Mathematical basis for evolution PDEs
La Habana CIMPA school - 2013
Lesson 1 - Variational solution for parabolic equation

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1 Introduction (June 24th and 25th, 2013)

In this first lesson we will focus on the question of existence (and uniqueness) of a solution $f = f(t,x)$ to the evolution PDE of "parabolic type"

$$\partial_t f = \Lambda f \quad \text{on} \quad (0,\infty) \times \mathbb{R}^d, \tag{1.1}$$

where Λ is the following integro-differential operator

$$(\Lambda f)(x) = \Delta f(x) + a(x) \cdot \nabla f(x) + c(x)\, f(x) + \int_{\mathbb{R}^d} b(y,x)\, f(y)\, dy, \tag{1.2}$$

that we complement by an initial condition

$$f(0,x) = f_0(x) \quad \text{in} \quad \mathbb{R}^d. \tag{1.3}$$

Here $t \ge 0$ stands for the *"time"* variable, $x \in \mathbb{R}^d$, $d \in \mathbb{N}^*$, stands for the *"position"* variable.

In order to develop the variational approach for the equation (1.1)-(1.2), we make the strong assumption that

$$f_0 \in L^2(\mathbb{R}^d) =: H, \quad \text{which is an Hilbert space,}$$

and that the coefficients satisfy

$$a \in W^{1,\infty}(\mathbb{R}^d), \quad c \in L^\infty(\mathbb{R}^d), \quad b \in L^2(\mathbb{R}^d \times \mathbb{R}^d).$$

The **main result** we will present in this chapter is the existence of a weak (variational) solution (which sense will be specified bellow)

$$f \in X_T := C([0,T); L^2) \cap L^2(0,T; H^1) \cap H^1(0,T; H^{-1}), \quad \forall T,$$

to the evolution equation (1.1), (1.3). We mean variational solution because the space of "test functions" is the same as the space in which the solution lives. It also refers to the associated stationary problem which is of "variational type" (see Chapter VIII & IX in the book "Functional Analysis" by H. Brézis).

The existence of solutions issue is tackled by following a scheme of proof that we will repeat for all the other evolution equations that we will consider in the next chapters.

(1) We look for **a priori estimates** by performing (formal) differential and integral calculus.

(2) We deduce a possible natural **functional space** in which lives a solution and we propose a **definition of a solution**, that it a (weak) sense in which we may understand the evolution equation.

(3) We state and prove the associated **existence theorem**. For the existence proof we typically argue as follows : we introduce a *"regularized problem"* for which we are able to construct a solution and we are allowed to rigorously perform the calculus leading to the *"a priori estimates"* and then we pass to the limit in the sequence of regularized solutions.

2 A priori estimates

Define $V = H^1(\mathbb{R}^d)$. We first observe that for any $f \in V$

$$\begin{aligned} \langle \Lambda f, f \rangle &= -\int |\nabla f|^2 + \int a \cdot \nabla_x \frac{f^2}{2} + \int c\, f^2 + \iint b(y,x)\, f(x) f(y)\, dx dy. \\ &\le -\|f\|_V^2 + \Big(1 + \frac{1}{2} \|(\operatorname{div} a)_+\|_{L^\infty} + \|c_+\|_{L^\infty} + \|b_+\|_{L^2}\Big) \|f\|_H^2. \end{aligned}$$

We also observe that for any $f, g \in V$

$$\begin{aligned} |\langle \Lambda f, g \rangle| &\le \|\nabla f\|_{L^2} \|\nabla g\|_{L^2} + \|a\|_{L^\infty} \|\nabla f\|_{L^2} \|g\|_{L^2} + (\|c\|_\infty + \|b\|_{L^2})) \|f\|_{L^2} \|g\|_{L^2} \\ &\le (1 + \|a\|_\infty + \|c\|_\infty + \|b\|_{L^2}) \|f\|_V \|g\|_V. \end{aligned}$$

We easily deduce from the two preceding estimates that our parabolic operator falls in the following abstract variational framework.

Abstract variational framework. We consider an Hilbert space H endowed with the scalar product $(\cdot,\cdot)$ and the norm $|\cdot|$. We identify H with its dual $H' = H$. We consider another Hilbert space V endowed with a norm $\|\cdot\|_V$ and we denote $\langle .,. \rangle$ the duality product on V. We assume $V \subset H$ with dense and bounded embedding so that $V \subset H \subset V'$.

We consider a linear operator $\Lambda : V \to V'$ which is bounded (or continuous), which means

(i) $\exists\, M > 0$ such that

$$|\langle \Lambda g, h \rangle| \le M\, \|g\|\, \|h\| \qquad \forall\, g, h \in V;$$

and which is *"(strongly/variationally) dissipative"* in the sense

(ii) $\exists\, \alpha > 0$, $b \in \mathbb{R}$ such that

$$\langle \Lambda g, g \rangle \le -\alpha\, \|g\|^2 + b\, |g|^2 \qquad \forall\, g \in V;$$

and we consider the abstract evolution equation

$$\frac{dg}{dt} = \Lambda g \quad \text{in} \quad (0,T), \tag{2.4}$$

for a solution $g : [0,T) \to H$, with prescribed initial value

$$g(0) = g_0 \in H. \tag{2.5}$$

A priori bound in the abstract variational framework. With the above assumptions and notations, any solution g to the abstract evolution equation (2.4) (formally) satisfies the following estimate

$$|g(T)|_H^2 + 2\alpha \int_0^T \|g(s)\|_V^2\, ds \le e^{2bT}\, |g_0|_H^2 \quad \forall\, T. \tag{2.6}$$

We (formally) prove (2.6). Using just the *dissipativity* assumption (ii), we have

$$\frac{d}{dt} \frac{|g(t)|_H^2}{2} = \langle \Lambda g, g \rangle \le -\alpha \|g(t)\|_V^2 + b\, |g(t)|_H^2,$$

and we conclude thanks to the Gronwall lemma, that we recall now.

Lemma 2.1 (Gronwall) *Consider* $0 \le u \in C^1([0,T])$, $0 \le v \in C([0,T])$ *and* $\alpha, b \ge 0$ *such that*

$$u' + 2\alpha v \le 2bu \quad \textit{in a point wise sense on } (0,T), \tag{2.7}$$

or more generally $0 \le u \in C([0,T])$ *and* $0 \le v \in L^1(0,T)$ *which satisfies* (2.7) *in the distributional sense, namely*

$$u(t) + 2\alpha \int_0^t v(s)\, ds \le 2b \int_0^t u(s)\, ds + u(0) \quad \forall\, t \in (0,T). \tag{2.8}$$

Then, the following estimate holds true

$$u(t) + 2\alpha \int_0^t v(s)\, ds \le e^{2bt}\, u(0) \quad \forall\, t \in (0,T). \tag{2.9}$$

Proof of Lemma 2.1. Since (2.7) clearly implies (2.8), we just have to prove that (2.8) implies (2.9). We introduce the C^1 function

$$w(t) := 2b \int_0^t u(s)\, ds + u(0).$$

Differentiate w, we get thanks to (2.8)

$$w'(t) = 2bu(t) \le 2b\, w(t)$$

so that

$$w(t) \le e^{2bt}\, w(0) = e^{2bt}\, u(0).$$

We conclude by coming back to (2.8). ◻

From the formal/natural/physics estimate (2.6) together with equation (2.4) and the continuity estimate (i) on Λ, we deduce

$$\Big\| \frac{dg}{dt} \Big\|_{V'} = \|\Lambda g\|_{V'} \le M\, \|g\|_V \in L^2(0,T),$$

and we conclude with

$$g \in L^\infty(0,T;H) \cap L^2(0,T;V) \cap H^1(0,T;V'). \tag{2.10}$$

3 Variational solutions

Definition 3.1 *For any given* $g_0 \in H$, $T > 0$, *we say that*

$$g = g(t) \in X_T := C([0,T];H) \cap L^2(0,T;V) \cap H^1(0,T;V')$$

is a **variational solution** *to the Cauchy problem* (2.4), (2.5) *on the time interval* $[0,T]$ *if it is a solution in the following weak sense*

$$(g(t), \varphi(t))_H = (g_0, \varphi(0))_H + \int_0^t \big\{ \langle \Lambda g(s), \varphi(s) \rangle_{V',V} + \langle \varphi'(s), g(s) \rangle_{V',V} \big\}\, ds \tag{3.1}$$

for any $\varphi \in X_T$ *and any* $0 \le t_1 \le t_2 \le T$.
We say that g *is a global solution if it is a solution on* $[0,T]$ *for any* $T > 0$.

Theorem 3.2 (J.L. Lions) *With the above notations and assumptions for any* $g_0 \in H$, *there exists a unique global variational solution to the Cauchy problem* (2.4), (2.5).
As a consequence, any solution satisfies (2.6) *and the application* $g_0 \mapsto g(t)$ *defines a* C_0*-semigroup on* H.

We start with some remarks and we postpone the proof of the existence part of Theorem 3.2 to the next section.

3.1 Parabolic equation.

As a consequence of Theorem 3.2, for any $f_0 \in L^2(\mathbb{R}^d)$ there exists a unique function

$$f = f(t) \in C([0,T]; L^2) \cap L^2(0,T; H^1) \cap H^1(0,T; H^{-1}), \quad \forall T > 0,$$

which is a solution to the parabolic equation (1.1)-(1.2) in the variational sense.

Exercice 3.3 *Prove that $f \geq 0$ if $f_0 \geq 0$. (Hint. Show that the sequence (g_k) defined in step 2 of the proof of the existence part is such that $g_k \geq 0$ for any $k \in \mathbb{N}$).*

3.2 About the functional space.

The space obtained thanks to the a priori estimates established on g is nothing but X_T as consequence of the following result.

Lemma 3.4 *The following inclusion*

$$L^2(0,T;V) \cap H^1(0,T;V') \subset C([0,T];H)$$

holds true. Moreover, for any $g \in L^2(0,T;V) \cap H^1(0,T;V')$ there holds

$$t \mapsto |g(t)|_H^2 \in W^{1,1}(0,T)$$

and

$$\frac{d}{dt}|g(t)|_H^2 = 2\,\langle g'(t), g(t)\rangle_{V',V} \quad \text{a.e. on} \quad (0,T).$$

Proof of Lemma 3.4. Step 1. We define $g_\varepsilon(t) := g *_t \rho_\varepsilon$ for a mollifier (ρ_ε) with compact support included in $(0,\infty)$ so that $g_\varepsilon \in C^1([0,T-\delta];V)$ for any $\delta \in (0,T)$ and for any $\varepsilon > 0$ small enough. We fix $\varepsilon, \varepsilon' > 0$ and we compute

$$\frac{d}{dt}|g_\varepsilon(t) - g_{\varepsilon'}(t)|_H^2 = 2\,\langle g'_\varepsilon - g'_{\varepsilon'}, g_\varepsilon - g_{\varepsilon'}\rangle_{V',V},$$

so that for any $t_1, t_2 \in [0,T)$

$$|g_\varepsilon(t_2) - g_{\varepsilon'}(t_2)|_H^2 = |g_\varepsilon(t_1) - g_{\varepsilon'}(t_1)|_H^2 + 2\int_{t_1}^{t_2} \langle g'_\varepsilon - g'_{\varepsilon'}, g_\varepsilon - g_{\varepsilon'}\rangle ds. \tag{3.2}$$

Since $g_\varepsilon \to g$ in $L^2(0,T;V)$ and in V a.e. on $[0,T)$, we may fix $t_1 \in [0,T)$ such that

$$g_\varepsilon(t_1) \to g(t_1) \quad \text{in } H. \tag{3.3}$$

As a consequence of (3.2), (3.3) as well as $g_\varepsilon \to g$ in $L^2(0,T;V)$ and $g'_\varepsilon \to g'$ in $L^2(0,T;V')$, we have

$$\limsup_{\varepsilon,\varepsilon' \to 0} \sup_{[0,T)} |g_\varepsilon(t) - g_{\varepsilon'}(t)|_H^2 \leq \lim_{\varepsilon,\varepsilon' \to 0} \int_0^T \|g'_\varepsilon - g'_{\varepsilon'}\|_{V'}\, \|g_\varepsilon - g_{\varepsilon'}\|_V\, ds = 0,$$

so that (g_ε) is a Cauchy sequence in $C([0,T);H)$, and then g_ε converges in $C([0,T);H)$ to a limit $\bar{g} \in C([0,T);H)$. That proves $g = \bar{g}$ a.e. and $g \in C([0,T);H)$.

Step 2. Similarly as for (3.2), we have

$$|g_\varepsilon(t_2)|_H^2 = |g_\varepsilon(t_1)|_H^2 + 2\int_{t_1}^{t_2} \langle g'_\varepsilon, g_\varepsilon\rangle ds,$$

and passing to the limit $\varepsilon \to 0$ we get

$$|g(t_2)|_H^2 = |g(t_1)|_H^2 + 2\int_{t_1}^{t_2} \langle g', g\rangle ds.$$

Using again that $\langle g', g\rangle \in L^1(0,T)$, we easily deduce from the above identity the two remaining claims of the Lemma. ◻

3.3 A posteriori estimate, uniqueness and C_0-semigroup.

Taking $\varphi = g \in X_T$ as a test function in (3.1), we deduce from Lemma 3.4,

$$\begin{aligned}\frac{1}{2}|g(t)|_H^2 - \frac{1}{2}|g_0|_H^2 &= |g(t)|_H^2 - |g_0|_H^2 - \int_0^t \langle g'(s), g(s)\rangle\, ds \\ &= \int_0^t \langle \Lambda g, g\rangle\, ds \\ &\le \int_0^t (-\alpha\, \|g\|_V^2 + b\, |g|_H^2)\, ds,\end{aligned}$$

and we then obtain (2.6) as an **a posteriori estimate** thanks to the Gronwall lemma 2.1.

Let us prove now the **uniqueness of the variational solution** g associated to a given initial datum $g_0 \in H$. In order to do so, we consider two variational solutions g and f associated to the same initial datum. Since the equation (2.4), (2.5) is linear, or more precisely, the variational formulation (3.1) is linear in the solution, the function $g - f$ satisfies the same variational formulation (3.1) but associated to the initial datum $g_0 - f_0 = 0$. The a posteriori estimate (2.6) then holds for $g - f$ and implies that $g - f = 0$.

We finally explain how we may associate a C_0-semigroup to the evolution equation (2.4), (2.5) as a mere consequence of the linearity of the equation and of the existence and uniqueness result.

Definition 3.5 *Consider X a Banach space, and denote by $\mathscr{B}(X)$ the set of linear and bounded operators on X. We say that $S = (S_t)_{t\ge 0}$ is a strongly continuous semigroup of linear operators on X, or just a C_0-semigroup on X, we also write $S(t) = S_t$, if*

(i) $\forall t \ge 0, \quad S_t \in \mathscr{B}(X)$ (one parameter family of operators);

(ii) $\forall f \in X, \quad t \mapsto S_t\, f \in C([0,\infty), X)$ (continuous trajectories);

(iii) $S_0 = I\,; \quad \forall\, s, t \ge 0 \quad S_{t+s} = S_t\, S_s$ (semigroup property).

For any $g_0 \in H$, we define $S_t g := g(t)$ where $g(t)$ is the unique variational solution associated to g_0.

• *S satisfies (i).* By linearity of the equation and uniqueness of the solution, we clearly have

$$S_t(g_0 + \lambda f_0) = g(t) + \lambda f(t) = S_t g_0 + \lambda S_t f_0$$

for any $g_0, f_0 \in H$, $\lambda \in \mathbb{R}$ and $t \ge 0$. Thanks to estimate (2.6) we also have $|S_t g_0| \le e^{bt}\,|g_0|$ for any $g_0 \in H$ and $t \ge 0$. As a consequence, $S_t \in \mathscr{B}(H)$ for any $t \ge 0$.

• *S satisfies (ii).* Thanks to lemma 3.4 we have $t \mapsto S_t g_0 \in C(\mathbb{R}_+; H)$ for any $g_0 \in H$.

• *S satisfies (iii).* For $g_0 \in H$ and $t_1, t_2 \ge 0$ denote $g(t) = S_t g_0$ and $\tilde g(t) := g(t + t_1)$. Making the difference of the two equations (3.1) written for $t = t_1$ and $t = t_1 + t_2$, we see that $\tilde g$ satisfies

$$\begin{aligned}(\tilde g(t_2), \tilde\varphi(t_2)) &= (g(t_1), \varphi(t_1)) + \int_{t_1}^{t_1+t_2} \big\{\langle \Lambda g(s), \varphi(s)\rangle + \langle \varphi'(s), g(s)\rangle\big\}\, ds \\ &= (\tilde g(0), \tilde\varphi(0)) + \int_0^{t_2} \big\{\langle \Lambda \tilde g(s), \tilde\varphi(s)\rangle + \langle \tilde\varphi'(s), \tilde g(s)\rangle\big\}\, ds,\end{aligned}$$

for any $\varphi \in X_{t_1+t_2}$ with the notation $\tilde\varphi(t) := \varphi(t+t_1) \in X_{t_2}$. Since the equation on the functions $\tilde g$ and $\tilde\varphi$ is nothing but the variational formulation associated to the equation (2.4), (2.5), we obtain

$$S_{t_1+t_2} g_0 = g(t_1 + t_2) = \tilde g(t_2) = S_{t_2} \tilde g(0) = S_{t_2} g(t_1) = S_{t_2} S_{t_1} g_0.$$

4 Proof of the existence part of Theorem 3.2.

We first prove thanks to a compactness argument in step 1 to step 3 that there exists a function $g \in L^2(0,T;V)$ such that

$$\langle g_0, \varphi(0)\rangle + \int_0^t \left\{ \langle \Lambda g(s), \varphi(s)\rangle_{V',V} + \langle \varphi'(s), g(s)\rangle_{V',V} \right\} ds = 0 \tag{4.4}$$

for any $\varphi \in C^1_c([0,T);V)$. We then deduce by some "regularization tricks" in step 4 and step 5 that the above weak solution is a variational solution.

Step 1. For a given $g_0 \in H$ and $\varepsilon > 0$, we seek $g_1 \in V$ such that

$$g_1 - \varepsilon \Lambda g_1 = g_0. \tag{4.5}$$

We introduce the bilinear form $a : V \times V \to \mathbb{R}$ defined by

$$a(u,v) := (u,v) - \varepsilon\,\langle \Lambda u, v\rangle.$$

Thanks to the assumptions made on Λ, we have

$$|a(u,v)| \le |u|\,|v| + \varepsilon\, M\,\|u\|\,\|v\|,$$

and

$$a(u,u) \ge |u|^2 + \varepsilon\,\alpha\,\|u\|^2 - \varepsilon\, b\,|u|^2 \ge \varepsilon\,\alpha\,\|u\|^2,$$

whenever $\varepsilon\, b < 1$, what we assume from now. On the other hand, the mapping $v \in V \mapsto (g_0, v)$ is a linear and continuous form. We may thus apply the Lax-Milgram theorem, and we get

$$\exists!\, g_1 \in V \qquad (g_1, v) - \varepsilon\langle \Lambda g_1, v\rangle = (g_0, v) \quad \forall\, v \in V.$$

Step 2. Fix $\varepsilon > 0$ as in the preceding step and build by induction the sequence (g_k) in $V \subset H$ defined by the family of equations

$$\forall\, k \qquad \frac{g_{k+1} - g_k}{\varepsilon} = \Lambda\, g_{k+1}. \tag{4.6}$$

Observe that from the identity

$$(g_{k+1}, g_{k+1}) - \varepsilon\,\langle \Lambda g_{k+1}, g_{k+1}\rangle = (g_k, g_{k+1}),$$

we deduce

$$|g_{k+1}|^2 + \varepsilon\,\alpha\,\|g_{k+1}\|^2 - \varepsilon\, b\,|g_{k+1}|^2 \le |g_k|\,|g_{k+1}|.$$

As a consequence, we have

$$|g_k| \le \frac{1}{1-\varepsilon b}\,|g_{k-1}| \le \frac{1}{(1-\varepsilon b)^k}\,|g_0| \le A^{k\,\varepsilon}\,|g_0| \quad \forall\, k \ge 0,$$

with $A := e^{2b}$ if $\varepsilon b \le 1/2$, and then

$$\begin{aligned}
\alpha \sum_{j=1}^k \varepsilon\,\|g_j\|^2 \quad &\le \quad \sum_{j=1}^k \frac{1}{2}(|g_{j-1}|^2 - |g_j|^2) + b \sum_{j=1}^k \varepsilon\,|g_j|^2 \\
&\le \quad \frac{1}{2}\,|g_{k-1}|^2 + b \sum_{j=1}^k \varepsilon\,|g_j|^2 \\
&\le \quad \frac{A^{2(k-1)\varepsilon}}{2}\,|g_0|^2 + \varepsilon\, b\,\frac{A^{2\varepsilon(k+1)} - 1}{A^{2\varepsilon} - 1}\,|g_0|^2 \quad \forall\, k \ge 0.
\end{aligned}$$

We fix $T > 0$, $n \in \mathbb{N}^*$ and we define

$$\varepsilon := T/n, \quad t_k = k\,\varepsilon, \quad g^\varepsilon(t) := g_k \text{ on } [t_k, t_{k+1}).$$

The two precedent estimates write then

$$\sup_{[0,T]} |g^\varepsilon|_H^2 + \alpha \int_0^T \|g^\varepsilon\|_V^2 \, dt \le \frac{b}{2} \, A^{2(T+1)} + \frac{3}{2} \, A^{2T}. \tag{4.7}$$

Step 3. Consider a test function $\varphi \in C_c^1([0,T);V)$ and define $\varphi_k := \varphi(t_k)$. Multiplying the equation (4.6) by φ_k and summing up from $k=0$ to $k=n$, we get

$$-\langle \varphi_0, g_0\rangle - \sum_{k=1}^n \langle \varphi_k - \varphi_{k-1}, g_k\rangle = \sum_{k=0}^n \varepsilon \, \langle \varphi_k, \Lambda g_{k+1}\rangle.$$

Introducing the two functions $\varphi^\varepsilon, \varphi_\varepsilon : [0,T) \to V$ defined by

$$\varphi^\varepsilon(t) := \varphi_{k-1} \quad \text{and} \quad \varphi_\varepsilon(t) := \frac{t_{k+1}-t}{\varepsilon} \, \varphi_k + \frac{t-t_k}{\varepsilon} \, \varphi_{k+1} \qquad \text{for} \quad t \in [t_k, t_{k+1}),$$

in such a way that

$$\varphi_\varepsilon'(t) = \frac{\varphi_{k+1} - \varphi_k}{\varepsilon} \qquad \text{for} \quad t \in (t_k, t_{k+1}),$$

the above equation also writes

$$-\langle \varphi(0), g_0\rangle - \int_0^T \langle \varphi_\varepsilon', g^\varepsilon\rangle \, dt = \int_0^T \langle \varphi^\varepsilon, \Lambda g^\varepsilon\rangle \, dt. \tag{4.8}$$

On the one hand, from (4.7) we know that up to the extraction of a subsequence, there exists $g \in Y_T$ such that $g^\varepsilon \to g$ weakly in $L^2(0,T;V)$. On the other hand, from the above construction, we have $\varphi_\varepsilon' \to \varphi'$ and $\varphi_\varepsilon \to \varphi$ both strongly in $L^2(0,T;V)$. We may then pass to the limit as $\varepsilon \to 0$ in (4.8) and we get (4.4).

Step 4. We prove that $g \in X_T$. Taking $\varphi := \chi(t)\,\psi$ with $\chi \in C_c^1((0,T))$ and $\psi \in V$ in equation (4.4), we get

$$\Big\langle \int_0^T g\chi' dt, \psi \Big\rangle = \int_0^T \langle g, \psi\rangle \chi' \, dt = -\int_0^T \langle \Lambda g, \psi\rangle \chi \, dt = \Big\langle -\int_0^T \Lambda g \chi \, dt, \psi \Big\rangle.$$

This equation holding true for any $\psi \in V$, it is equivalent to

$$\int_0^T g\chi' dt = -\int_0^T \Lambda g \chi \, dt \quad \text{in} \quad V' \quad \text{for any} \quad \chi \in \mathcal{D}(0,T),$$

or in other words

$$g' = \Lambda g \ \text{ in the sense of distributions in } V'.$$

Since $g \in L^2(0,T;V)$, we get that $\Lambda g \in L^2(0,T;V')$ and the above relation precisely means that $g \in H^1(0,T;V')$. We conclude thanks to Lemma 3.4 that $g \in X_T$.

Step 5. Assume first $\varphi \in C_c([0,T);H) \cap L^2(0,T;V) \cap H^1(0,T;V')$. We define $\varphi_\varepsilon(t) := \varphi *_t \rho_\varepsilon$ for a mollifier (ρ_ε) with compact support included in $(0,\infty)$ so that $\varphi_\varepsilon \in C_c^1([0,T);V)$ for any $\varepsilon > 0$ small enough and

$$\varphi_\varepsilon \to \varphi \quad \text{in} \quad C([0,T];H) \cap L^2(0,T;V) \cap H^1(0,T;V').$$

Writing the equation (4.4) for φ_ε and passing to the limit $\varepsilon \to 0$ we get that (4.4) also holds true for φ.

Assume next that $\varphi \in X_T$. We fix $\chi \in C^1(\mathbb{R})$ such that $\operatorname{supp} \chi \subset (-\infty, 0)$, $\chi' \le 0$, $\chi' \in C_c(]-1,0[)$ and $\int_{-1}^0 \chi' = -1$, and we define $\chi_\varepsilon(t) := \chi((t-T)/\varepsilon)$ so that $\varphi_\varepsilon := \varphi\chi_\varepsilon \in C_c([0,T);H)$ and $\chi_\varepsilon \to \mathbf{1}_{[0,T]}$, $\chi_\varepsilon' \to -\delta_T$ as $\varepsilon \to 0$. Equation (4.4) for the test function φ_ε writes

$$-\langle g_0, \varphi(0)\rangle - \int_0^t \chi_\varepsilon' \langle \varphi, g\rangle \, ds = \int_0^T \chi_\varepsilon \big\{ \langle \Lambda g, \varphi\rangle + \langle \varphi', g\rangle \big\} \, ds,$$

and we obtain the variational formulation (3.1) for $t_1 = 0$ and $t_2 = T$ by passing to the limit $\varepsilon \to 0$ in the above equation. ◻