ORDINARY DIFFERENTIAL EQUATIONS

Still a draft - part of the text is written in french
notes à prendre en compte pour une future version

Contents

First half course:
- Examples: gradient flow, Hamiltonian flow; others;
- Cauchy-Lipschitz theorem;

Second half course:
- Gronwall lemma;
- Smooth dependence by perturbations;

Third half course:
- Linear stability;

Fourth half course:
- Nonlinear stability and Lyapunov function;
- Volume preserving flow, Hamiltonian system;

Not considered here
- Variations calculus and Euler-Lagrange equation.

1. Examples

Example 1. $u' = au$, $a \in \mathbb{R}$. The solution is $u(t) = e^{at}u_0$. Three proofs.
(1) Separation of variables: $\frac{du}{u} = adt$ implies $\ln(u/u_0) = at$ and $e^s := (\ln)^{-1}(s)$.
(2) Power series. We set $u(t) := \sum_{k=0}^{\infty}(at)^k/k!u_0$ and we differentiate.
(3) We define e^{at} as the solution of the ODE (existence through the CL theorem).

Example 2. $u' = a(t)u + b(t)$. We define A as the primitive of a such that $A(0) = 0$. We observe that

$$(ue^{-A})' = be^{-A}$$

and we integrate in order to find

$$u(t) = e^{A(t)}u_0 + \int_0^t b(s)e^{A(t)-A(s)}\,ds.$$

Example 3. $u' = Ku^{1+\alpha}$, $K \in \mathbb{R}^*$, $\alpha > -1$, $\alpha \neq 0$, $u(0) = u_0 \geq 0$. We use the separation of variables method:

$$\frac{du}{u^{1+\alpha}} = Kdt, \quad u^{-\alpha} - u_0^{-\alpha} = -\alpha Kt,$$

and finally

$$u = (u_0^{-\alpha} - \alpha Kt)^{-1/\alpha}.$$

(1) $u' = u^{1+\alpha}$, $\alpha > 0$ ($K = 1$), then u blows up in $T_{\max} = u_0^{-\alpha}/\alpha$.
(2) $u' = -u^{1+\alpha}$, $\alpha > 0$ ($K = -1$), then u decays with rate $\mathcal{O}(1/t^{1/\alpha})$.
(3) $u' = u^{1-\gamma}$, $\gamma \in (0,1)$, $u_0 = 0$, meaning $\alpha = -\gamma \in (-1,0)$, $K = 1$, so that

$$u = (\gamma t)^{1/\gamma}, \quad \text{and also} \quad u(t) = C(t-t_0)_+^{1/\gamma},\ \forall\, t_0 \geq 0.$$

$u \mapsto u^{1-\gamma}$ is not a Lipschitz function : no uniqueness of the solution starting from $u(0) = 0$.
(4) $u' = -u^{1-\gamma}$, $\gamma \in (0,1)$, $u_0 > 0$, meaning $\alpha = -\gamma \in (-1,0)$, $K = -1$, so that

$$u = (u_0^\gamma - t)^{1/\gamma} \text{ on } (0, u_0^\gamma)$$

and thus

$$\bar{u} := (u_0^\gamma - t)_+^{1/\gamma} \text{ on } (0, \infty)$$

is a solution (which vanishes in finite time).

Example 4. $u' = u(1-u)$ the logistic equation, with $u_0 \in (0,1)$. We compute

$$\frac{du}{u} + \frac{du}{1-u} = \frac{du}{u(1-u)} = dt,$$

so that

$$\ln\frac{u}{1-u} - \ln\frac{u_0}{1-u_0} = t, \quad \frac{u}{1-u} = \frac{u_0}{1-u_0}e^t$$

and finally

$$u = \frac{1}{1 + \frac{1-u_0}{u_0}e^{-t}} \quad \underset{t\to-\infty}{\longrightarrow} 0, \quad \underset{t\to+\infty}{\longrightarrow} 1.$$

Example 5. $u' = Au$, $A \in M_d(\mathbb{R})$. The power series

$$u(t) = \sum_{k=0}^{\infty} \frac{1}{k!}(tA)^k u_0 =: e^{tA}u_0$$

is a solution.

Example 6. $u'' + au' + bu = 0$, $a, b \in \mathbb{R}$. We look for exponential solutions $u = e^{rt}$. The coefficient r must satisfy

$$r^2 + ar + b = 0; \quad \text{and define } \Delta := a^2 - 4b.$$

Depending of $\Delta \neq 0$ or $= 0$, we have two solutions $r_1, r_2 \in \mathbb{C}$ to the algebraic equation and thus two solutions $t \mapsto e^{r_i t}$ to the ODE or one solution $r \in \mathbb{R}$ to the algebraic equation and thus two solutions $t \mapsto e^{rt}$ and $t \mapsto te^{rt}$ to the ODE.

Example 7. Gradient flow $u' = -\nabla\phi(u)$, $\phi : \mathbb{R}^d \to \mathbb{R}$ (convex, $\phi(u) \to \infty$ at infinity). We observe that

$$\frac{d}{dt}\,\phi(u(t)) = \langle \nabla\phi(u), u' \rangle = -|\nabla\phi(u)|^2 \le 0.$$

Example 8. Fundamental Principle of Dynamics and Hamiltonian systems

$$\begin{cases} \dot x = v \\ \dot v = -\nabla\phi(x) \end{cases} \quad \text{and} \quad \begin{cases} \dot p = \nabla_q \mathcal{H}(p,q) \\ \dot q = -\nabla_p \mathcal{H}(p,q). \end{cases}$$

For the Fundamental Principle of Dynamics equation, we compute

$$\frac{d}{dt}\,[\frac{|v|^2}{2} + \phi(x)] = v \cdot \dot v + \nabla\phi(x)\dot x = 0.$$

For the Hamiltonian system, we compute

$$\frac{d}{dt}\,\mathcal{H}(p,q) = \nabla_p \mathcal{H} \cdot \dot p + \nabla_q \mathcal{H} \cdot \dot q = 0,$$

and the FPD equation is a particular HS taking $\mathcal{H}(x,v) := |v|^2/2 + \phi(x)$.

Example 9. Prey-Predator equation of Lotka-Voltera (Ecology)

$$\begin{cases} \dot u = \alpha u - \beta uv \\ \dot v = \delta uv - \gamma v, \end{cases} \quad \alpha, \beta, \gamma, \delta > 0.$$

We observe that $V(u,v) := \delta u - \gamma \ln u + \beta v - \alpha \ln v$ is such that

$$\frac{d}{dt} V = \big(\delta - \frac{\gamma}{u}\big)\dot u + \big(\beta - \frac{\alpha}{v}\big)\dot v = 0.$$

Example 10. FitzHugh-Nagumo equation (Neuroscince)

$$\begin{cases} \dot u = u - u^3/3 - v \\ \tau \dot v = u + a - bv, \end{cases} \quad a, b, \tau > 0.$$

Relaxation oscillator in spiking neuron dynamic.

Example 11. Chemical reaction equation (Chemistry)

$$\begin{cases} \dot u = v^k - u \\ \dot v = u - v^k, \end{cases} \quad k \in \mathbb{N}.$$

We observe that

$$\frac{d}{dt}(u + v) = 0.$$

General framework. An ODE is on "normal form" when

$$y^{(n)} = f(t, y, y', ..., y^{(n-1)}), \quad f : \mathbb{R}^{n+1} \to \mathbb{R}.$$

Such an equation, may be written as a system of first order

$$Y' = F(t, Y), \quad F : \mathbb{R}^{n+1} \to \mathbb{R}^n. \tag{1.1}$$

By introducing $U(t) := (t, Y(t)) \in \mathbb{R}^{n+1}$ and $\mathcal{F}(U) := (1, F_1(u_2, ..., u_{n+1}), ..., F_n(u_2, ..., u_{n+1}))$, a system of first order reduces to an autonomous system of first order

$$U' = \mathcal{F}(U). \tag{1.2}$$

In the sequel, we will consider a system of first order (1.1) or an autonomous system of first order (1.2).

2. The Cauchy-Lipschitz Theorem

We will always assume

$$F \in C^1(\mathbb{R} \times \mathbb{R}^d; \mathbb{R}^d), \quad d \geq 1, \tag{2.1}$$

and we consider the first order system

$$u' = F(t, u), \quad u(t_0) = u_0. \tag{2.2}$$

However:

(1) Exactly the same Cauch-Lipschitz theory holds without any change in the statement and in the proof when $F : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ and

$$|F(t, u_2) - F(t, u_2)| \leq C_R |u_2 - u_1|,$$

for any $(t, u_1, u_2) \in [-R, R] \times B_R \times B_R$ and any $R > 0$.

(2) We may adapt the result when $F : \mathcal{O} \to \mathbb{R}^d$ with $\mathcal{O} \subset \mathbb{R} \times \mathbb{R}^d$ open.

(3) We may generalize the Cauch-Lipschitz theory to the case when F enjoys Sobolev regularity (DiPerna-Lions theory).

(4) Some other extensions are possible: Osgood condition, log-Lipschitz condition, Nagumo uniqueness condition, ...

We call solution to the ODE (2.2) a couple (J, V) such that J is an open interval containing t_0, $v : J \to \mathbb{R}^d$ is a C^1 function and the equation (2.2) holds punctually, namely

$$v(t_0) = u_0, \quad \forall\, t \in J, \ v'(t) = F(t, v(t)).$$

We say that $(\mathcal{I}, u)$ is a maximal solution if for any solution (J, v), there holds $J \subset \mathcal{I}$ and $u_{|J} = v$. We say that $(\mathcal{I}, u)$ is a globall solution if $\mathcal{I} = \mathbb{R}$.

Theorem 2.1. *For any $(t_0, u_0) \in \mathbb{R} \times \mathbb{R}^d$, there exists a unique maximal solution $(\mathcal{I}, u)$, and denoting $\mathcal{I} =]T_-, T_+[$, the following alternatives hols*

$$T_- = -\infty \quad \textit{or} \quad |u(t)| \to \infty, \ t \searrow T_-,$$
$$T_+ = +\infty \quad \textit{or} \quad |u(t)| \to \infty, \ t \nearrow T_+.$$

Because of (2.1) and the equivalence between (1.1) and (1.2), we may only consider the autonomous case $F = F(u)$.

Sorry : the proof below of Theorem 2.1 is written in french!

La preuve repose sur les deux lemmes suivants.

Lemma 2.2. *Soient $t_1 \in \mathbb{R}$, $v_1 \in \mathbb{R}^d$ et J un intervalle ouvert de $\mathbb{R}$ contenant t_1. Un couple (J,v), $v \in C^1(J,\mathbb{R}^d)$, est solution de l'équation différentielle*

$$v'(t) = F(v(t)),\ \forall\, t \in J, \quad v(t_1) = v_1, \tag{2.3}$$

si, et seulement si, le couple (J,v), $v \in C(J,\mathbb{R}^d)$, est solution de l'équation intégrale

$$v(t) = v_1 + \int_{t_1}^{t} F(v(s))\, ds, \quad \forall\, t \in J. \tag{2.4}$$

Preuve du Lemme 2.2. Il suffit d'intégrer l'équation (2.3) pour montrer l'implication directe et de dériver l'équation (2.4) pour montrer l'implication réciproque (ce qui est possible puisqu'une solution de (2.4) est automatiquement de classe C^1). □

Pour $R, \tau > 0$, $t_1 \in \mathbb{R}$ et $v_1 \in \mathbb{R}^d$, $|v_1| \le R$, on définit
- l'intervalle fermé $J := [t_1 - \tau, t_1 + \tau]$;
- l'espace $E = C(J,\mathbb{R}^d)$ que l'on munit de la norme de la convergence uniforme

$$\|v\| := \sup_{t \in J} |v(t)|, \quad \forall\, v \in C(J,\mathbb{R}^d),$$

ce qui en fait un espace de Banach;
- l'ensemble complet

$$\mathcal{C} := \{v \in C(J,\mathbb{R}^d);\ v(t_1) = v_1,\ \|v\| \le 2R\}. \tag{2.5}$$

Lemma 2.3. *Pour tout $R > 0$, il existe $\varepsilon > 0$, tel que pour tout $t_1 \in \mathbb{R}$ et $v_1 \in \mathbb{R}^d$, $|v_1| \le R$,*
- existence locale: il existe une solution (J_1, v_1) à l'équation intégrale (2.4) pour $J_1 := [t_1 - \varepsilon, t_1 + \varepsilon]$;
- unicité locale: pour tout $\tau \in (0,\varepsilon]$, il y a unicité de la solution $v \in \mathcal{C}$ de l'équation intégrale (2.4), et $v = v_{1|J}$.

Preuve du Lemme 2.3. Etape 1: Préliminaires. On se donne $\tau > 0$ et on écrit (2.4) sous la forme

$$v \in E,\ v = \mathcal{F}(v), \tag{2.6}$$

avec $\mathcal{F} : E \to E$, $v \mapsto \mathcal{F}(v) := w$, où

$$w(t) := v_1 + \int_{t_0}^{t} F(v(s))\, ds, \quad \forall\, t \in J,$$

et on rappelle que J, E et $\mathcal{C}$ ont été définis juste avant l'énoncé du Lemme. On vérifie qu'il existe $\varepsilon > 0$, assez petit, tel que si $\tau \in]0,\varepsilon]$, on a

$$\mathcal{F} : \mathcal{C} \to \mathcal{C} \tag{2.7}$$

et

$$\forall\, v_1, v_2 \in \mathcal{C}, \quad \|\mathcal{F}(v_2) - \mathcal{F}(v_1)\| \le \frac{1}{2}\|v_2 - v_1\|. \tag{2.8}$$

Pour (2.7), on écrit

$$\begin{aligned} |w(t)| &\le |u_0| + \int_{[t_0,t]} \sup_{s \in J} |F(v(s)|\, ds \\ &\le R + |t - t_0| \sup_{|x| \le 2R} |F(x)| \le 2R, \end{aligned}$$

lorsque $\varepsilon \in (0,\varepsilon_1]$, avec $\varepsilon_1 > 0$ assez petit.

Pour (2.8), on pose

$$L_R := \sup_{|x|,|y|\leq 2R,\, x\neq y} \frac{|F(x)-F(y)|}{|x-y|},$$

et on écrit

$$\begin{aligned} |w_2(t)-w_1(t)| &= \Big| \int_{t_0}^{t} (F(v_2(s)) - F(v_1(s)))\, ds \Big| \\ &\leq \int_{[t_0,t]} \Big\{ \sup_{|x-y|\leq 4R,\ x\neq y} \frac{|F(x)-F(y)|}{|x-y|} \Big\} |v_2(s)-v_1(s)|\, ds \\ &\leq |t-t_0| L_R \|v_2 - v_1\| \leq \frac{1}{2}\|v_2-v_1\|, \end{aligned}$$

lorsque $\varepsilon \in (0,\varepsilon_2]$, avec $\varepsilon_2 > 0$ assez petit. On a (2.7) et (2.8) en prenant $\varepsilon := \min(\varepsilon_1, \varepsilon_2)$.

Etape 2: Existence locale. On peut appliquer le théorème de point fixe de Banach à la fonction $\mathcal{F}$ définie sur $C(J_1;\mathbb{R}^d)$, ou plus simplement, on définit la suite $v^{n+1} := \mathcal{F}(v^n)$, $v^0 = u_0$. On a alors $v^n \in \mathcal{C}$ pour tout $n \geq 0$ d'après (2.7). Par récurrence, on a également

$$\|v^{n+1} - v^n\| = \|\mathcal{F}(v^n) - \mathcal{F}(v^{n-1})\| \leq \frac{1}{2}\|v^n - v^{n-1}\| \leq \frac{1}{2^n}\|v^1 - v^0\|.$$

Il s'ensuit que (v^n) est une suite de Cauchy dans l'espace complet $\mathcal{C}$, et qu'il existe donc $v_1 \in \mathcal{C}$ tel que $v_1 = \lim v^n$. Par continuité de $\mathcal{F}$ (qui est 1/2-Lipschitzienne dans $\mathcal{C}$), on a également $\mathcal{F}(v_1) = \lim \mathcal{F}(v^n) = \lim v^n = v_1$, ce qui conclut que (J_1, v_1) est solution de (2.6).

Etape 3: Unicité locale. Considérons (J,v) une solution de l'équation intégrale (2.4). Plus précisément, on suppose que $J = [t_1 - \tau, t_1 + \tau] \subset J_1$, et on suppose que $v \in \mathcal{C}$, cet ensemble étant défini par (2.5). D'après (2.8), et en notant $\|\cdot\|$ la norme de la convergence uniforme de $C(J,\mathbb{R}^d)$, on a

$$\|v - v_1\| = \|\mathcal{F}(v) - \mathcal{F}(v_1)\| \leq \frac{1}{2}\|v - v_1\|,$$

ce qui implique bien $v = v_{1|J}$. □

Preuve du Théorème 2.1. Etape 1: construction. On note $\mathcal{S}$ l'ensemble des solutions (J,v) à l'équation différentielle (2.2) avec J intervalle ouvert contenant t_0 et $v \in C^1(J,\mathbb{R}^d)$. D'après le lemme 2.3 appliqué à $t_1 = t_0$, il existe une solution (J_1, v_1) à l'équation intégrale (2.4). D'après le lemme 2.2, cela donne une solution (J,v) à l'équation différentielle (2.2), en posant $J := \mathrm{Int}(J_1)$ et $v := v_{1|J}$. Cela démontre que $\mathcal{S} \neq \emptyset$.

Soient deux éléments (J_1, v_1) et (J_2, v_2) de $\mathcal{S}$.

- Commençons par démontrer que $v_1 = v_2$ sur l'intervalle $J_1 \cap J_2$. Pour cela, observons que $v_1(t_0) = v_2(t_0) = u_0$ de sorte que le plus grand intervalle $\mathcal{Z}$ contenu dans $J_1 \cap J_2$, contenant t_0 et tel que $v_{1|\mathcal{Z}} = v_{2|\mathcal{Z}}$ est bien défini. Il est fermé puisque v_1 et v_2 sont continus. Il est également ouvert. En effet, si $t_1 \in \mathcal{Z}$, en posant $R := |v_1(t_1)| = |v_2(t_2)|$, il existe $\tau > 0$ tel que

$$J' := [t_1 - \tau, t_1 + \tau] \subset J_1 \cap J_2, \quad \sup_{t\in J'} |v_i(t)| \leq 2R.$$

En notant $\varepsilon = \varepsilon(R) > 0$ le nombre donné par le Lemme 2.3, et en posant $J'' := J' \cap [t_1 - \varepsilon, t_1 + \varepsilon]$, le résultat d'unicité locale du Lemme 2.3, nous dit que $v_{1|J''} = v_{2|J''}$.

Cela implique bien que $J'' \subset \mathcal{Z}$ et $\mathcal{Z}$ est ouvert. Par connexité, nous avons établi que $\mathcal{Z} = J_1 \cap J_2$.
- On peut alors définir $J := J_1 \cup J_2$ et la fonction $v \in C^1(J, \mathbb{R}^d)$, en posant

$$v(t) := v_1(t) \text{ si } t \in J_1, \quad v(t) := v_1(t) \text{ si } t \in J_2.$$

Le couple (J, v) est ainsi une solution telle que $J_i \subset J$ et $v_{|J_i} = v_i$.

Pour conclure, on pose

$$I := \bigcup_{(J,v)\in\mathcal{S}} J, \quad u(t) = v(t) \text{ si } t \in J, \ (J, v) \in \mathcal{S}.$$

Il est alors clair que (I, u) est solution maximale.

Etape 2: alternative. On note $I =]T_-, T_+[$ et traitons le critère d'exposition en l'extrémité T_+ (l'extrémité T_- se traite de la m me manière). Supposons $T_+ < \infty$ et supposons par l'absurde qu'il existe $R < \infty$ tel que

$$\liminf_{t \nearrow T^+} |u(t)| \le R.$$

En notant $\varepsilon = \varepsilon(R) > 0$ le nombre donné par le Lemme 2.3, et en choisissant $t_1 \in (T_+ - \varepsilon, T_+)$, le résultat d'existence locale du Lemme 2.3, nous dit qu'il existe v_1 solution de l'équation intégrale (2.4) associée à la donné initiale $v_1(t_1) = u(t_1)$ et définie sur l'intervalle $]t_1 - \varepsilon, t_1 + \varepsilon[$. D'après l'Etape 1 appliquée à la condition initiale $u(t_1)$ en t_1, nous obtenons une solution $\tilde{u}$ de l'équation différentielle

$$\tilde{u}' = f(\tilde{u}), \quad \tilde{u}(t_1) = u(t_1),$$

définie sur l'intervalle $\tilde{J}_1 = I \cup J_1$. Comme $\tilde{J}_1$ contient strictement I et que $\tilde{u}$ est également une solution différentielle (2.2), cela contredit la maximalité de (I, u). On a donc démontré que $|u(t)| \to +\infty$ lorsque $t \nearrow T_+$. □

When F is globaly Lipschitz, we may adapt the proof and obtain a global solution.

Lemma 2.4. *When $F \in \mathrm{Lip} \cap C^1$ there exists a unique global solution $u \in C^1(\mathbb{R}; \mathbb{R}^d)$ to the ODE* (2.2)*.*

Preuve du Lemme 2.4. Pour simplifier on prend $t_0 = 0$. On note $a > L := \|F'\|_{L^\infty}$. Pour une fonction $v \in C(\mathbb{R}; \mathbb{R}^d)$, on définit

$$\|v\| := \sup_{t\in\mathbb{R}} e^{-a|t|} |v(t)| \in [0, +\infty].$$

On définit enfin l'espace

$$\mathcal{E} := \{v \in C(\mathbb{R}, \mathbb{R}^d); \ \|v\| < \infty\},$$

qui muni de la norme $\|\cdot\|$ est un espace de Banach. Nous allons montrer que l'application

$$w(t) := (\mathcal{F}(v))(t) := u_0 + \int_0^t F(v(s))\, ds$$

est définie de $\mathcal{E}$ dans lui-m me et est une contraction, ce qui nous permettra de conclure comme dans la preuve du Lemme 2.3.

D'une part, pour $v \in \mathcal{E}$ et $t \in \mathbb{R}$, on a

$$\begin{aligned} |w(t)|e^{-a|t|} &\le |u_0|e^{-a|t|} + e^{-a|t|} \int_{-|t|}^{|t|} \big(|F(v(s)) - F(0)| + |F(0)|\big) ds \\ &\le (|u_0| + 2|tF(0)|)e^{-a|t|} + e^{-a|t|} \int_{-|t|}^{|t|} L|v(s)| ds \\ &\le |u_0| + \frac{2}{ae}|F(0)| + L\|v\| e^{-a|t|} \int_{-|t|}^{|t|} e^{a|s|} ds \\ &\le |u_0| + \frac{2}{ae}|F(0)| + 2\frac{L}{a}\|v\|(1 - e^{-a|t|}), \end{aligned}$$

de sorte que $w \in \mathcal{E}$.

D'une part, pour $v_i \in \mathcal{E}$, $t \in \mathbb{R}_+$ et en notant $w_i = \mathcal{F}(v_i)$, $w := w_2 - w_1$, $v := v_2 - v_1$, on a

$$\begin{aligned} |w(t)|e^{-a|t|} &= e^{-a|t|} \Big| \int_0^t \big(F(v_2(s)) - F(v_1(s))\big) ds \Big| \\ &\le e^{-a|t|} \int_0^t L|v(s)| ds \le L\|v\| e^{-a|t|} \int_0^t e^{as} ds \le \frac{L}{a}\|v\|. \end{aligned}$$

En effectuant un calcul semblable pour $t \in \mathbb{R}_-$, on obtient que $\mathcal{F}$ est Lipschitzienne de contante de Lipschitz $L/a < 1$. □

3. Gronwall Lemma ans first applications

We refer to *Chapter 1 - On the Gronwall lemma* to the course *"An introduction to evolution PDEs"* for more material about this issue. Gronwall Lemma is a useful tool.

Lemma 3.1 (classical differential version of Gronwall lemma)**.** *We assume that* $u \in C([0,T);\mathbb{R})$*,* $T \in (0,\infty)$*, satisfies the differential inequality*

$$u' \le a(t)u + b(t) \quad on \quad (0,T), \tag{3.1}$$

for some $a, b \in L^1(0,T)$*. Then,* u *satisfies the pointwise estimate*

$$u(t) \le e^{A(t)}u(0) + \int_0^t b(s)e^{A(t)-A(s)}\, ds, \quad \forall\, t \in [0,T), \tag{3.2}$$

where we have defined the primitive function

$$A(t) := \int_0^t a(s)\, ds.$$

Some examples and important special cases of the Gronwall lemma are

$$u' \le a(t)u \implies u(t) \le u(0)e^{A(t)}, \tag{3.3}$$

$$u' \le au + b \implies u(t) \le u(0)e^{at} + \frac{b}{a}(e^{at} - 1), \tag{3.4}$$

$$u' \le au + b(t) \implies u(t) \le u(0)e^{at} + \int_0^t e^{a(t-s)}\, b(s)\, ds, \tag{3.5}$$

$$u' + b(t) \le a(t)u,\ a,\, b \ge 0 \implies u(t) + \int_0^t b(s)\, ds \le u(0)e^{A(t)}. \tag{3.6}$$

Proof of Lemma 3.1. The differential inequality (3.1) means

$$-\langle u, \varphi' \rangle \le \langle au + b, \varphi \rangle$$

for any $0 \le \varphi \in \mathcal{D}(0,T)$. We set

$$v(t) = u(t)\, e^{-A(t)} - \int_0^t b(s) e^{-A(s)}\, ds,$$

and we observe that

$$v' \le 0 \quad \mathcal{D}'(0,T), \qquad v \in C([0,T]).$$

• When furthermore $v \in C^1$ (or even $v \in W^{1,1}$), we immediately conclude

$$v(t) = v(0) + \int_0^t v'(s)\, ds \le v(0) = u(0),$$

from what (3.2) follows.

• In the general case when $v \in C([0,T])$, we proceed as follows. We fix $\varepsilon > 0$ and $\varrho \in C^1_c(0,\varepsilon)$ such that $\varrho \ge 0$, $\int \varrho = 1$. For any function $0 \le w \in C^1_c(\varepsilon, T)$, the function $\psi := -w + (\int_0^T w)\, \varrho$ belongs to $C_c(0,T)$ and $\int_0^T \psi = 0$. As a consequence ψ has a primitive φ such that $\varphi(0) = \varphi(T) = 0$. The function φ thus enjoys the following properties $\varphi \in C^1_c(0,T)$, $\varphi \ge 0$ and $\varphi' = \psi$. We deduce

$$\begin{aligned} 0 \ge \langle v', \varphi \rangle &= \int_0^T v \left\{ w - \left(\int_0^T w \right) \varrho \right\} dt \\ &= \int_0^T w \left\{ v - \int_0^T v\, \varrho \right\} dt. \end{aligned}$$

Because the above inequality is true for any $w \in C^1_c(\varepsilon, T)$, $w \ge 0$, it comes

$$v \le \int_0^T v\, \varrho \quad \text{on} \quad (\varepsilon, T).$$

Taking $\varrho = \varrho_\alpha$ for a mollifier sequence (ϱ_α) (i.e. $\varrho_\alpha \rightharpoonup \delta_0$ as $\alpha \to 0$) and letting $\alpha \to 0$, we deduce again $v \le v(0)$ on $(0,T)$. □

In many situations, it is not easy to deal with differential inequalities and it is much more natural to start from the associated integral inequality. The conclusion can be however the same.

Lemma 3.2 (integral version of Gronwall lemma)**.** *We assume that* $u \in C([0,T);\mathbb{R})$*,* $T \in (0,\infty)$*, satisfies the integral inequality*

$$u(t) \le u_0 + \int_0^t a(s)u(s)\,ds + \int_0^t b(s)\,ds \quad \text{on} \quad [0,T), \tag{3.7}$$

for some $0 \le a \in L^1(0,T)$ *and* $b \in L^1(0,T)$*. Then,* u *satisfies the same pointwise estimate*

$$u(t) \le u_0\, e^{A(t)} + \int_0^t b(s) e^{A(t)-A(s)}\,ds, \quad \forall\, t \in (0,T).$$

Remark 3.3. *Lemma 3.2 can be seen as an integral (and thus weak) version of Lemma 3.1, but we emphasize that we additionally need to assume* $a \ge 0$ *here.*

Some examples and important special cases of the Gronwall lemma are

$$u(t) \le \int_0^t a(s)u(s)\,ds \implies u(t) \equiv 0, \tag{3.8}$$

$$u(t) \le u_0 + \int_0^t a(s)u(s)\,ds \implies u(t) \le u_0\, e^{A(t)}, \tag{3.9}$$

$$u(t) + \int_0^t |b(s)|ds \le u_0 + \int_0^t a(s)u(s)\,ds \implies u(t) + \int_0^t |b(s)|ds \le u(0)e^{A(t)}. \tag{3.10}$$

Proof of Lemma 3.2. Step 1. We first assume that $b \equiv 0$*.* We set $v(t) = u(t) - u_0\, e^{A(t)}$ and we compute

$$\begin{aligned} v(t) &\le \int_0^t a(s)\, u(s)\, ds + u_0\,(1 - e^{A(t)}) \\ &= \int_0^t a(s)\,(v(s) + u_0\, e^{A(s)})\, ds + u_0\,(1 - e^{A(t)}) \\ &= \int_0^t a(s)\, v(s)\, ds. \end{aligned}$$

Because a is not negative, it yields

$$v^+(t) \le \int_0^t a(s)\, v^+(s)\, ds =: w(t). \tag{3.11}$$

The function $w \in W^{1,1}(0,T)$ satisfies

$$w'(t) = a(t)\, v^+(t) \le a(t)\, w(t) \quad \text{on} \quad (0,T),$$

and we may use Lemma 3.1 in order to deduce $w(t) \le w(0) = 0$, next $v(t) \le v^+(t) \le w(t) \le 0$ and the conclusion.

Step 2. We do not assume $b \equiv 0$ *anymore.* We define

$$v(t) := u(t) - u_0 e^{A(t)} - \int_0^t b(s) e^{A(t)-A(s)}\,ds.$$

We observe that we have again

$$v(t) \le \int_0^t a(s)\, v(s)\, ds,$$

and we conclude as in the first step. □

We mention one possible discrete version of the Gronwall lemma.

Lemma 3.4 (discrete version of Gronwall lemma)**.** *We consider a real numbers sequence* (u_n) *such that*

$$u_{n+1} \le a_{n+1} u_n + b_{n+1}, \quad \forall\, n \ge 0, \tag{3.12}$$

where (a_n) *and* (b_n) *are two given real numbers sequences and* (a_n) *is furthermore positive. Then*

$$u_n \le A_n u_0 + \sum_{k=1}^{n} A_{k,n} b_k, \quad \forall\, n \ge 0, \tag{3.13}$$

where we have defined

$$A_n := \prod_{k=1}^{n} a_k, \quad A_{k,n} = A_n / A_k = \prod_{i=k+1}^{n} a_i.$$

Proof of Lemma 3.4. We define

$$v_n := A_n u_0 + \sum_{k=1}^{n} A_{k,n} b_k,$$

and we observe that

$$\begin{aligned} v_{n+1} &= A_{n+1} u_0 + \sum_{k=1}^{n+1} A_{k,n+1} b_k \\ &= a_{n+1} A_n u_0 + \sum_{k=1}^{n} a_{n+1} A_{k,n} b_k + b_{n+1} \\ &= a_{n+1} v_n + b_{n+1}. \end{aligned}$$

We then easily check by induction that $u_n \le v_n$ for any $n \ge 0$. □

Some particularly interesting special cases of that discrete Gronwall lemma are

$$u_{n+1} \le a u_n + b_{n+1} \implies u_n \le a^n u_0 + \sum_{k=1}^{n} a^{n-k} b_k, \tag{3.14}$$

$$u_{n+1} \le (1+\alpha) u_n + b_{n+1},\ \alpha, b_{n+1} \ge 0 \implies u_n \le e^{n\alpha} u_0 + e^{n\alpha} \sum_{k=1}^{n} b_k, \tag{3.15}$$

$$u_{n+1} + b_{n+1} \le a u_n,\ a \ge 1, b_{n+1} \ge 0 \implies u_n + \sum_{k=1}^{n} b_k \le a^n u_0. \tag{3.16}$$

Lemma 3.5. *When* F *is* C^1 *and satisfies*

$$|F(t,u)| \le B(T)(1+|u|), \quad \forall\, t \in [-T,T],\ \forall\, u \in \mathbb{R}^d \tag{3.17}$$

for some function $B \in C(\mathbb{R}_+)$*, what is the case when* F *is globally Lipschitz, then the maximal solution given by the Cauchy-Lipschitz theorem is a global solution.*

Proof of Lemma 3.5. We only show $\sup I = +\infty$ when $B(T) = B$. From the integrale equation (2.4), we get

$$|u(t)| \le |u_0| + \int_0^t B(1+|u(s)|)\,ds, \quad \forall\, t \in I_+ := I \cap \mathbb{R}_+.$$

From the Gronwall Lemma 3.2, we deduce

$$|u(t)| \le (|u_0|+1) e^{Bt} - 1, \quad \forall\, t \in I_+.$$

The function u cannot blow up in finite time and the alternatives formulated in the statement of Theorem 2.1 imply $\sup I = +\infty$. □

Lemma 3.6 (of continuous dependance)**.** *Assume $F \in C^1$ and satisfies* (3.17) *so that any maximal solution is a global solution. For any $T, R \geq 0$, there exist a constant $C_{T,R} \geq 1$ such that*

$$|u_2(t) - u_1(t)| \leq C_{T,R}|u_2(0) - u_1(0)|, \quad \forall\, t \in [0,T],$$

if $|u_i(0)| \leq R$.

Proof of Lemma 3.6. We only again consider the case $B(T) = B$. From Lemma 3.5, we know that

$$|u_i(t)| \leq B(T,R) := (R+1)e^{BT} \quad \forall\, t \in [0,T].$$

We set

$$L := \sup_{t\in[0,T]} \sup_{|u|\leq B(T,R)} \|D_x F(t,u)\|.$$

From the integral equation (2.4), we get

$$\begin{aligned} |u_2(t) - u_1(t)| &= \left| u_2(0) - u_1(0) + \int_0^t [F(s,u_2(s)) - F(s,u_1(s))]\, ds \right| \\ &\leq |u_2(0) - u_1(0)| + L \int_0^t |u_2(s) - u_1(s)|\, ds, \end{aligned}$$

for any $t \in [0,T]$. From the Gronwall Lemma 3.2, we then obtain the result with $C_{T,R} = e^{LT}$. □

We next present a generalization of the Gronwall lemma to a nonlinear differential inequality framework.

Lemma 3.7 (nonlinear version of Gronwall lemma)**.** *Let $f \in C^1((0,T) \times \mathbb{R})$ and consider $u, v \in C([0,T];\mathbb{R})$ such that*

$$u' \leq f(t,u), \quad v' \geq f(t,u), \quad u(0) \leq v(0), \tag{3.18}$$

(in a distributional sense). Then $u \leq v$ on $[0,T]$.

Proof of Lemma 3.7. We set $w(t) := u(t) - v(t) \in C([0,T])$ and

$$a(t) := \partial_u f(t,u(t)) \text{ if } v(t) = u(t); \quad a(t) := \frac{f(t,u(t)) - f(t,v(t))}{u(t) - v(t)} \text{ otherwise,}$$

and we observe that $a \in C([0,T])$. We compute

$$w' = u' - v' \leq f(t,u) - f(t,v) = a\, w, \quad w(0) \leq 0.$$

We may apply Lemma 3.1 (or more precisely (3.3)) which implies $w \leq 0$ and that gives our conclusion. □

Exercise 3.8. *Establish a integral version of Lemma 3.7. More precisely, consider $f \in C^1((0,T)\times\mathbb{R})$ increasing with respect to the second variable and $u, v \in C([0,T])$ such that*

$$u(t) \leq u_0 + \int_0^t f(s,u(s))\, ds, \quad v(t) \geq v_0 + \int_0^t f(s,v(s))\, ds, \quad u_0 \leq v_0,$$

and prove that $u \leq v$ on $[0,T]$. [Hint. Observe that $w_t := u(t) - v(t)$ satisfies

$$w_t \leq L \int_0^t w_s\, ds$$

for some constant L which may depend on u, v and f].

A consequence of Lemma 3.7 is the following useful result.

Lemma 3.9. *Let $0 \le u \in C([0,\infty))$ satisfy (in the sense of distributions)*

$$u' + k_1\, u^{\theta_1} \le k_2\, u^{\theta_2} + k_3,$$

with $\theta_1 > 0$, $\theta_2 \ge 0$, $\theta_2/\theta_1 < 1$, $k_1 > 0$ and k_2, $k_3 \ge 0$. Then, there exists $C_0 = C_0(k_i, \theta_i) \ge 0$ such that

$$\sup_{t\ge 0} u(t) \le \max(C_0, u(0)). \tag{3.19}$$

Assume moreover $\theta_1 > 1$. For any $\tau > 0$, there exists $C_\tau = C_\tau(k_i, \theta_i, \tau) \ge 0$ such that

$$\sup_{t\ge \tau} u(t) \le C_\tau. \tag{3.20}$$

Proof of Lemma 3.9. Step 1. We set $f(u) := k_2\, u^{\theta_2} + k_3 - k_1\, u^{\theta_1}$ for any $u \ge 0$ and we observe that there exists $C_0 > 0$ (large enough) such that $f(u) \le 0 = f(C_0)$ for any $u \ge C_0$. As a consequence, u is a subsolution and $v := \max(u_0, C_0)$ is a supersolution to the ODE $w' = f(w)$ both with same initial datum. We conclude thanks to Lemma 3.7.

Step 2. For (any) $C_0' > C_0$ there exists $k_1' > 0$ such that $f(u) \le -k_1'\, u^{\theta_1}$ for any $u \ge C_0'$. We consider the solution v to the ODE

$$v' = -k_1' v^{\theta_1}, \quad v(0) = u_0.$$

From Example 3 and Lemma 3.7, we immediately deduce that v satisfies

$$v(t) \le (k_1' t)^{-\frac{1}{\theta_1 - 1}}, \quad \forall\, t > 0.$$

We conclude that (3.20) holds for any $\tau > 0$ with $C_\tau := \max(C_0', (k_1'\tau)^{-\frac{1}{\theta_1-1}})$. □

Corollary 3.10. *Consider $y \in C^1(I;\mathbb{R}^n)$ such that*

$$|y'(t)| \le \lambda(t)\, |y(t)| + \mu(t) \quad \forall\, t \in I$$

with $\lambda, \mu \in C(I)$ et $\lambda \ge 0$. Define $v, w \in C^1(I)$ the solutions to

$$v' = \lambda\, v + \mu, \quad w' = -\lambda\, w - \mu, \quad v(t_0) = w(t_0) = |y(t_0)|.$$

Then

$$\forall\, t \in I,\ t \ge t_0 \quad |y(t)| \le v(t), \qquad \forall\, t \in I,\ t \le t_0 \quad |y(t)| \le w(t).$$

Sorry, the proof of this result is written is french!
Preuve du Corollaire 3.10. On suppose $t_0 = 0$ et on pose $u(t) = |y(t)|$ de sorte que

$$\begin{aligned} u(t) &= \left| y(t_0) + \int_{t_0}^t y'(s)\, ds \right| \le |y(t_0)| + \int_{t_0}^t |y'(s)|\, ds \\ &\le |y(t_0)| + \int_{t_0}^t [\lambda(s)\, |y(s)| + \mu(s)]\, ds \end{aligned}$$

et on peut appliquer Lemma 3.7 à la fonction u, de sorte que $u(t) \le v(t)$ pour $t \ge 0$.

Pour les temps $t \le t_0$ on fait un raisonnement semblable en posant $u(t) := |y(t_0 - t)|$. Par le même calcul que ci-dessus on obtient

$$u(t) \le |y(t_0)| + \int_0^t [\lambda(t_0 - s)\, u(s) + \mu(t_0 - s)]\, ds.$$

La version 3 implique que $u \le z$ sur $[0, T]$ où $T := t_0 - \inf I$ et z est la solution de l'équation

$$z' = \tilde{\lambda}\, z + \tilde{\mu}, \quad z(0) = |y(t_0)|, \quad \tilde{\lambda}(s) = \lambda(t_0 - s), \;\; \tilde{\mu}(s) = \mu(t_0 - s).$$

Or justement $z_*(t) := w(t_0 - t)$ définie sur $[0, T]$ satisfait $z_*(0) = |y(t_0)|$ et

$$z_*'(t) = -w'(t_0 - t) = (\lambda\, w - \mu)(t_0 - t) = \tilde{\lambda}\, z_* + \tilde{\mu}.$$

Par unicité de la solution on a donc $z \equiv z_*$ et donc $|y(t)| = u(t_0 - t) \le z(t_0 - t) = w(t)$ pour tout $t \in I$, $t \le t_0$. □

4. Local stability, smooth dependance and flow

4.1. Local stability, smooth dependance.

In this section, we first consider the family of Cauchy problems

$$\dot{y} = f(t, y; \lambda), \qquad y(t_0) = y_0(\lambda), \tag{4.1}$$

with $f : \mathbb{R} \times \mathbb{R}^d \times \Lambda \to \mathbb{R}^d$ and $y_0 : \Lambda \to \mathbb{R}^d$ of class C^1, where $\Lambda \subset \mathbb{R}^m$ is an open set. We additionally assume that there exists $M \ge 0$ and for any $R \in (0, \infty)$ there exists $L_R \ge 0$ such that

$$|f(t, x, \lambda) - f(t, y, \lambda)| \le L_R |x - y| \tag{4.2}$$

$$|f(t, x, \lambda)| \le M(1 + |x|) \tag{4.3}$$

for any t, x, y and λ.

Remark 4.1. *One could also consider a local formulation of the same result by just assuming $f : \mathcal{O} \to \mathbb{R}^d$ of class C^1, where $\mathcal{O}$ is an open set of $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^m$ such that $(t_0, y_{0,\lambda}, \lambda) \in \mathcal{O}$. In that case, one should bother with maximal solutions which are possibly not global solutions.*

Theorem 4.2. *Assume (4.2) and (4.3).*
(1) $\exists!$ global solution $y = y(t; \lambda)$ to (4.1) for any $(\lambda, y_0(\lambda)) \in \Lambda \times \mathbb{R}^d$.
(2) Trajectoires are bounded on $[-T, T]$ unifomly in $\lambda \in \Lambda$.
(3) $\lambda \mapsto y(t, \lambda)$ is continuous.
(4) $\lambda \mapsto y(t, \lambda)$ is Lipschitz.
(5) $y \in C^1(\mathbb{R} \times \Lambda; \mathbb{R}^d)$ and $w := D_\lambda y \in C(\mathbb{R}; M_{m,d}(\mathbb{R}))$ is the unique global solution to the linear ODE

$$\dot{w} = D_x f(t, y_\lambda(t), \lambda) w + D_\lambda f(t, y_\lambda(t), \lambda), \quad w(0) = D_\lambda y_0(\lambda).$$

The proof of Theorem 4.2 is very similar to the proof of Theorem 4.4 to which we refer. Preuve en commentaire à rédiger pour une future version.

4.2. **The flow associated to an ODE.**

In this section, we consider the ODE

$$\dot{x}(t) = a(t, x(t)), \quad x(s) = x, \tag{4.4}$$

with regularity and uniform growth bound

$$a \in C^1(\mathbb{R} \times \mathbb{R}^d; \mathbb{R}^d), \quad a/\langle x \rangle \in L^\infty(\mathbb{R} \times \mathbb{R}^d; \mathbb{R}^d). \tag{4.5}$$

Remark 4.3. *One could also consider a much more general framework as mentionded in the previous section. In particular, the second condition on the vector field a may be relaxed as*

$$|a(t, x)| \le \Gamma(t)\langle x \rangle, \quad \Gamma \in C(\mathbb{R}).$$

Theorem 4.4. *Assume* (4.5).
(1) For any $(x, s) \in \mathbb{R} \times \mathbb{R}^d$, there exists $(x(t))_{t\in\mathbb{R}}$ a unique global solution to (4.4)*. We then define*

$$\Phi : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d, \quad (t, s, x) \mapsto \Phi_{t,s}(x) := x(t).$$

(2) Φ satisfies the group property

$$\Phi_{s,s} = I, \quad \Phi_{t_3,t_2} \circ \Phi_{t_2,t_1} = \Phi_{t_3,t_1}, \quad \forall\, s, t_i \in \mathbb{R}.$$

(3) Φ is C^1 and $D_x\Phi_{t,s} := A$ is the unique solution to the linear ODE

$$A' = D_x a(t, \Phi_{t,s}(x))A, \quad A(s) = I.$$

(4) In particular $\Phi_{t,s}$ is a C^1 diffeomorphism for any $(t, s) \in \mathbb{R}^2$ and the mapping $[0, T] \times [0, T] \times B(0, R) \to \mathbb{R}^d$, $(s, t, x) \mapsto \Phi_{s,t}(x)$ is Lipschitz for any $T, R > 0$ (we denote by $L_{T,R}$ the associated Lipschitz constant).

Remark 4.5. *(1) From (2), we have*

$$\Phi_{t,s} \circ \Phi_{s,t} = \Phi_{t,t} = I, \quad \forall\, s, t \in \mathbb{R},$$

so that $\Phi_{t,s}^{-1} = \Phi_{s,t}$.
(2) In the autonomous case $a = a(x)$, we set $X_t = \Phi_{t,0}$ and we have

$$\Phi_{t,s} = X_{t-s}, \quad X_t \circ X_s = X_{t+s}, \quad X_0 = Id, \quad X_t^{-1} = X_{-t}.$$

(3) Introducing the variable $z := (t, y, \lambda) \in \mathcal{O} \subset \mathbb{R}^{d+1+m}$, the autonomous vector field $g : \mathcal{O} \to \mathbb{R}^{d+1+m}$, $z \mapsto g(z) := (1, f(t, y, \lambda), 0)$ and the function $z(t) := (t, y(t), \lambda)$, we observe that

$$y'(t) = f(t, y(t), \lambda) \quad \Leftrightarrow \quad z'(t) = g(z(t)).$$

Theorem 4.2 and Theorem 4.4 are thus equivalent (in this $C^1_{x,\lambda}$ version).

Proof of Theorem 4.4.
(1) Well-posedness is a consequence of the Cauchy-Lipschitz and the fact that maximal solutions are global solutions is a consequence of the Gronwall lemma (and more precisely from Lemma 3.5).
(2) For $t_1, t_2 \in \mathbb{R}$ and $x \in \mathbb{R}^d$, we define the two functions

$$x(t) := \Phi_{t,t_1}(x) \quad \text{and} \quad y(t) := \Phi_{t,t_2}(x(t_2)).$$

By definition of Φ, we have

$$\dot x = a(t,x(t)), \quad x(t_1) = x$$
$$\dot y = a(t,y(t)), \quad y(t_2) = x(t_2).$$

That provides two solutions to the ODE (4.4) starting from the same point $x(t_2)$ at time t_2. From the uniqueness part of the Cauchy-Lipschitz theorem, we have thus $y(t) \equiv x(t)$. In other words, we have establsihed that

$$\Phi_{t,t_2} \circ \Phi_{t_2,t_1}(x) = \Phi_{t,t_1}(x).$$

(3) In order to simplify notations we assume from now on that a is independ of time and Lipschiz. The general case can be easily handed by adapting the arguments. From Lemma 3.6, we already know that for any $T \ge 0$, there exist a constant $C_T \ge 1$ such that

$$\sup_{[0,T]} |z(t) - y(t)| \le C_T |z_0 - y_0|, \quad \forall\, y_0, z_0 \in \mathbb{R}^d. \tag{4.6}$$

We next define $A(t) : \mathbb{R}^d \to M_d(\mathbb{R})$ by $A(t) w_0 := w(t)$, where $w(t)$ is the global solution to the linear equation

$$w' = (Da)(y(t))\, w, \quad w(0) = w_0. \tag{4.7}$$

We want to show that $D_x \Phi_t = A(t)$, what will be true if we are able to show that

$$\begin{aligned} \xi(t) &:= \Phi_t(z_0) - \Phi_t(y_0) - A(t)(z_0 - y_0) \\ &:= z(t) - y(t) - w(t) = o(|z_0 - y_0|) \end{aligned}$$

when $w_0 := z_0 - y_0$ in (4.7). Because $a \in C^1$, for any $T > 0$, we have

$$a(z) - a(y) = Da(y)(z - y) + o_K(|z - y|),$$

for any $y, z \in K$ a compact set of $\mathbb{R}^d$. From Lemma 3.5, we have

$$y(t), z(t) \in K_{T,R} := \{u \in \mathbb{R}^d; |u| \le (R+1) e^{BT}\},$$

for any $t \in [0,T]$, and together with (4.6), we deduce

$$a(z(t)) - a(y(t)) = Da(y(t))(z(t) - y(t)) + o_{T,R}(|z_0 - y_0|),$$

for any $t \in [0,T]$ and any $y_0, z_0 \in B_R$. We next compute

$$\begin{aligned} \dot\xi(t) &= \dot z(t) - \dot y(t) - \dot w(t) \\ &= a(z(t)) - a(y(t)) - Da(y(t))(w(t) \\ &= Da(y(t))(z(t) - y(t) - w(t)) + o_{T,R}(|z_0 - y_0|) \\ &= Da(y(t))\xi(t) + o_{T,R}(|z_0 - y_0|). \end{aligned}$$

From the Gronwall lemma and denoting $L := \|Da\|_{L^\infty}$, we immediately deduce that

$$|\xi(t)| \le e^{Lt} |\xi_0| + \frac{e^{Lt}}{L}\, o_{T,R}(|z_0 - y_0|) = o_{T,R}(|z_0 - y_0|),$$

because $\xi(0) = 0$. That concludes the proof of $D\Phi_t(y_0) = A(t)$.

We omit the last step which would consist in proving that $(t, s, y_0) \to D\Phi_{t,s}(y_0)$ is continuous from $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^d$ into $M_d(\mathbb{R})$. □

Note : Preuve en commentaire à rédiger pour une future version plus complète.

5. Linear stability

In this section, we are concerned with the autonomous system

$$y' = F(y), \quad y(0) = y_0, \tag{5.1}$$

with a vector field $F : \mathbb{R}^d \to \mathbb{R}^d$ which satisfies the condition (4.5).

Definition 5.1. *We say that $a \in \mathbb{R}^d$ is a singular point if $F(a) = 0$, we note $a \in \mathcal{S}$. For $y_0 \in \mathbb{R}^d$, we call orbit or trajectory the set/function $(\Phi_t(y_0))$.*

If a is a singular point then $\Phi_t(a) \equiv a$ is a stationary solution to equation (5.1). If $y_0 \notin \mathcal{S}$ then $\Phi_t(y_0)$ is never a singular pont (from the uniqueness of solutions in the Cauchy-Lipschitz Theorem).

Lemma 5.2. *Let y be a solution to* (5.1)*. If $y(t) \to a$ as $t \to \infty$ or $y(t) \to a$ as $t \to -\infty$, then $a \in \mathcal{S}$.*

Proof of Lemma 5.2. We prove the first claim. We write

$$y(n+1) - y(n) = \int_0^1 F(y(s+n))\, ds, \quad \forall\, n \in \mathbb{N}.$$

Passing to the limit $n \to \infty$, we get

$$0 = \int_0^1 F(a)\, ds = F(a),$$

so that $a \in \mathcal{S}$. □

Definition 5.3. *Take $a \in \mathcal{S}$.*
(1) We define $W^s(a) := \{y_0 \in \mathbb{R}^d;\ \Phi_t(y_0) \to a \text{ as } t \to +\infty\}$. We say that a is attractive or is asymptotically stable when $B(a, \delta) \subset W^s(a)$ for some $\delta > 0$.
(2) We define $W^u(a) := \{y_0 \in \mathbb{R}^d;\ \Phi_t(y_0) \to a \text{ as } t \to -\infty\}$. We say that a is repulsive or is asymptotically unstable when $B(a, \delta) \subset W^u(a)$ for some $\delta > 0$.

Lemma 5.4. *If a is attractive then $W^s(a)$ is an open set and $W^u(a) = \emptyset$. If a is repulsive then $W^u(a)$ is an open set and $W^s(a) = \{a\}$.*

Preuve en commentaire à rédiger pour une future version plus complète.

The linearized equation around a singular point a is particularly simple and writes

$$w' = DF(a)w. \tag{5.2}$$

Definition 5.5. *Take $a \in \mathcal{S}$.*
(1) We say that a is a sink or is a stable node if $\Re e\lambda < 0$ for any eigenvalue λ of $DFa)$.
(2) We say that a is a source or is an unstable node if $\Re e\lambda > 0$ for any eigenvalue λ of $DF(a)$.

Theorem 5.6. *Take $a^* \in \mathcal{S}$.*
(1) If a^* is a sink then a^* is attractive.
(2) If a^* is a source then a^* is repulsive.

Sorry, the proof of this result is written is french!

A first proof of Theorem 5.6. We only deal with the attractif case.
On établit une première borne exponentielle en $t \to \infty$ lorsque $DF(a^*)$ est diagonalisable et ses valeurs propres sont toutes de partie réelle strictement négatives

$\Re e\lambda_i \le \Re e\lambda_1 = -\alpha < 0$. Comme $DF(a^*)$ est diagonalisable, il existe un système de coordonnées telle que

$$\langle w, DF(a^*)w\rangle \le -\alpha |w|^2, \quad \forall\, w \in \mathbb{R}^d,$$

où $\langle\cdot,\cdot\rangle$ désigne le produit scalaire usuel et $|\cdot|$ la norme euclidienne associée. Comme F est de classe C^1, il existe une fonction $\varepsilon_\delta \to 0$ lorsque $\delta \to 0$ telle que

$$|F(a^* + w) - DF(a^* + w)w| \le \varepsilon_R |w|, \quad \forall\, w \in B(0,\delta).$$

Combinant les deux inégalités, pour tout $w \in B(0,\delta)$, on a

$$\begin{aligned}\langle w, F(a^*+w)\rangle &= \langle w, DF(a^*)w\rangle + \langle w, F(a^*+w) - DF(a^*)w\rangle \\ &\le (\varepsilon_\delta - a)|w|^2.\end{aligned}$$

On fixe $\delta > 0$ tel que $\varepsilon_R - a < 0$. Consider y a solution to the ODE (5.1). On note $w(t) := y(t) - a^*$ et on suppose que $|w(T)| < \delta$. On définit

$$T^* := \sup\{\tau > T;\ |w(t)| < \delta,\ \forall\, t \in]t,\tau[\,\}.$$

Par continuité, on a $T^* > T$. Par ailleurs, on calcule

$$\frac{1}{2}\frac{d}{dt}|w|^2 = \langle w, F(a^*+w)\rangle \le (\varepsilon_R - a)|w|^2,$$

sur $]T,T^*[$. Cela implique $|w(t)| \le e^{(\varepsilon_\delta - a)(t-T)}|w(T)| \le |w(T)| < \delta$ pour tout $t \in]T,T^*[$. On conclut que $T^* = \infty$, puis que pour tout $\varepsilon \in]0,\alpha[$, il existe C_ε tel que

$$|y(t) - a^*| \le C_\varepsilon\, e^{(\varepsilon-\alpha)t}, \quad \forall\, t \ge 0.$$

De plus, on peut prendre $C_\varepsilon := |y(0) - a^*|$ lorsque cette dernière quantité est assez petite.

Une deuxième preuve du même résultat. On ne fait plus l'hypothèse que $DF(a^*)$ est diagonalisable, mais on écrit $F(a^*+w) = Aw + G(w)$ avec $A := DF(a^*)$ et on suppose que $|G(w)| \le K|w|^2$ pour tout $w \in B(0,1)$. Avec les notations introduites précédemment, on a

$$w' = Aw + G(w),$$

soit donc

$$w(t) = e^{At}w_0 + \int_0^t e^{A(t-s)}G(w(s))\,ds.$$

Par hypothèse, on a $|e^{At}w_0| \le C_1 e^{-\alpha t}|w_0|$ pour tout $t \ge 0$, avec $C_1 \ge 1$ et $\Re e\lambda_i < -\alpha < 0$.

En notant $u(t) := |w(t)|$, $u_0 := w_0$, $C_2 := C_1 K$, on en déduit

$$u(t) \le C_1 e^{-\alpha t}\, u_0 + C_2 \int_0^t e^{-\alpha(t-s)} u(s)^2\,ds, \quad \forall\, t > 0.$$

Note: Ajouter le résultat qui suit dans la liste des lemmes de Gronwall dans une prochaine version. En supposant

$$4C_2C_1u_0 < \alpha,$$

on va montrer le retour exponentiel de u vers 0. Pour cela, on fixe $C \in]C_1u_0, 2C_1u_0[$ et on note

$$T := \sup\{\tau \ge 0, u(t) \le C,\ \forall\, t \in [0,\tau]\}.$$

Par continuité de $u(t)$, on a $T > 0$. On a alors

$$u(t) \le C_1 e^{-\alpha t}\, u_0 + 2C_1C_2 u_0 \int_0^t e^{-\alpha(t-s)} u(s)\, ds, \quad \forall\, t \in [0, T[.$$

Le lemme de Gronwall appliqué à la fonction $t \mapsto u(t)e^{\alpha t}$ et la condition de petitesse implique

$$u(t) \le C_1\, u_0\, e^{(C_2 2C_1 u_0 - \alpha)t} \le C_1\, u_0 < C, \quad \forall\, t \in [0, T[.$$

Par un argument de continuation (connexité) cette borne implique que $T = \infty$ et que cette m me borne est vraie avec $C := C_1 u_0$. En utilisant cette première borne dans l'inégalité intégrale de départ, on trouve

$$u(t) \le C_1 e^{-\alpha t}\, u_0 + C_2 C_1^2\, u_0^2 e^{-\alpha t} \int_0^t e^{(4C_2C_1u_0 - \alpha)s}\, ds, \quad \forall\, t > 0,$$

ce qui suffit pour conclure. □

In some case, we can be a bit more accurate.

Theorem 5.7 (Spiral node in dimension $d = 2$)**.** *Assume $d = 2$, that a^* is a node for the ODE* (5.1) *and that the two eigenvalues of $DF(a^*)$ write*

$$\lambda_\pm = \alpha \pm i\beta, \quad \alpha, \beta \in \mathbb{R}\backslash\{0\},$$

so that a^* is a sink if $\alpha < 0$ and a^* is a source if $\alpha > 0$.
(1) Assume $\alpha < 0$. For any solution y to the ODE (5.1) *such that $y(0) \in W^s(a^*)$, there exists a solution $w(t)$ to the linearized system* (5.2) *such that $y(t) \sim a^* + w(t)$ when $t \to \infty$.*
(2) Assume $\alpha > 0$. For any solution y to the ODE (5.1) *such that $y(0) \in W^u(a^*)$, there exists a solution $w(t)$ to the linearized system* (5.2) *such that $y(t) \sim a^* + w(t)$ when $t \to -\infty$.*

Sorry, the proof of this result is written is french!

Preuve du Théorème 5.7 dans le cas d'une spirale stable. On écrit $y = a^* + z$ de sorte que z solution de

$$z' = Az + g(z)$$

avec $A := DF(a^*)$ et on suppose $|g(u)| \le K|u|^2$. En notant $V(t) := e^{-At}z(t)$, on a

$$V' = e^{-At}g(z),$$

puis

$$V(t) = z_0 + \int_0^t e^{-As} g(z(s))\, ds.$$

Grâce aux estimations établies dant la preuve du Théorème 5.6, pour tout $\gamma \in (\alpha, 0)$, il existe $C_\gamma > 0$ tel que

$$|z(t)| \le C_\gamma e^{\gamma t}, \quad \forall\, t \ge 0.$$

On fixe $\gamma := 3\alpha/4$, de sorte que

$$|e^{-As} g(z(s))| \le C e^{s\alpha/2} \in L^1(\mathbb{R}_+).$$

On peut donc écrire

$$z(t) = e^{At}\big(z_0 + \int_0^\infty e^{-As} g(z(s))\, ds\big) - e^{At} \int_t^\infty e^{-As} g(z(s)) ds,$$

ce qui implique le résultat. □

We end mentionning a last (more general) result.

Theorem 5.8 (Structural stability)**.** *Assume that a^* is a node for the EDO* (5.1)*: $\Re e \lambda \neq 0$ for any eigenvalue λ of $DF(a^*)$. Then, there exists an homeomorphism between the solutions of the nonlinear ODE* (5.1) *in a neighborhood of a^* and the solutions of the linearized system* (5.2) *in a neighborhood of* 0*. In particular, if $y(t) \to a^*$ as $t \to \pm\infty$, then $y(t) \sim a^* + \exp(DF(a^*)t) y^\sharp$, for some $y^\sharp$.*

6. Nonlinear stability

In this section, we are still concerned with the autonomous system

$$y' = F(y), \quad y(0) = y_0, \tag{6.1}$$

with a vector field $F : \mathbb{R}^d \to \mathbb{R}^d$ which satisfies the condition (4.5).

6.1. Invariant set.

We say that a set $\mathcal{O} \subset \mathbb{R}^d$ is invariant for the flow if

$$x \in \mathcal{O} \quad \text{implies} \quad \Phi_t(x) \in \mathcal{O}, \ \forall\, t \geq 0.$$

Definition 6.1. *We say that $\mathcal{G} : \mathbb{R}^d \to \mathbb{R}$ of class C^1 is a Lyapunov function if*

$$D\mathcal{G}(y) F(y) \leq 0, \quad \forall\, y \in \mathbb{R}^d.$$

We say that $\mathcal{G}$ is coercive if $\mathcal{G}(y) \to \infty$ as $|y| \to \infty$.

For a gradient flow $F = -\nabla V$, $\mathcal{G} := V$ is a Lyapunov function. For a Hamiltonian flow $F = (\nabla_p \mathcal{H}, -\nabla_q \mathcal{H})$, $\mathcal{G} := \mathcal{H}$ is a Lyapunov function.

Lemma 6.2. *If $\mathcal{G}$ is a Lyapunov function, then $t \mapsto \mathcal{G}(y(t))$ is decreassing for any solution y to the autonomous system* (6.1)*. In particular,*

$$[\mathcal{G} \leq \lambda] = \{y \in \mathbb{R}^d;\ \mathcal{G}(y) \leq \lambda\}$$

is an invariant set for any $\lambda \in \mathbb{R}$. Furthermore, this set is boounded if $\mathcal{G}$ is coercive.

Proof of Lemma 6.2. We compute

$$\frac{d}{dt} \mathcal{G}(y(t)) = D\mathcal{G}(y(t))\, y'(t) = D\mathcal{G}(y(t))\, F(y(t)) \leq 0.$$

Si $f(y) = -DV(y)$ avec $V : \mathbb{R}^d \to \mathbb{R}$ de classe C^2 alors on peut prendre $G(y) := V(y)$. Si f est un champ associé à un Hamiltonien H, on peut prendre $G = \nabla H$. G est coercive ssi $\{G(z) \leq \lambda\}$ est compact (ou vide) pour tout $\lambda \in \mathbb{R}$ et donc les trajectoires $\{y(t)\}_{t \geq 0} \subset \{G(z) \leq G(y_0)\}$ sont bornées. □

Sorry, here come two examples/exercises written is french!

Exemple 1: système proie-prédateur de Lokta-Voltera. $F(t)$ représente le nombre de proies (F =food) qui utilisent une ressource naturelle (plancton) pour se nourrir et $P(t)$ ne nombre de prédateurs qui mangent les proies (le terme quadratique prend en compte la probabilité de rencontre):

$$\dot{F} = \alpha\, F - \beta\, F\, P$$
$$\dot{P} = \gamma\, P\, F - \mu\, P.$$

On effectue le changement de variables $u = \ln F$, $v = \ln P$ et le système de Lokta-Voltera se met sous la forme d'un système Hamiltonien

$$\dot{u} = \alpha - \beta\, e^v = (\partial_v H)(u,v)$$
$$\dot{v} = \gamma\, e^u - \mu = -(\partial_u H)(u,v),$$

avec $H(u,v) = \alpha\, v + \mu\, u - \beta\, e^v - \gamma\, e^u$.

Exemple/Exercice 2. On considère le système suivant (avec $k \in \mathbb{N}$)

$$\dot{n}(t) + n(t) = p(t)^k, \quad n(0) = n^0 > 0$$
$$\dot{p}(t) + p(t)^k = n(t), \quad p(0) = p^0 > 0.$$

1. Montrer que les solutions de (2) vérifient $n(t) > 0$ et $p(t) > 0$. Pour k pair, on pourra considérer l'équation $\dot{w} + w = 0$ et appliquer deux fois le lemme de comparaison et une fois le résultat d'unicité. Pour k qui n'est pas pair, soit on raisonne sur l'ensemble où $p \geq 0$ et par connexité, soit on introduit l'équation modifiée où on a remplacé $p(t)^k$ par $(p(t)_+)^k$).

2. Trouver une loi de conservation et en déduire une borne globale. C'est $n(t) + p(t) = n^0 + p^0$.

3. Montrer que $n(t) \to n_\infty$ et $p(t) \to p_\infty$ lorsque $t \to \infty$ avec $n_\infty + p_\infty = n^0 + p^0$ et $n_\infty = p_\infty^k$. Ces relations définissent des valeurs positives uniques n_∞, p_∞. On a alors

$$\frac{d}{dt}(n(t) - n_\infty) + n(t) - n_\infty = p(t)^k - (p_\infty)^k = (p(t) - p_\infty)\, R(t) = (n_\infty - n(t))\, R(t).$$

On a donc

$$\frac{d}{dt}\frac{(n(t) - n_\infty)^2}{2} + (n(t) - n_\infty)^2 \leq 0.$$

Note : Definitions de la stabilité en commentaire à rédiger pour une future version plus complète.

6.2. Dynamical system, steady states and ω-limit set.

Definition 6.3. *We say that $(S_t)_{t\geq 0}$ is a dynamical system on a metric space $(\mathcal{Z}, d)$ if*

- *(S1) $\forall t \geq 0$, $S_t \in C(\mathcal{Z}, \mathcal{Z})$ (continuously defined on $\mathcal{Z}$);*
- *(S2) $\forall x \in \mathcal{Z}$, $t \mapsto S_t\, x \in C([0,\infty), \mathcal{Z})$ (trajectories are continuous);*
- *(S3) $S_0 = I$; $\forall\, s, t \geq 0$, $S_{t+s} = S_t\, S_s$ (semigroup property).*

We say that $\bar{z} \in \mathcal{Z}$ is invariant (or is a steady state, a stationary point) if $S_t \bar{z} = \bar{z}$ for any $t \geq 0$. We denote by $\mathcal{E}$ the set of all steady states,

$$\mathcal{E} := \{y \in \mathcal{Z};\ S_t y = y\ \forall\, t \geq 0\}.$$

We remark that $\mathcal{E}$ is closed by definition ($\mathcal{E} = \cap_{t\geq 0}(S_t - I)^{-1}(\{0\})$).

The flow (Φ_t) associated to the autonomous system (6.1) for a force field F satisfying (4.5) is a Dynamical system on $\mathbb{R}^d$, and thus on any invariant set $\mathcal{O} \subset \mathbb{R}^d$.

Theorem 6.4. (Dynamic system and steady state). *Consider a compact and convex subset $\mathcal{Z}$ of a Banach space X. Then any dynamical system $(S_t)_{t\geq 0}$ on $\mathcal{Z}$ admits at least one steady state, that is $\mathcal{E} \neq \emptyset$.*

We accept and use the following fixed point of Brouwer (finite dimension) and Schauder (infinite dimension).

Theorem 6.5 (**Brouwer & Schauder**). *Consider a compact and convex subset $\mathcal{Z}$ of a Banach space X and a function $S \in C(\mathcal{Z},\mathcal{Z})$. Then S admits at least one fixed point $z^* \in \mathcal{Z}$, that is $Sz^* = z^*$.*

Proof of Theorem 6.4. For any $n \geq 1$, there exists $z_n \in \mathcal{Z}$ such that $S_{2^{-n}} z_n = z_n$ thanks to the Brouwer or the Schauder fixed point Theorem 6.5. On the one hand, from the semigroup property *(S3)*

$$S_{i\,2^{-m}} z_n = z_n \quad \text{for any} \quad i,n,m \in \mathbb{N}, \quad m \leq n. \tag{6.2}$$

On the other hand, by compactness of $\mathcal{Z}$, we may extract a subsequence $(z_{n_k})_k$ which converges to a limit $\bar z \in \mathcal{Z}$. By the continuity assumption *(S1)* on S_t, we may pass to the limit $n_k \to \infty$ in (6.2) and we obtain $S_t\,\bar z = \bar z$ for any dyadic time $t \geq 0$. We conclude that $\bar z$ is a stationary point by the trajectorial continuity assumption *(S2)* on S_t and the density of the dyadic real numbers in the real line. □

Note: ajouter le corollaire classique de Brouwer sur les champs rentrant pour une future version plus complètes: celui-ci s'obtient aussi comme corollaire de Theorem 6.4.

Consider a dynamical system $(S_t)_{t\geq 0}$ on a metric space $(\mathcal{Z},d)$. For any given $z \in \mathcal{Z}$, we define the associated *omega-limit set* as

$$\omega(z) = \{y \in \mathcal{Z};\ \exists t_n \nearrow \infty \ \text{ and } \ S_{t_n}\, z \to y\},$$

or equivalently

$$\omega(z) := \bigcap_{T>0} \omega_T(z), \quad \omega_T(z) := \overline{\{S_t z;\ t \geq T\}}. \tag{6.3}$$

We observe that

$$\mathcal{E}_z := \{y \in \omega_0(z);\ S_t y = y,\ \forall\, t \geq 0\} \subset \omega(z).$$

For $y \in \mathcal{E}_z$, there is indeed the alternative $y \in \{S_t z;\ t \geq 0\}$, so that there exists $T \geq 0$ such that $y = S_T z$ and then $y = S_t z$ for any $t \geq T$, so that $y \in \omega(z)$, or $y \in \omega_0(z) \backslash \{S_t z;\ t \geq 0\} = \omega(z)$. We also observe that $\omega(z) = \{\bar z\}$ if $S_t z \to \bar z$ when $t \to \infty$.
and $\mathcal{E}_z = \{\bar z\}$ if $S_t z \to \bar z$ when $t \to \infty$.

Theorem 6.6. (Dynamic system and ω-limit set). *Consider a dynamical system $(S_t)_{t\geq 0}$ on a metric space $(\mathcal{Z},d)$ which trajectories are relatively compact. More precisely, we assume*

(S4) $\omega_0(z)$ is compact for some fixed $z \in \mathcal{Z}$.

Then there hold

(i) $S_t(\omega(z)) = \omega(z)$ $\forall t \geq 0$;

(ii) $\omega(z)$ is a nonempty connected and compact subset of $\mathcal{Z}$. More precisely, any $y \in \omega(z)$ belongs to an eternal trajectory in the sense that there exists $(y_t)_{t\in\mathbb{R}} \subset \omega(z)$ such that $y_0 = y$ and $y_{s+t} = S(s) y_t$ for any $t \in \mathbb{R}$ and any $s \geq 0$ and there exists (t_n) such that $S(t+t_n) z \to y_t$ as $n \to \infty$ for any $t \in \mathbb{R}$;

(iii) $d(S_t\, z, \omega(z)) \to 0$ as $t \to \infty$;

(iv) When furthermore $\omega(z)$ is a discrete set, we have $\omega(z) = \mathcal{E}_z = \{\bar z\}$ for some $\bar z \in \mathcal{Z}$.

Proof of Theorem 6.6. (i) On the one hand, for any $y \in \omega(z)$, there exists (t_n) such that $S_{t_n} z \to y$, so that $S_{t_n+t} z \to S_t y$ and $S_t y \in \omega(z)$. That proves $S_t(\omega(z)) \subset \omega(z)$. On the other hand, given $y \in \omega(z)$ and $t_n \to \infty$ such that $S_{t_n} z \to y$, there exists $w \in \mathcal{Z}$ and a subsequence (t_{n_k}) such that $S_{t_{n_k}-t} z \to w$ because of assumption *(S4)*, and then $w \in \omega(z)$. We deduce

$$S_t w = S_t(\lim S_{t_{n_k}-t} z) = \lim S_{t_{n_k}} z = y.$$

That proves the reverse inclusion $\omega(z) \subset S_t(\omega(z))$.

(ii) For any $n \geq 0$, the set $\omega_n(z)$ is a nonempty connected and compact subset of $\mathcal{Z}$ by assumption *(S4)*. The sequence $(\omega_n(z))$ being decreasing, we have $\omega(z) = \lim \omega_n(z)$ which is nothing but (6.3) and thus (ii). More precisely, consider $y \in \omega(z)$ and (t_n) such that $S(t_n)z \to y$. For any $t \in \mathbb{R}$, we may extract a subsequence of $S(t_n+t)z$ which converges to a limit y_t. Better, thanks to Cantor's diagonal process, there exists one subsequence (t_{n_k}) such that for any $t \in \mathbb{Z}$ there holds $S(t_{n_k}+t)z \to y_t$, and next, for any $t \in \mathbb{R}$

$$S(t_{n_k}+t)z = S(t-[t])S(t_{n_k}+[t])z \to S(t-[t])y_{[t]} =: y_t.$$

As a consequence, $y_t \in \omega(z)$, $y_0 = y$ and $y_{t+s} = \lim S(t_{n_k}+s+t)z = \lim S(s)S(t_{n_k}+t)z = S(s)y_t$ for any $t \in \mathbb{R}$ and $s \geq 0$.

(iii) We argue by contradiction. Assume that there exist a sequence $t_n \to \infty$ and a real number $\epsilon > 0$ such that $d(S_{t_n} z, \omega(z)) \geq \epsilon$. From assumption *(S4)*, there exists a subsequence (t_{n_k}) such that $S_{t_{n_k}} z \to w \in \omega(z)$ and then $d(S_{t_{n_k}} z, \omega(z)) \to 0$, which is absurd.

(iv) First, $\omega(z)$ is a singleton as a discrete and connected nonempty set, we then have $\omega(z) = \{\bar z\}$. Next, by uniqueness of the possible limits, we deduce $S_t z \to \bar z$ as $t \to \infty$. Finally, for any $t \geq 0$, we have $S_t \bar z \in \omega(z)$ and thus $S_t \bar z = \bar z \in \mathcal{E}_z$. □

6.3. **Strict Lyapunov functional and La Salle invariance principle.**

Definition 6.7. *We say that $\mathcal{G} : \mathbb{R}^d \to \mathbb{R}$ of class C^1 is a strict Lyapunov function for the autonomous system* (6.1) *if*
(1) $F(a) = 0$ implies $D_a \mathcal{G} = 0$;
(2) $F(y) \neq 0$ implies $D_y \mathcal{G} F(y) < 0$.
In that case, $t \mapsto \mathcal{G}(y(t))$ is strictly decreassing for any solution $y(t)$ to the autonomous system (6.1).

Definition 6.8. *We say that $\mathcal{G}$ is a Lyapunov functiion for a dynamical system $(S_t)_{t\geq 0}$ on a metric space $(\mathcal{Z}, d)$ if*
(1) $\mathcal{G} \in C(\mathcal{Z}, \mathbf{R})$;
(2) $t \mapsto \mathcal{G}(S_t z)$ is decreasing.
For a given $z \in \mathcal{Z}$, we then define

$$\mathcal{G}_z := \inf_{y \in \omega_0(z)} \mathcal{G}(y),$$

and next

$$\omega_{\mathcal{G}}(z) := \{y \in \omega_0(z);\ \mathcal{G}(S_t y) = \mathcal{G}_z\ \forall\, t \geq 0\}.$$

Theorem 6.9. (La Salle invariance principle). *Consider a dynamical system* $(S_t)_{t\geq 0}$ *on a metric space* $(\mathcal{Z}, d)$ *and* $z \in \mathcal{Z}$. *Assuming that*

(S4) $(S_t z)_{t\geq 0}$ *is relatively compact;*

(H2) $\mathcal{G}$ *is a Lyapunov functional;*

there holds $\omega(z) \subset \omega_{\mathcal{G}}(z)$, *and more precisely*

$$\mathcal{G}_z \in \mathbb{R}, \quad \mathcal{G}(S_t z) \searrow \mathcal{G}_z \text{ as } t \to \infty \quad \text{and} \quad d(S_t z, \omega_{\mathcal{G}}(z)) \to 0 \text{ as } t \to \infty.$$

If furthermore $\omega_{\mathcal{G}}(z)$ *is discrete, we have* $\omega(z) = \mathcal{E}_z = \{\bar{z}\}$ *for some* $\bar{z} \in \mathcal{Z}$.

Proof of Theorem 6.9. On the one hand, $\mathcal{G}(S_t z)$ is bounded (because the trajectories are relatively compact) so that $\mathcal{G}_z \in \mathbb{R}$ and it is decreasing so that $\lim \mathcal{G}(S_t z) = \mathcal{G}_z$. On the other hand, for any $y \in \omega(z)$ there exists $t_n \to \infty$ such that $S_{t_n} z \to y$ which in turns implies $\mathcal{G}_z = \lim \mathcal{G}(S_{t_n+s} z) = \mathcal{G}(\lim S_{t_n+s} z) = \mathcal{G}(S_s y)$ for any $s \geq 0$. In other words, we have $\omega(z) \subset \omega_{\mathcal{G}}(z)$ and the second convergence result is a consequence of (iii) in Theorem 6.6.

When furthermore $\omega_{\mathcal{G}}(z)$ is discrete the same holds for $\omega(z)$ and thus $\omega(z)$ is a singleton since it is also connected and nonempty. We may use (iv) in Theorem 6.6 in order to conclude. □

We may apply now the previous general result to the autonomous system (6.1).

Theorem 6.10. (Lyapunov). *Consider the autonomous ODE* (6.1) *for a force field* F *satisfying* (4.5). *Assume that*
(1) there exists a coercive and strict Lyapunov function $\mathcal{G}$;
(2) the set of singular values $\mathcal{S}$ *is discrete.*
Then, for any $y_0 \in \mathbb{R}^d$, *the associated solution* y *to the autonomous ODE* (6.1) *satisfies* $y(t) \to a$ *as* $t \to +\infty$ *for some* $a \in \mathcal{S}$.

Proof of Theorem 6.10. Take $y_0 \in \mathbb{R}^d$ and define the set $\mathcal{Z} := [\mathcal{G} \leq \mathcal{G}(y_0)]$. By continuity and coercivity of $\mathcal{G}$ the set $\mathcal{Z}$ is compact. Since $\mathcal{Z}$ is invariant, (Φ_t) may be considered as a dynamical system on $\mathcal{Z}$. By hypothesis, the function $\mathcal{G}$ is a Lyapunov function for this dynamical system. Furthermore, if $x_0 \in \omega_{\mathcal{G}}(y_0)$, denoting $x(t) := \Phi_t(x_0)$, we have

$$0 = \mathcal{G}(x(t)) - \mathcal{G}(x_0) = \int_0^t D\mathcal{G}_{x(s)} F(x(s))\, ds.$$

By the definition of a strict Lyapunov function, we have $D\mathcal{G}_{x(s)} F(x(s)) = 0$ for any $s \geq 0$, and thus $x(s) \in \mathcal{S}$ for any $s \geq 0$. In partciular $\omega_{\mathcal{G}}(x_0) \subset \mathcal{S}$, and thus $\omega_{\mathcal{G}}(x_0)$ is discrete. We immediately conclude thanks to Theorem 6.9. □

Note : Quelques complements en commentaire à rédiger pour une future version.

We end mentionning a last (more general) result.

Theorem 6.11 (Poincaré-Bendixson)**.** *In dimension* $d = 2$, *any bounded trajectory of the autonomous system* (6.1) *converges to a singular point or rolls up on a limit cycle (it is asymptotically close to a periodic orbit).*

6.4. **Hamiltonian system.**

We start with a result concerning the general ODE

$$\dot{y}(t) = b(t, y(t)), \quad y(0) = y, \tag{6.4}$$

with a force field b satisfying the regularity and uniform growth bound (4.5). We remind the notation $\Phi_t(y) = y(t)$ for the unique associated solution. We also write below $D\Phi_t(y) := D_y \Phi_t(y)$.

Theorem 6.12. (Liouville). *Consider the ODE* (6.4) *with a force field satisfying* (4.5). *The function*

$$J = J(t, y) := \det D\Phi_t(y)$$

satisfies the ODE

$$\frac{d}{dt} J = (\operatorname{div} b(t, \Phi_t(y)))J, \quad J(0, y) = 1. \tag{6.5}$$

The proof use the two following elementary linear algebra results.

Lemma 6.13. *The mapping* $\det : M_d(\mathbb{R}) \to \mathbb{R}$ *is* C^1 *and* $D\det(I) = \operatorname{tr}$. *In other words, for any* $B \in M_d(\mathbb{R})$ *and* $h \in \mathbb{R}$, *there holds*

$$\det(I + hB) = 1 + h\, \operatorname{tr} B + \mathcal{O}(h^2).$$

Preuve (sorry it is in french). Cela se fait par récurrence sur la dimension d. Pour $d = 2$ on a $\det(I + hB) = 1 + h \operatorname{tr} B + h^2 \det B$ (calcul direct). Pour $d > 2$, on a

$$\det(I_d + hB_d) = (1 + hb_{11}) \det(I_{d-1} + hB_{d-1}) + \sum_{i=2}^{d} h^2\, b_{i1} \det(\tilde{B}_j),$$

et on utilise l'hypothèse de récurrence pour le premier terme et une majoration pour le second terme. □

Lemma 6.14. *If* $A, B \in C^1((0, T); M_d(\mathbb{R}))$ *satisfy*

$$\frac{d}{dt} A(t) = B(t) A(t),$$

then

$$\frac{d}{dt}(\det A(t)) = (\operatorname{tr} B(t))(\det A(t)).$$

Preuve (sorry it is in french). On écrit un DL à l'ordre 2

$$A(t + h) = A(t) + h\, B(t)\, A(t) + O(h^2) = (I + h\, B(t))\, A(t) + O(h^2),$$

de sorte que

$$\begin{aligned} \det A(t + h) &= (1 + h \operatorname{tr} B(t)) \det A(t) + O(h^2) \\ &= \det A(t) + h \operatorname{tr} B(t) \det A(t) + O(h^2), \end{aligned}$$

et on conclut en divisant par h et en laissant tendre $h \to 0$. □

Proof of Theorem 6.12. Let us remind from Theorem 4.4-(3) that the evolution of the differential $D\Phi_t(y)$ is given by the equation

$$\frac{d}{dt} D\Phi_t(y) = Db(t, \Phi_t(y))\, D\Phi_t(y), \quad D\Phi_0(y) = I.$$

Using Lemma 6.14 with t $A(t) := D\Phi_t(y)$ and $B(t) := Db(t, \Phi_t(y))$, we immediately deduce that (6.5) holds. □

Theorem 6.15. (Liouville). *Consider the ODE* (6.4) *with a force field satisfying* (4.5) *and assume furthermore that*

$$\mathrm{div} b = 0.$$

Then the associated flow is volume preserving. More precisely, for any open set $\Omega \subset \mathbb{R}^d$ *and any* $t \in \mathbb{R}$*, the open set* $\Omega(t) := \Phi_t(\Omega) = \{\Phi(t, y); y \in \Omega\}$ *satisfies*

$$\int_{\Omega(t)} dy = \int_{\Omega} dy.$$

Proof of Theorem 6.15. From (6.5), there holds

$$\frac{d}{dt} J = 0, \quad J(0, \cdot) = 1,$$

and thus $\det D\Phi_t = J \equiv 1$. We then may compute

$$\int_{\Omega(t)} dy = \int_{\Phi_t(\Omega)} |\mathrm{div}(D\Phi_t(y))| \, dy = \int_{\Omega} dy,$$

thanks to the change of variables theorem. □

We finally consider an Hamiltonian system

$$\begin{cases} \dot{p} = \nabla_q \mathcal{H}(p, q) \\ \dot{q} = -\nabla_p \mathcal{H}(p, q), \end{cases}$$

for some Hamiltonian function $\mathcal{H}$ of class C^2. It is worth emphasizing that the associated maximal solutions are global solutions.

Theorem 6.16. *An Hamiltonian system has no sink nor source.*

Proof of Theorem 6.16. Defining the force field $b(y) := (\nabla_q \mathcal{H}(y), -\nabla_p \mathcal{H}(y))$, $y := (p, q)$, we compute

$$\mathrm{div} b = \mathrm{div}_p \nabla_q \mathcal{H} - \mathrm{div}_q \nabla_p \mathcal{H} = 0,$$

so that the Theorem 6.15 claims that the Hamiltonian flow is volume preserving. We next argue by contradiction. If for instance the Hamiltonian system has a sink $a \in \mathbb{R}^d$, there is a neighborhood $\mathcal{V}$ of a and some constants $M \geq 1$, $\alpha < 0$ such that

$$|\Phi_t(y) - a| \leq M \, e^{\alpha t} |y - a|, \quad \forall \, y \in \mathcal{V}, \ t \geq 0,$$

from Theorem 5.6. We deduce

$$\int_{\Phi_t(\mathcal{V})} dy \leq \int_{B(a, M e^{\alpha t})} dy = C e^{\alpha d t} \to 0,$$

as $t \to \infty$, and that is in contradiction with the volume preserving property of the flow. □

Note : Quelques complements en commentaire à rédiger pour une future version.