EXERCISES ON TRANSPORT EQUATIONS

1. The Gronwall lemma

We recall the two following variants of the Gronwall lemma:

Lemma 1.1 (classical differential version of Gronwall lemma)**.** *We assume that $u \in C([0,T);\mathbb{R})$, $T \in (0,\infty)$, satisfies the differential inequality*

$$u' \le a(t)u + b(t) \quad on \quad (0,T), \tag{1.1}$$

for some $a, b \in L^1(0,T)$. Then, u satisfies pointwise the estimate

$$u(t) \le e^{A(t)}u(0) + \int_0^t b(s)e^{A(t)-A(s)}\,ds \quad on \quad (0,T), \tag{1.2}$$

where we have defined the primitive function

$$A(t) := \int_0^t a(s)\,ds. \tag{1.3}$$

Lemma 1.2 (integral version of Gronwall lemma)**.** *We assume $u \in \mathcal{L}^\infty(0,T;\mathbb{R})$, $T \in (0,\infty)$, satisfies pointwise the integral inequality*

$$u(t) \le u_0 + \int_0^t a(s)u(s)\,ds + \int_0^t b(s)\,ds \quad on \quad (0,T), \tag{1.4}$$

for some $0 \le a \in L^1(0,T)$ and $b \in L^1(0,T)$. Then, u satisfies pointwise the estimate

$$u(t) \le u_0\, e^{A(t)} + \int_0^t b(s)e^{A(t)-A(s)}\,ds \quad on \quad (0,T). \tag{1.5}$$

Exercise 1.3. *(1) Prove Lemma 1.1 under the additional assumptions $a, u \ge 0$ as a consequence of Lemma 1.2. (*Hint. Pass to the limit $\varphi \to \mathbf{1}_{[0,t]}$ in the distrubtional formulation of (1.1)*).*
*(2) Prove Lemma 1.1 in full generality. (*Hint. Take φ as a primitive of $\psi := -w + (\int_0^T w)\,\varrho$ for arbirary $0 \le w \in C^1_c(\varepsilon,T)$ and $\varrho \in C_c(0,\varepsilon)$ a probability measure*).*

Exercise 1.4. *Let $f \in C^1((0,T)\times\mathbb{R})$ and consider $u, v \in C([0,T];\mathbb{R})$ such that*

$$u' \le f(t,u), \quad v' \ge f(t,u), \quad u(0) \le v(0), \tag{1.6}$$

(in a distributional sense). Prove that $u \le v$ on $[0,T]$.

2. The characteristics method for smooth data

We consider the transport equation

$$\partial_t f + b\cdot\nabla f = 0 \quad \text{in} \quad (0,\infty)\times\mathbb{R}^d, \quad f(0,x) = f_0(x) \quad \text{in} \quad \mathbb{R}^d. \tag{2.1}$$

for a drift force field $b = b(t,x) : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ which is C^1 and globally Lipschitz. We denote $t \mapsto x(t) = \Phi_{t,s}(x) \in C^1(\mathbb{R}_+;\mathbb{R}^d)$. the unique solution to the ODE

$$\dot x(t) = b(t, x(t)), \quad x(s) = x. \tag{2.2}$$

Exercise 2.1. *Make explicit the construction and formulas in the three following cases:*
(1) $b(x) = b \in \mathbb{R}^d$ is a constant vector. (Hint. One must find $f(t,x) = f_0(x - bt)$).
(2) $b(x) = x$. (Hint. One must find $f(t,x) = f_0(e^{-t}x)$).
(3) $b(x,v) = v$, $f_0 = f_0(x,v) \in C^1(\mathbb{R}^d\times\mathbb{R}^d)$ and look for a solution $f = f(t,x,v) \in C^1((0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d)$. (Hint. One must find $f(t,x,v) = f_0(x - vt, v)$).
(4) Assume that $b = b(x)$ and prove that (S_t) is a group on $C(\mathbb{R}^d)$, where

$$\forall\, f_0 \in C(\mathbb{R}^d),\ \forall\, t\in\mathbb{R},\ \forall\, x\in\mathbb{R}^d \quad (S_t f_0)(x) = f(t,x) := f_0(\Phi_t^{-1}(x)). \tag{2.3}$$

Exercise 2.2. *(1) Show that*

$$f(t,x,v) := f_0(x - vt, v)e^{-t}, \quad t \ge 0,\ x \in \mathbb{R}d,\ v\in\mathbb{R}^d,$$

is a solution to the dampted free transport equation

$$\partial_t f + v\cdot\nabla_x f = -f, \quad f(0,\cdot) = f_0.$$

(2) Show that

$$f(t,x) := f_0(\Phi_{0,t}(x))\, e^{-\int_0^t c(\tau,\Phi_{\tau,t}(x))\,d\tau} + \int_0^t G(s,\Phi_{s,t}(x))\, e^{-\int_s^t c(\tau,\Phi_{\tau,t}(x))\,d\tau}\,ds$$

is a solution to the transport equation with source term

$$\partial_t f + b\cdot\nabla f + c\,f = G, \quad f(0) = f_0, \tag{2.4}$$

with $b = b(t,x)$, $c = c(t,x)$ and $G = G(t,x)$ smooth functions. (Hint. Compute the time derivative of $f(t,\Phi_t(x))\,\exp\int_0^t c(s,\Phi_s(x))\,ds$).

Exercise 2.3. *1) Consider the transport equation with vanishing boundary condition*

$$\begin{cases} \partial_t f + \partial_x f = 0 \\ f(t,0) = 0, \quad f(0,x) = f_0(x), \end{cases} \tag{2.5}$$

where $f = f(t,x)$, $t \ge 0$, $x \ge 0$. Assume $f_0 \in C^1_c(]0,\infty[)$. Establish that $\bar f(t,x) := f_0(x-t)$ provides a solution to equation (2.5).
2) Consider the transport equation with boundary condition

$$\begin{cases} \partial_t f + \partial_x f + af = 0 \\ f(t,0) = b(t), \quad f(0,x) = f_0(x), \end{cases} \tag{2.6}$$

where $f = f(t,x)$, $t \ge 0$, $x \ge 0$. Assume $a \in L^\infty(\mathbb{R}_+)$, $f_0 \in C^1_c(]0,\infty[)$ and $b \in C^1_c(]0,T[)$. Show that the characteristics method provides a unique smooth solution f given by $f = \bar f$, with

$$\bar f(t,x) := e^{A(x-t)-A(x)} f_0(x-t)\mathbf{1}_{x>t} + e^{-A(x)} b(t-x)\mathbf{1}_{t>x}, \quad A(x) := \int_0^x a(u)\,du.$$

(Hint. When $f \in C^1([0,T] \times \mathbb{R}_+)$, observe that both

$$\frac{d}{dt}(e^{A(t+x)} f(t,t+x)) = 0, \quad \frac{d}{dx}(e^{A(x)} f(t+x,x)) = 0, \quad A(x) := \int_0^x a(u)\,du,$$

and then $f = \bar f$. Also observe that $\bar f \in C^1([0,T] \times \mathbb{R}_+)$ in that case and conclude).

3. The characteristics method for non-smooth data

We recall that the solution to the transport equation is given by

$$\bar f(t,x) := f_0(\Phi_{-t}(x)). \tag{3.1}$$

We recall the Liouville theorem which tells us that the Jacobian function $J := \det D\Phi_t(y)$ satisfies the ODE

$$\frac{d}{dt} J = (\mathrm{div} b(t,\Phi_t(y)))J, \quad J(0,y) = 1,$$

so that

$$\det D\Phi_t(y) = e^{\int_0^t (\mathrm{div} b(s,\Phi_s(y)))ds}. \tag{3.2}$$

Exercise 3.1. *Prove that* (3.1) *does not depend of the choice of the function $f_0 \in \mathcal{L}^p(\mathbb{R}^d)$ in the class $\{f_0\} \in L^p(\mathbb{R}^d)$. (Hint. Take $g_0 \in \{f_0\}$ and compute $\|f_0 \circ \Phi_{-t} - g_0 \circ \Phi_{-t}\|_{L^p}$).*

Exercise 3.2. *(1) For any matrix $B \in M_d(\mathbb{R})$ and $h \in \mathbb{R}$, prove that*

$$\det(I + hB) = 1 + h\,\mathrm{tr} B + \mathcal{O}(h^2).$$

(2) Consider $A, B \in C^1((0,T); M_d(\mathbb{R}))$ which satisfy

$$\frac{d}{dt} A(t) = B(t)A(t),$$

and prove that

$$\frac{d}{dt}(\det A(t)) = (\mathrm{tr}\, B(t))(\det A(t)).$$

(3) Establish the Liouville theorem (3.2).

Exercise 3.3. *Prove that when $f_0 \in L^\infty(\mathbb{R}^d)$, formula* (3.1) *provides a function $f \in L^\infty(\mathcal{U}) \cap C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^d))$ which is a solution to the transport equation* (2.1) *in the distributional sense.*

Exercise 3.4. *Prove that the function f given by formula* (3.1) *satisfies*

$$\int_0^T \int_{\mathbb{R}^d} f\, L^*\psi\, dxdt = \int_{\mathbb{R}^d} f_0\, \psi(0)\, dx - \int_{\mathbb{R}^d} f(T)\, \psi(T)\, dx,$$

with $L^*\psi := -\partial_t \psi - div(b\,\psi)$, for any $\psi \in C^1_c([0,T] \times \mathbb{R}^d)$. (Hint. (1) Prove first the result for $\psi \in C^1_c((0,T) \times \mathbb{R}^d)$ by intoducing the function $\psi_\varepsilon := \psi *_{t,x} \rho_\varepsilon$, for a mollifier (ρ_ε) on $\mathbb{R}^{d+1}$. *(2)* Introduce next the function $\psi_\varepsilon := \psi\chi_\varepsilon$, with $\chi_\varepsilon \in C^1_c(\mathbb{R})$, $\mathbf{1}_{(\varepsilon,T-\varepsilon)} \le \psi_\varepsilon \le 1$, $\psi'_\varepsilon \rightharpoonup \delta_0 - \delta_T$. We may take $\psi'_\varepsilon(t) := \rho_\varepsilon(t) - \rho_\varepsilon(t-T)$ for a mollifier (ρ_ε) on $\mathbb{R}$.

Exercise 3.5. *Consider the dampted free transport equation with source term*

$$\left\{ \begin{array}{l} \partial_t f + v\cdot\nabla_x + f = G \\ \quad f(0,\cdot) = f_0, \end{array} \right. \tag{3.3}$$

where $f = f(t,x,v)$, $t \geq 0$, $x, v \in \mathbb{R}^d$, $f_0 \in L^1(\mathbb{R}^{2d})$ *and* $G \in L^1((0,T)\times\mathbb{R}^{2d})$. *Establish that*

$$f(t,x,v) := f_0(x-vt,v)e^{-t} + \int_0^t G(s,x+(s-t)v,v)e^{s-t}ds \tag{3.4}$$

belongs to $C([0,T];L^1(\mathbb{R}^{2d}))$ *and provides a weak solution.*

Exercise 3.6. *Consider the transport equation with boundary condition*

$$\left\{ \begin{array}{l} \partial_t f + \partial_x f + a f = 0 \\ f(t,0) = b(t), \quad f(0,x) = f_0(x), \end{array} \right. \tag{3.5}$$

where $f = f(t,x)$, $t \geq 0$, $x \geq 0$, $a \in L^\infty(\mathbb{R}_+)$, $f_0 \in L^1(\mathbb{R}_+)$ *and* $b \in L^1([0,T])$.
(a) Establish the a priori estimate

$$\sup_{[0,T]} \|f(t,\cdot)\|_{L^1} \;\leq (\|b\|_{L^1(0,T)} + \|f_0\|_{L^1})e^{t\|a\|_{L^\infty}}, \quad \forall\, t \geq 0.$$

(Hint. Use the Gronwall lemma).
(b) Establish the existence of a weak solution $f \in C([0,T];L^1(\mathbb{R}_+))$.

4. Duhamel formula and perturbation argument (bis)

Exercise 4.1. *Consider the renewal equation*

$$\left\{ \begin{array}{l} \partial_t f + \partial_x f + a f = 0 \\ f(t,0) = \rho_{f(t)}, \quad f(0,x) = f_0(x), \end{array} \right. \tag{4.1}$$

where $f = f(t,x)$, $t \geq 0$, $x \geq 0$, *and*

$$\rho_g := \int_0^\infty g(y)\,a(y)\,dy.$$

Assume $a \in L^\infty(\mathbb{R}_+)$ *and* $f_0 \in L^1(\mathbb{R}_+)$. *Establish that there exists a unique mild solution* $f \in C([0,T];L^1(\mathbb{R}_+))$ *to equation* (4.1).