# Sheet of exercises 4 about chapters 1 to 6

## 1. Evolution equation and semigroup (Chapter 5)

**Exercise 1.1.** For $h \in E_T := C([0,T]; D(\Lambda)) \cap C^1([0,T]; X)$ prove that $S_\Lambda h \in E_T$ and

$$\frac{d}{dt}[S_\Lambda(t)h(t)] = S_\Lambda(t)\Lambda h(t) + S_\Lambda(t)h'(t).$$

*(Hint. Write*

$$\begin{aligned}\frac{S_\Lambda(t+s)h(t+s) - S_\Lambda(t)h(t)}{s} &= \frac{S_\Lambda(t+s) - S_\Lambda(t)}{s}h(t) + S_\Lambda(t+s)h'(t) \\ &\quad + S_\Lambda(t+s)\Big(\frac{h(t+s)-h(t)}{s} - h'(t)\Big)\end{aligned}$$

*and pass to the limit $s \to 0$).*

## 2. Problem I - Local in time estimates (2017-2018)

Consider a smooth and fast decaying initial datum $f_0$, the associated solution $f = f(t,x)$, $t \geq 0$, $x \in \mathbb{R}^d$, to heat equation

$$\partial_t f = \frac{1}{2}\Delta f, \quad f(0,.) = f_0,$$

and for a given $\alpha \in \mathbb{R}^d$, define

$$g := f\, e^{\psi}, \quad \psi(x) := \alpha \cdot x.$$

(1) Establish that

$$\partial_t g = \frac{1}{2}\Delta g - \alpha \cdot \nabla g + \frac{1}{2}|\alpha|^2 g.$$

(2) Establish that $\|g(t,.)\|_{L^1} \leq e^{\alpha^2 t/2}\, \|g_0\|_{L^1}$ for any $t \geq 0$.

(3) Establish that

$$\|g(t)\|^2_{L^2}\, e^{-\alpha^2 t} \leq \frac{\|g_0\|^2_{L^1}}{(2/d\, C_N\, t)^{d/2}}, \quad \forall\, t > 0.$$

(4) Denoting by $T(t)$ the semigroup associated to the parabolic equation satisfies by $g$, prove successively that

$$T(t) : L^1 \to L^2, \quad L^2 \to L^\infty, \quad L^1 \to L^\infty,$$

for some constants $Ct^{-d/4}e^{\alpha^2 t/2}$, $Ct^{-d/4}e^{\alpha^2 t/2}$ and $Ct^{-d/2}e^{\alpha^2 t/2}$.

(5) Denoting by $S$ the heat semigroup and by $F(t,x,y) := (S(t)\delta_x)(y)$ the fundamental solution associated to the heat equation when starting from the Dirac function in $x \in \mathbb{R}^d$, deduce

$$F(t,x,y) \leq \frac{C}{t^{d/2}}\, e^{\alpha\cdot(x-y) + \alpha^2 t/2}, \quad \forall\, t > 0, \forall\, x, y, \alpha \in \mathbb{R}^d,$$

and then

$$F(t,x,y) \leq \frac{C}{t^{d/2}}\, e^{-\frac{|x-y|^2}{2t}}, \quad \forall\, t > 0, \forall\, x, y \in \mathbb{R}^d.$$

(6) May we prove a similar result for the parabolic equation

$$\partial_t f = \mathrm{div}_x(A(x)\nabla_x f), \quad 0 < \nu \leq A \in L^\infty\, ?$$

## 3. Problem III (2016-2017)

We consider the relaxation equation

$$\partial_t f = \mathcal{L} f := -v \cdot \nabla f + \rho_f M - f \quad \text{in} \quad (0,\infty) \times \mathbb{R}^{2d}, \tag{3.1}$$

on the unknown $f = f(t,x,v)$, $t \geq 0$, $x, v \in \mathbb{R}^d$, with

$$\rho_f(t,x) = \int_{\mathbb{R}^d} f(t,x,v)\, dv, \quad M(v) := \frac{1}{(2\pi)^{d/2}} \exp(-|v|^2/2).$$

We complement the equation with an initial condition

$$f(0,x,v) = f_0(x,v) \quad \text{in} \quad \mathbb{R}^{2d}. \tag{3.2}$$

*Question 1. A priori estimates and associated semigroup.* We denote by $f$ a nice solution to the relaxation equation (3.1)–(3.2).

(a) Prove that $f$ is mass conserving.

(b) Prove that

$$|\rho_g| \leq \|g\|_{L^2_v(M^{-1/2})}, \quad \forall\, g = g(v) \in L^2_v(M^{-1/2})$$

and deduce that

$$\|f(t,\cdot)\|_{L^2_{xv}(M^{-1/2})} \leq \|f_0\|_{L^2_{xv}(M^{-1/2})}.$$

(c) Consider $m = \langle v \rangle^k$, $k > d/2$. Prove that there exists a constant $C \in (0,\infty)$ such that

$$|\rho_g| \leq C \|g\|_{L^p_v(m)}, \quad \forall\, g = g(v) \in L^p_v(m), \quad p = 1, 2,$$

and deduce that

$$\|f(t,\cdot)\|_{L^p_{xv}(m)} \leq e^{\lambda t} \|f_0\|_{L^p_{xv}(m)},$$

for a constant $\lambda \in [0,\infty)$ that we will express in function of $C$.

(d) What strategy can be used in order to exhibit a semigroup $S(t)$ in $L^p_{xv}(m)$, $p = 2$, $p = 1$, which provides solutions to (3.1) for initial date in $L^p_{xv}(m)$? Is the semigroup positive? mass conservative? a contraction in some spaces?

**The aim of the problem is to prove that the associated semigroup $S_{\mathcal{L}}$ to (3.1) is bounded in $L^p(m)$, $p = 1, 2$, without using the estimate proved in question (1b).**

In the sequel we will not try to justify rigorously the a priori estimates we will establish, but we will carry on the proofs just as if there do exist nice (smooth and fast decaying) solutions.

We define

$$f := \rho_f M, \quad \mathcal{B} f = \mathcal{L} f - f.$$

*Question 2.* Prove that $S_{\mathcal{B}}$ satisfies a growth estimate $\mathcal{O}(e^{-t})$ in any $L^p_{xv}(m)$ space. Using the Duhamel formula

$$S_{\mathcal{L}} = S_{\mathcal{B}} + S_{\mathcal{B}} * S_{\mathcal{L}}$$

prove that $S_{\mathcal{L}}$ is bounded in $L^1_{xv}(m)$.

*Question 3.* Establish that $: L^1_{xv}(m) \to L^1_x L^\infty_v(m)$ where

$$\|g\|_{L^1_x L^p_v(m)} := \int_{\mathbb{R}^d} \|g(x,)\|_{L^p(m)}\, dx.$$

Prove that

$$\frac{d}{dt} \int \Big( \int f^p \, dx \Big)^{1/p} dv = \int \Big( \int (\partial_t f) f^{p-1}\, dx \Big) \Big( \int f^p\, dx \Big)^{1/p-1} dv.$$

Deduce that $S_{\mathcal{B}}$ satisfies a growth estimate $\mathcal{O}(e^{-t})$ in any $L^1_x L^p_v(m)$ space for $p \in (1,\infty)$, and then in $L^1_x L^\infty_v(m)$. Finally prove that $S_{\mathcal{B}}(t)$ is appropriately bounded in $\mathscr{B}(L^1, L^1_x L^\infty_v(m))$ and that $S_{\mathcal{L}}$ is bounded in $L^1_x L^\infty_v(m)$.

*Question 4.* We define $u(t) := S_{\mathcal{B}}(t)$. Establish that

$$(u(t) f_0)(x,v) = M(v) e^{-t} \int_{\mathbb{R}^d} f_0(x - v_* t, v_*)\, dv_*.$$

Deduce that

$$\|u(t)f_0\|_{L^\infty_{xv}(m)} \le C\,\frac{e^{-t}}{t^d}\,\|f_0\|_{L^1_x L^\infty_v(m)}.$$

*Question 5.* Establish that there exists some constants $n \ge 1$ and $C \in [1,\infty)$ such that

$$\|u^{(*n)}(t)\|_{L^1_{xv}(m)\to L^\infty_{xv}(m)} \le C\,e^{-t/2}.$$

Deduce that $S_{\mathcal{L}}$ is bounded in $L^\infty_{xv}(m)$.

*Question 6.* How to prove that $S_{\mathcal{L}}$ is bounded in $L^2_{xv}(m)$ in a similar way? How to shorten the proof of that last result by using question (1b)? Same question for the space $L^\infty_{xv}(m)$.