le R} + (pR^{p-1} + 2\theta(s-R))\mathbf{1}_{s>R}$, $\beta_R(s) = s^p\mathbf{1}_{s\le R} + (R^p + pR^{p-1}(s-R) + \theta(s-R)^2)\mathbf{1}_{s>R}$. Observe that $s\beta'_R(s) \le p\beta_R(s)$ because $s\beta''_R(s) \le (p-1)\beta'_R(s)$ and $\beta_R(s) \le \beta(s)$ because $\beta''_R(s) \le \beta''(s)$. Pass to the limit $p \to \infty$ in order to deal with the case $p = \infty$).

5) Prove that for any $f_0 \in L^p(\mathbb{R}^d)$, $1 \le p \le \infty$, there exists at least one weak (in the sense of distributions) solution to the linear parabolic equation (1.3). (Hint: Consider $f_{0,n} \in L^1 \cap L^\infty$ such that $f_{0,n} \to f_0$ in L^p, $1 \le p < \infty$, and prove that the associate variational solution $f_n \in X_T$ is a Cauchy sequence in $C([0,T];L^p)$. Conclude the proof by passing to the limit $p \to \infty$).

6) Prove that if $0 \le f_0 \in L^p(\mathbb{R}^d)$, $p \in (1,\infty)$, there exists a weak solution $f \in C([0,T];L^p(\mathbb{R}^d))$ such that $f(t,\cdot) \ge 0$ for any $t \in (0,T)$.

7) Prove the existence of a weak solution to the McKean-Vlasov equation for any initial datum $f_0 \in L^1(\mathbb{R}^d)$.

Exercise 1.6. (McKean-Vlasove equation) Consider the linear parabolic equation

$$\partial_t f = \mathcal{L}_g f := \Delta f + \operatorname{div}(a_g f), \quad f(0) = f_0, \tag{1.4}$$

with

$$a_g := a * g, \quad a \in L^\infty(\mathbb{R}^d)^d, \tag{1.5}$$

associated to the nonlinear McKean-Vlasov equation.

1) Defining $F := f\langle x\rangle^{2k}$, establish that F is a solution to the linear parabolic equation

$$\partial_t F = \mathcal{M}_g F := \Delta f + \operatorname{div}(a_g f) + b\cdot\nabla F + c_g F, \tag{1.6}$$

with b and c_g to be determined. *(Hint.* $b := -4kx/\langle x\rangle^2$, $c_g := \langle x\rangle^{2k}\Delta\langle x\rangle^{-2k} - a_g \cdot b$*)*

2) Establish that for any $F_0 \in L^2$ and $g \in L^\infty(0,T;L^1)$, there exists a unique variational solution $F \in X_T$ to the parabolic equation (1.6).

3) Establish that for $f \in L^2_k$ and $g \in L^\infty(0,T;L^1)$, there exists a unique variational solution $f \in Y_T$ to the parabolic equation (1.4) with $Y_T = C([0,T];H) \cap L^2(0,T;V) \cap H^1(0,T;V')$, $H := L^2_k$, $V := H^1_k$.

Exercise 1.7. (Aubin-Lions Lemma) Prove that in the Aubin-Lions Lemma we may assume

$$(F_n),\ (G_n) \text{ are bounded in } L^1((0,T)\times B_R),\ \forall\, R>0,$$

and obtain the same conclusion.