EXERCISES ON THE HEAT EQUATION

1. The heat equation and the Fourier transform

Exercise 1.1. *Apply the Fourier technique to the heat equation with a source term*

$$\partial_t f = \Delta f + G \quad \text{on} \quad \mathscr{U}, \quad f(0,\cdot) = f_0 \quad \text{on} \quad \mathbb{R}^d, \tag{1.1}$$

with $f_0 \in L^2(\mathbb{R}^d)$ and $G \in L^2(\mathscr{U})$. Build a solution f which
(1) satisfies $f \in C([0,T]; L^2(\mathbb{R}^d))$;
(2) satisfies $f \in L^2(0,T; H^1(\mathbb{R}^d))$ and more precisely

$$\|f\|^2_{L^2(0,T;H^1)} \leqslant \frac{1}{2}\|f_0\|^2_{L^2(\mathbb{R}^d)} + (\frac{2}{3}T^{3/2} + \frac{1}{2}T)\|G\|^2_{L^2(\mathscr{U})}. \tag{1.2}$$

(3) Establish the representation formula

$$f(t,\cdot) = \gamma_t * f_0 + \int_0^t \gamma_{t-s} * G(s,\cdot)ds, \tag{1.3}$$

where we recall that we have defined the heat kernel

$$\gamma_t(x) := \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x|^2}{4t}}.$$

Exercise 1.2. *Use the Fourier transform method in order to solve*
(1) The wave equation

$$\partial^2_{tt} f - c^2 \partial^2_{xx} f = 0 \quad \text{on} \quad (0,T) \times \mathbb{R}, \quad f(0,\cdot) = f_0, \ \partial_x f(0,\cdot) = g_0 \quad \text{on} \quad \mathbb{R},$$

with $f = f(t,x)$ and $c > 0$. [Hint. One has to find

$$f(t,x) = \frac{1}{2}(f_0(x+ct) + f_0(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g_0(y)dy.]$$

(2) The Shrödinger equation on $f = f(t,x)$

$$i\partial_t f = \Delta f \quad \text{on} \quad (0,T) \times \mathbb{R}^d, \quad f(0,\cdot) = f_0 \quad \text{on} \quad \mathbb{R}^d.$$

(3) The Kolmogorov equation on $f = f(t,x,v)$

$$\partial_t f + v \cdot \nabla_x f = \Delta_v f \quad \text{on} \quad (0,T) \times \mathbb{R}^d \times \mathbb{R}^d, \quad f(0,\cdot) = f_0 \quad \text{on} \quad \mathbb{R}^d \times \mathbb{R}^d.$$

2. The heat equation and the heat kernel

Exercise 2.1.
(1) Show that $\gamma_{t+s} = \gamma_t * \gamma_s$ *for any* $t, s > 0$.
(2) Show that

$$\|\nabla_x \gamma_t\|_{L^r} = \frac{C_{d,r}}{t^{\frac{d}{2}(1-\frac{1}{r})+\frac{1}{2}}} \tag{2.1}$$

(3) Prove the Young inequality for convolution products

$$\|g * h\|_{L^p} \leqslant \|g\|_{L^q} \|h\|_{L^r}, \quad \frac{1}{p} = \frac{1}{q} + \frac{1}{r} - 1, \tag{2.2}$$

for any functions f, g *and any (compatible) exponents* $p, q, r \in [1, \infty]$.
(4) Recover the regularization estimate

$$\|f(t,\cdot)\|_{\dot H^1(\mathbb{R}^d)} \leqslant \frac{C}{t^{1k/2}} \|f_0\|_{L^2}. \tag{2.3}$$

Exercise 2.2 (variation of parameters formula)**.** *Consider the heat equation with a source term* (1.1) *with* $f_0 \in L^2(\mathbb{R}^d)$ *and* $G \in L^2(\mathscr{U})$. *Established (directly) that the function*

$$f(t,\cdot) := \gamma_t * f_0 + \int_0^t \gamma_{t-s} * G(s,\cdot) ds \tag{2.4}$$

(1) is a solution to the heat equation with source term (1.1)*;*
(2) satisfies $f \in C([0,T]; L^2(\mathbb{R}^d))$.
(3) Why this solution is nothing but the one provided by Exercise 1.1?
(4) When furthermore $f_0 = 0$, *establish (directly) that* $f \in L^2(0,T; H^1(\mathbb{R}^d))$ *and more precisely* (1.2).
(5) Establish (directly) (1.2) *for* $f_0 \in L^2(\mathbb{R}^d)$ *and* $G \in L^2(\mathscr{U})$.

Exercise 2.3. *For* $G \in L^1(\mathscr{U})$ *establish that the solution* f *to the heat equation with source term given by* (2.4) *satisfies* $f \in L^p(\mathscr{U})$ *for any* $1 < p < 1 + 2/d$. *More generally and more precisely, establish that*

$$\|f\|_{L^p(\mathscr{U})} \lesssim C T^{1-(1+\frac{d}{2})(\frac{1}{q}-\frac{1}{p}-1)} \|G\|_{L^q(\mathscr{U})},$$

under the condition $1 \leqslant q < p$, $(1+\frac{d}{2})(\frac{1}{q}-\frac{1}{p}) < 1$.

Exercise 2.4. *Consider the heat equation with source term*

$$\frac{\partial f}{\partial t} = \Delta f + G \quad \text{in } \mathbb{R} \times \mathbb{R}^d, \tag{2.5}$$

with $f, G \in L^2(\mathbb{R}^{d+1})$. *Using the Fourier transform in both variables, establish that*

$$\|f\|^2_{L^p} \lesssim \|f\|^2_{H^1} \lesssim \|f\|^2_{L^2} + \|G\|^2_{L^2},$$

with $p := 2(d+1)/(d-1) > 2$. *[Hint. Write the equation on the Fourier side, use the Sobolev embedding in* $\mathbb{R}^{d+1}$*, the Fourier definition of the Sobolev space in* $\mathbb{R}^{d+1}$ *and the Plancherel identity.]*

3. The heat equation and the energy method

Exercise 3.1. *(1) Consider* $f \in L^1(\mathbb{R}^d)$ *such that* $\mathrm{div} f \in L^1(\mathbb{R}^d)$. *Show that*

$$\int_{\mathbb{R}^d} \mathrm{div} f \, dx = 0.$$

[Hint. That is true for $f \in C^1_c(\mathbb{R}^d)$. *For* $f \in L^1(\mathbb{R}^d)$ *we introduce a mollifier* (ρ_ε), *a truncation fiunction* χ_M *and* $\rho_\varepsilon * (f\chi_M) \in C^1_c(\mathbb{R}^d)$.*]*
(2) Deduce that for $f \in H^1(\mathbb{R}^d)$ *such that* $\Delta f \in L^2(\mathbb{R}^d)$ *and* $g \in H^1(\mathbb{R}^d)$, *there holds*

$$\int_{\mathbb{R}^d} g \, \Delta f = - \int_{\mathbb{R}^d} \nabla g \cdot \nabla f.$$

Exercise 3.2. *Apply the energy method to the heat equation with a source term* (1.1) *with* $f_0 \in L^2(\mathbb{R}^d)$ *and* $G \in L^2(\mathscr{U})$.

Exercise 3.3. *Consider the parabolic equation*

$$\partial_t f = \mathrm{div}(A \nabla f) + \mathrm{div}(a f) + b \cdot \nabla f + c\, f \quad \text{in} \quad \mathscr{U},$$

for some coefficients $A, a, b, c \in L^\infty(\mathbb{R}^d)$ *with* $A \geqslant A_0 I$, $A_0 > 0$. *We complement that equation with the initial condition*

$$f(0,\cdot) = f_0 \quad \text{on} \quad \mathbb{R}^d, \tag{3.1}$$

for an initial datum $f_0 \in L^2(\mathbb{R}^d)$.
(1) Establish formally the energy estimate which implies that $f \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1)$.
(2) Same thing when we only assume $a, b \in L^d(\mathbb{R}^d)$ *and* $c \in L^{d/2}(\mathbb{R}^d)$.