Sheet of exercises 3 about chapters 1, 2, 3 & 4

1. Lebesgue spaces

Exercise 1.1. Consider two sequences (u_n) in $L^1(E)$, (v_n) in $L^\infty(E)$.

(1) Asume $u_n \to u$ strongly in $L^1(E)$, $v_n \to u$ a.e. in E and uniformly bounded. Prove that $u_n v_n \to uv$ strongly in $L^1(E)$.

Hint. Use the reciprocal version of the dominated convergence theorem.

(2) Asume $u_n \rightharpoonup u$ weakly in $L^1(E)$, $v_n \to u$ a.e. in E and uniformly bounded. Prove that $u_n v_n \to uv$ strongly in $L^1(E)$,

Hint. Use the Egorov theorem and (1).

Exercise 1.2. Consider u such that $u \in L^p$ for any $p \in (1, \infty)$.

(1) If $\sup_{p>1} \|u\|_{L^p} < \infty$, prove that $u \in L^1$ and $\|u\|_{L^1} \le \liminf \|u\|_{L^p}$ in the limit $p \to 1$.

(1) If $\sup_{p<\infty} \|u\|_{L^p} < \infty$, prove that $u \in L^\infty$ and $\|u\|_{L^\infty} \le \limsup \|u\|_{L^p}$ in the limit $p \to \infty$.

2. Variational solutions (Chapter 1)

Exercise 2.1. Let $T > 0$ be fixed. For a.e. $t \in [0, T]$, we are given a bilinear form $a(t; \cdot, \cdot) : V \times V \to \mathbb{R}$ such that for some constants $\alpha, M > 0$, $\kappa \in \mathbb{R}$

(i) For every $f, g \in V$ the function $t \mapsto a(t; f, g)$ is measurable;
(ii) $|a(t; f, g)| \le M \|f\| \|g\|$ for a.e. $t \in [0, T]$, for any $f, g \in V$;
(iii) $a(t; f, f) \ge \alpha \|f\|^2 - \kappa |f|^2$ for a.e. $t \in [0, T]$, for any $f \in V$.

Establish that for any $\mathfrak{F} \in L^2(0, T; V')$ and $f_0 \in H$, there exists a unique function $f \in X_T$ such that

$$\big\langle \frac{d}{dt} f, g \big\rangle + a(t; f, g) = \langle \mathfrak{F}, g \rangle, \quad \text{a.e. on } [0, T]$$

and $f(0) = f_0$.

Hint. For $f \in \mathscr{H} := L^2(0, T; V)$ and $\varphi \in \Phi := C^2([0, T); V)$, define

$$\mathscr{E}(f, \varphi) := \int_0^T \{a(t, f, g) + \kappa (f, g)_H - \langle f, g' \rangle\} dt, \quad \ell(\varphi) := \int_0^T \langle \mathfrak{F}_\kappa, \varphi \rangle dt + (\varphi(0), f_0)_H,$$

with $\mathfrak{F}_\kappa := \mathfrak{F} e^{-\kappa t}$, and apply the generalized version of the Lax-Milgram theorem.

3. The Poincaré inequality (Chapter 3)

Exercise 3.1. Establish that exists a function W such that $W \geq 1$ and there exist some constants $\theta > 0$, $b, R \geq 0$ such that

$$(L^* W)(x) := \Delta W(x) - \nabla V \cdot \nabla W(x) \leq -\theta\, W(x) + b\, \mathbf{1}_{B_R}(x), \qquad \forall\, x \in \mathbb{R}^d, \tag{3.1}$$

where $B_R = B(0,R)$ denotes the centered ball of radius R, in the following situations:

(i) $V(x) := \langle x \rangle^\alpha$ with $\alpha \geq 1$;

(ii) there exist $\alpha > 0$ and $R \geq 0$ such that

$$x \cdot \nabla V(x) \geq \alpha \qquad \forall\, x \notin B_R;$$

(iii) there exist $a \in (0,1)$, $c > 0$ and $R \geq 0$ such that

$$a\, |\nabla V(x)|^2 - \Delta V(x) \geq c \qquad \forall\, x \notin B_R;$$

(iv) V is convex (or it is a compact supported perturbation of a convex function) and satisfies $e^{-V} \in L^1(\mathbb{R}^d)$.

Exercise 3.2. Generalize the Poincaré inequality to a general superlinear potential $V(x) = \langle x \rangle^\alpha/\alpha + V_0$, $\alpha \geq 1$, in the following strong (weighted) formulation

$$\int |\nabla g|^2\, \mathcal{G} \geq \kappa \int |g - \langle g \rangle_{\mathcal{G}}|^2\, (1 + |\nabla V|^2)\, \mathcal{G} \qquad \forall\, g \in \mathcal{D}(\mathbb{R}^d),$$

where we have defined $\mathcal{G} := e^{-V} \in \mathbf{P}(\mathbb{R}^d)$ (for an appropriate choice of $V_0 \in \mathbb{R}$).

4. Transport equation (Chapter 4)

Exercise 4.1. Make explicit the construction and formulas in the three following cases:

(1) $a(x) = a \in \mathbb{R}^d$ is a constant vector. (Hint. One must find $f(t,x) = f_0(x - at)$).

(2) $a(x) = x$. (Hint. One must find $f(t,x) = f_0(e^{-t}x)$).

(3) $a(x,v) = v$, $f_0 = f_0(x,v) \in C^1(\mathbb{R}^d \times \mathbb{R}^d)$ and look for a solution $f = f(t,x,v) \in C^1((0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d)$. (Hint. One must find $f(t,x,v) = f_0(x - vt, v)$).

(4) Assume that $a = a(x)$ and prove that (S_t) is a group on $C(\mathbb{R}^d)$, where

$$\forall\, f_0 \in C(\mathbb{R}^d),\ \forall\, t \in \mathbb{R},\ \forall\, x \in \mathbb{R}^d \quad (S_t f_0)(x) = f(t,x) := f_0(\Phi_t^{-1}(x)). \tag{4.1}$$

Exercise 4.2. We consider the ODE

$$\dot{x}(t) = a(t, x(t)), \quad x(s) = x \in \mathbb{R}^d, \quad s \geq 0, \tag{4.2}$$

associated to a vector field $a : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ which is C^1 and satisfies the globally Lipschitz estimate

$$|a(t,x) - a(t,y)| \leq L\, |x - y|, \quad \forall\, t \geq 0,\ x, y \in \mathbb{R}^d, \tag{4.3}$$

for some constant $L \in (0,\infty)$. From the Cauchy-Lipschitz theorem we know that this one admits a unique solution $t \mapsto x(t) = \Phi_{t,s}(x) \in C^1(\mathbb{R}_+; \mathbb{R}^d)$. Moreover, for any $s, t \geq 0$, the vectors valued function $\Phi_{t,s} : \mathbb{R}^d \to \mathbb{R}^d$ is a C^1-diffeomorphism which satisfies the semigroup properties $\Phi_{0,0} = \mathrm{Id}$, $\Phi_{t_3,t_2} \circ \Phi_{t_2,t_1} = \Phi_{t_3,t_1}$ for any $t_3, t_2, t_1 \geq 0$. We denote $\Phi_t = \Phi_{t,0}$.

1) Establish that $|\Phi_t(y) - \Phi_t(x)| \leq e^{Lt}|x - y|$ for any $t \geq 0$, $x, y \in \mathbb{R}^d$ (Hint. Use the Gronwall lemma) and similarly $|\Phi_t(y) - \Phi_t(x)| \geq e^{-Lt}|x - y|$ for any $t \geq 0$, $x, y \in \mathbb{R}^d$. Deduce that $\|D\Phi_t\|_\infty \leq e^{Lt}$ and $\|(\det D\Phi_t^{-1})^{-1}\|_\infty \leq e^{Lt}$.

2) Establish that $|\Phi_t(x)| \leq (|x| + B(t))e^{tL}$, with $B(t) := \int_0^t |a(s,0)|\, ds$, for any $t \geq 0$, $x \in \mathbb{R}^d$, and similarly $|\Phi_t^{-1}(x)| \leq (|x| + B(t))e^{tL}$, for any $t \geq 0$, $x \in \mathbb{R}^d$.

3) Prove that for any $R > 0$, there exists R_t such that $\Phi_t(B_R) \subset B_{R_t}$ and deduce that if $\operatorname{supp} f_0 \subset B_R$ then the function $f(t,x) := f_0(\Phi_t^{-1}(x))$ is such that $\operatorname{supp} f(t,\cdot) \subset B_{R_t}$. *(Hint. Observe that $B_R \subset \Phi_t^{-1}(B_{R_t})$).*

Solution or Hint for Exercise 4.2. 1) Using (4.3), we have

$$-2L|y_t - x_t|^2 \le 2(a(y_t) - a(x_t)) \cdot (y_t - x_t) = \frac{d}{dt}|y_t - x_t|^2 \le 2L|y_t - x_t|^2$$

from what we deduce

$$e^{-L(t-s)}|x_s - y_s|^2 \le |x_t - y_t|^2 \le e^{L(t-s)}|x_s - y_s|^2, \quad \forall\, s,t \in (0,T),\ s < t.$$

We deduce the two first estimates.

We next write $D\Phi_t(x)h := \lim_{s\to 0}(\Phi_t(x+sh) - \Phi_t(x))/s$ and we deduce the third estimate from the first one.

We finally write $\det D\Phi_t^{-1} \det D\Phi_t = 1$, so that $(\det D\Phi_t^{-1})^{-1} = \det D\Phi_t$ and we bound the last term.

2) Write $x_t := \Phi_t(x)$ and use the Gronwall lemma applied to the inequality

$$|x_t| \le |x| + \int_0^t |a(s,x_s) - a(s,0)|\, ds + \int_0^t |a(s,0)|\, ds \le |x| + L\int_0^t |x_s|\, ds + B(t).$$

For the second estimate, we consider the backward problem. With the same notations, we have

$$|x_s| \le |x_t| + \int_s^t L|x_\tau| d\tau + \int_s^t |a(\tau,0)| d\tau = u(s).$$

We compute

$$u'(s) = -L|x_s| - |a(s,0)| \ge -Lu(s) - |a(s,0)| \quad \text{and} \quad \frac{d}{ds}\big(u(s)e^{Ls}\big) \ge -|a(s,0)|e^{Ls},$$

so that

$$x_0 = u(0) \le u(t)e^{Lt} + \int_0^t |a(s,0)|e^{Ls}\, ds \le (|x_t| + B(t))e^{tL}.$$

3) We observe successively that $\Phi_t^{-1}(B_R) \subset B_{R_t}$, $B_R \subset \Phi_t^{-1}(B_{R_t})$ and $\Phi_t^{-1}(B^c_{R_t}) \subset B^c_R$.

Exercise 4.3. (1) Show that for any characteristics solution f to the transport equation associated to an initial datum $f_0 \in C^1_c(\mathbb{R}^d)$, for any times $T > 0$ and radius R, there exists some constants $C_T, R_T \in (0,\infty)$ such that

$$\sup_{t\in[0,T]} \int_{B_R} |f(t,x)|\, dx \le C_T \int_{B_{R_T}} |f_0(x)|\, dx.$$

(Hint. Use the change of variable $x \mapsto y := \Phi_t^{-1}(x)$ in the characteristics formulation of the solution to the transport equation and use the property of finite speed propagation)

(2) Adapt the proof of existence and uniqueness to the case $f_0 \in L^\infty$.

(Hint. For the existence, use (1) and the L^∞ a priori estimate. For the uniqueness, observe that for the equation

$$\partial_t f + a\cdot\nabla f + c\, f = G, \quad f(0) = f_0, \tag{4.4}$$

with $a = a(t,x)$, $c = c(t,x)$ and $G = G(t,x)$ smooth functions, the solution is given through the explicit formula

$$f(t,x) := f_0(\Phi_{0,t}(x))\, e^{-\int_0^t c(\tau,\Phi_{\tau,t}(x))\, d\tau} + \int_0^t G(s,\Phi_{s,t}(x))\, e^{-\int_s^t c(\tau,\Phi_{\tau,t}(x))\, d\tau}\, ds. \tag{4.5}$$

and use a duality argument).

(3) Prove that for any $f_0 \in C_0(\mathbb{R}^d)$ there exists a global weak solution f to the transport equation which furthermore satisfies $f \in C([0,T]; C_0(\mathbb{R}^d))$.

Solution or Hint for Exercise 4.3. We consider the transport equation

$$\partial_t f + a\cdot\nabla f + c\, f = G, \quad f(0) = f_0, \tag{4.6}$$

with $a = a(t,x)$, $c = c(t,x)$ and $G = G(t,x)$ smooth functions. With the usual notations for the flow associated to the associated ODE (4.2), if a smooth solution f to the transport equation does exist, we must have

$$\frac{d}{dt}\Big[f(t,\Phi_t(x))\, e^{\int_0^t c(s,\Phi_s(x))\,ds}\Big] = G(t,\Phi_t(x))\; e^{\int_0^t c(s,\Phi_s(x))\,ds},$$

from which we deduce

$$f(t,\Phi_t(x)) = f_0(x)\, e^{-\int_0^t c(\tau,\Phi_\tau(x))\,d\tau} + \int_0^t G(s,\Phi_s(x))\, e^{-\int_s^t c(\tau,\Phi_\tau(x))\,d\tau}\,ds.$$

Using that $\Phi_t^{-1} = \Phi_{0,t}$ and the semigroup property of $\Phi_{s,t}$, we conclude.

Exercise 4.4. Consider a solution $g \in C([0,T]; L^p(\mathbb{R}^d))$, $1 \le p < \infty$, to the transport equation. Prove that $g(t,.) \ge 0$ for any $t \ge 0$ if If $g_0 \ge 0$.

Exercise 4.5. Consider the relaxation equation

$$\partial_t f + v\cdot\nabla_x f = M(v)\rho_f - f$$

on the function $f = f(t,x,v)$, $t \ge 0$, $x, v \in \mathbb{R}^d$, where we denote

$$\rho_f := \int_{\mathbb{R}^d} f\,dv, \quad M(v) := (2\pi)^{-d/2}\exp(-|v|^2/2).$$

Prove the existence and uniqueness of a solution $f \in C(\mathbb{R}_+; L^1(\mathbb{R}^d\times\mathbb{R}^d))$ to the relaxation equation for any initial datum $f_0 \in L^1(\mathbb{R}^d\times\mathbb{R}^d)$.

Exercise 4.6. 1) Consider the transport equation with boundary condition

$$\left\{\begin{array}{l} \partial_t f + \partial_x f + af = 0 \\ f(t,0) = b(t), \quad f(0,x) = f_0(x), \end{array}\right. \tag{4.7}$$

where $f = f(t,x)$, $t \ge 0$, $x \ge 0$. Assume $a \in L^\infty(\mathbb{R}_+)$, $f_0 \in L^1(\mathbb{R}_+)$ and $b \in L^1([0,T])$.

(a) Establish the a priori estimate

$$\sup_{[0,T]} \|f(t,\cdot)\|_{L^1} \le (\|b\|_{L^1(0,T)} + \|f_0\|_{L^1})e^{t\|a\|_{L^\infty}}, \quad \forall\, t \ge 0.$$

(Hint. Use the Gronwall lemma).

(b) When $f_0 \in C^1_c(]0,\infty[)$ and $b \in C^1_c(]0,T[)$, show that the characteristics method provides a unique smooth solution f given by $f = \bar f$, with

$$\bar f(t,x) := e^{A(x-t)-A(x)} f_0(x-t)\mathbf{1}_{x>t} + e^{-A(x)} b(t-x)\mathbf{1}_{t>x}, \quad A(x) := \int_0^x a(u)\,du.$$

(Hint. When $f \in C^1([0,T]\times\mathbb{R}_+)$, observe that both

$$\frac{d}{dt}(e^{A(t+x)} f(t,t+x)) = 0, \quad \frac{d}{dx}(e^{A(x)} f(t+x,x)) = 0, \quad A(x) := \int_0^x a(u)\,du,$$

and then $f = \bar f$. Also observe that $\bar f \in C^1([0,T]\times\mathbb{R}_+)$ in that case and conclude).

(c) Establish the existence and uniqueness of a weak solution $f \in C([0,T]; L^1(\mathbb{R}_+))$ to equation (4.7). *(Hint. For the existence use an approximation argument. For the uniqueness use a renormalization or a duality technique).*

2) Consider the renewal equation

$$\left\{\begin{array}{l} \partial_t f + \partial_x f + af = 0 \\ f(t,0) = \rho_{f(t)}, \quad f(0,x) = f_0(x), \end{array}\right. \tag{4.8}$$

where $f = f(t,x)$, $t \ge 0$, $x \ge 0$, and

$$\rho_g := \int_0^\infty g(y)\,a(y)\,dy.$$

Assume $a \in L^\infty(\mathbb{R}_+)$. Establish that there exists a unique weak solution $f \in C([0,T]; L^1(\mathbb{R}_+))$ associated to equation (4.8) for any $f_0 \in L^1(\mathbb{R}_+)$. *(Hint. Use the contraction mapping fixed point theorem).*