EXERCISES ON PARABOLIC EQUATIONS

1. The variational method

Exercise 1.1. *We consider the parabolic equation*

$$\partial_t f = \mathcal{L} f \quad on \quad (0,\infty)\times\mathbb{R}^d, \quad f(0,x) = f_0(x) \quad in \quad \mathbb{R}^d, \tag{1.1}$$

on the function $f = f(t,x)$, $t \ge 0$, $x \in \mathbb{R}^d$, where $\mathcal{L}$ is the operator

$$\mathcal{L} f := \operatorname{div}(A\nabla f) + \operatorname{div}(af) + b\cdot\nabla f + cf, \tag{1.2}$$

with

$$0 < \nu I \le A \in L^\infty, \quad a, b \in L^d + L^\infty, \quad c \in L^1_{\rm loc}, \quad c_+ \in L^{d/2} + L^\infty.$$

We define $V := \{f \in H^1,\ \sqrt{c_-} f \in L^2\}$. For $f_0 \in L^2$, establish the existence of a weak solution $f \in L^2(0,T;V)$.

[Hint. Observe that $f(|b-a|\mathbf{1}_{|b-a|>M} + \sqrt{c_+}\mathbf{1}_{c_+>M}) \to 0$ in L^2 when $M \to \infty$ and that $2/d + 2/2^* = 1$, where 2^* denotes the Sobolev exponent.]

Exercise 1.2. *We consider the Fokker-Planck equation*

$$\partial_t f = \Delta f + \operatorname{div}(xf) \quad on \quad (0,\infty)\times\mathbb{R}^d, \quad f(0,x) = f_0(x) \quad in \quad \mathbb{R}^d, \tag{1.3}$$

on the function $f = f(t,x)$, $t \ge 0$, $x \in \mathbb{R}^d$. We define $L^2_k := \{f \in L^2;\ \langle x\rangle^k f \in L^2\}$, $\langle x\rangle^2 := 1 + |x|^2$, and $H^1_k := \{f \in L^2_k;\ \nabla f \in L^2_k\}$. For $f_0 \in L^2_k$, $k > d/2$, establish the existence of a weak solution $f \in L^2(0,T;H^1_k)$.

Exercise 1.3. *We consider the transport equation with kernel term*

$$\partial_t f = \operatorname{div}(af) + b\cdot\nabla f + cf + \mathcal{K} f, \quad f(0) = f_0,$$

on the function $f = f(t,x)$, $t \ge 0$, $x \in \mathbb{R}^d$, with

$$(\mathcal{K} f)(x) := \int_{\mathbb{R}^d} k(x,y) f(y)\,dy, \quad a, b, c \in L^\infty((0,T)\times\mathbb{R}^d), \quad k \in L^2(\mathbb{R}^d\times\mathbb{R}^d).$$

For $f_0 \in L^2(\mathbb{R}^d)$, establish the existence of a weak solution $f \in L^2((0,T)\times\mathbb{R}^d)$ thanks to the variational method.

[Hint. Observe that $\mathcal{K} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$.]

Exercise 1.4. *We consider the kinetic Fokker-Planck (or Kolmogorov) equation*

$$\partial_t f = -v\cdot\nabla_x f + \Delta_v f, \quad f(0,\cdot) = f_0,$$

on the function $f = f(t,x,v)$, $t \ge 0$, $x, v \in \mathbb{R}^d$. For $f_0 \in L^2(\mathbb{R}^{2d})$, establish the existence of a weak solution $f \in L^2((0,T)\times\mathbb{R}^d_x; H^1(\mathbb{R}^d_v))$ thanks to the variational method.

Exercise 1.5. *Prove that*

$$L^2(0,T;H^{-1}(\mathbb{R}^d)) = \{F_0 + \sum_{i=1}^d \partial_{x_i} F_i,\ F_i \in L^2(\mathcal{U}),\ 0 \le i \le d\}.$$

[Hint. Consider the mapping $A : \mathscr{H} := L^2(0,T;H^1) \to \mathscr{E} := (L^2(\mathcal{U}))^{d+1}$, $f \mapsto (f, \nabla f)$, $\mathscr{F} := RA$ and $B := A^{-1} : \mathscr{F} \to L^2(0,T;H^1)$. For a linear form $T \in L^2(0,T;H^{-1}(\mathbb{R}^d)) = \mathscr{H}'$, define the linear form $S : \mathscr{F} \to \mathbb{R}$, $G \in \mathscr{F} \mapsto S(G) := \langle T, BG\rangle$ and prove that there exists $\bar S \in \mathscr{E}'$ and thus $F_i \in L^2(\mathcal{U})$ such that $\bar S_{|\mathscr{F}} = S$ and $\bar S(G) = \sum_i (F_i, G_i)_{L^2(\mathcal{U})}$ for any $G \in \mathscr{E}$. Deduce that $\langle T, f\rangle = S(Af)$ and conclude].

2. Around renormalization

Exercise 2.1. *(1) For $f \in L^2(\mathcal{U})$ prove that $f_\pm, |f| \in L^2(\mathcal{U})$. For $f \in L^2(0,T;H^1(\mathbb{R}^d))$ prove that $f_\pm, |f| \in L^2(0,T;H^1(\mathbb{R}^d))$ and $\nabla f_+ = \nabla f \mathbf{1}_{f>0}$ [Hint. Consider $\beta_\varepsilon(f)$ with $\beta_\varepsilon(s) := s_+^2(\varepsilon^2 + s^2)^{-1/2}$]. What about $f \in X_T$?*

(2) For $f \in H^1(\Omega)$ prove that $\nabla f = 0$ on $\{f = c\}$ for any $c \in \mathbb{R}$. [Hint. Consider $\beta_\varepsilon(f)$ and $\gamma_\varepsilon(f)$ with $\beta_\varepsilon(s) := (s+\varepsilon)_+^2(\varepsilon^2 + s^2)^{-1/2}$ and $\gamma_\varepsilon(s) := (s-\varepsilon)_+^2(\varepsilon^2 + s^2)^{-1/2}$].

Exercise 2.2 (Weak maximum principle)**.** *We consider the parabolic equation*

$$\partial_t f = \mathcal{L} f + \mathfrak{F} \quad \text{on} \quad (0,\infty) \times \mathbb{R}^d, \quad f(0,x) = f_0(x) \quad \text{in} \quad \mathbb{R}^d, \tag{2.1}$$

on the function $f = f(t,x)$, $t \ge 0$, $x \in \mathbb{R}^d$, with $\mathcal{L}$ given by (1.2) and

$$A, a, b, c \in L^\infty(\mathbb{R}^d), \quad A \ge \nu I,\ \nu > 0. \tag{2.2}$$

We assume $0 \le f_0 \in L^2(\mathbb{R}^d)$, $0 \le \mathfrak{F} \in L^2(0,T;H^{-1}(\mathbb{R}^d))$ (the order relation ≥ 0 has to be understood in the weak sense). Establish that the weak solution $f \in L^2(0,T;H^1)$ satisfies $f \ge 0$.

Exercise 2.3 (Maximum principle)**.** *We consider more or less the same equation as above and we aim to establish a maximum principle in the sense that $f \in L^\infty(\mathcal{U})$ under convenient (uniform) boundedness conditions on the data.*

(1) We first assume $a = b = c = \mathcal{F} = 0$ and $f_0 \in L^\infty(\mathbb{R}^d)$. Establish that $\|f\|_{L^\infty(\mathcal{U})} \le \|f_0\|_{L^\infty(\mathbb{R}^d)}$.
[Hint. For $k > \|f_0\|_{L^\infty}$, use the test function $\varphi := (f-k)_+$, observe that $\nabla f \nabla \varphi = \nabla f \nabla \varphi$ and establish that $f \le k$ on $\mathcal{U}$].

(2) Establish the same estimate when $a = c = \mathcal{F} = 0$ and $b \in L^\infty$ (or even $b \in L^d + L^\infty$).

Exercise 2.4. *Let (ρ_ε) be a mollifer on the real line, namely $0 \le \rho_\varepsilon \in C_c^\infty(\mathbb{R})$ such that $\|\rho_\varepsilon\|_{L^1} = 1$ and (for instance) $\operatorname{supp} \rho_\varepsilon \subset (-\varepsilon, \varepsilon)$. For $f \in L^1_{\rm loc}(\mathcal{U})$, $\mathcal{U} := (0,T) \times \mathbb{R}^d$, we define $f_\varepsilon := \rho_\varepsilon *_t f$.*

(1) For $f \in C([0,T];L^2(\mathbb{R}^d))$, prove that $f_\varepsilon \in C^1((0,T);L^2(\mathbb{R}^d))$ and $f_\varepsilon \to f$ in $C((0,T);L^2(\mathbb{R}^d))$.

(2) For $f \in L^2(\mathcal{U})$, prove that $f_\varepsilon \in C^1((0,T);L^2(\mathbb{R}^d))$ and $f_\varepsilon \to f$ in $L^2(\mathcal{U})$. [Hint. Use that for any $\eta > 0$ there exists $g \in C_c(\mathcal{U})$ such that $\|g - f\|_{L^2(\mathcal{U})} < \eta$.]

(3) For any $f \in X_T$, prove that $f_\varepsilon \in C^1((0,T);H^1(\mathbb{R}^d))$ and $f_\varepsilon \to f$ in X_T.

Exercise 2.5. (L^p estimates). *For $b, c \in L^\infty(\mathbb{R}^d)$, $(\operatorname{div} b)_- \in L^\infty(\mathbb{R}^d)$, $f_0 \in L^p(\mathbb{R}^d)$, $1 \le p \le \infty$, we consider the linear parabolic equation*

$$\partial_t f = \Lambda f := \Delta f + b \cdot \nabla f + c f, \quad f(0) = f_0. \tag{2.3}$$

We introduce the usual notations $H := L^2$, $V := H^1$ and X_T the associated space for some given $T > 0$.

1) Consider a convex function $\beta \in C^2(\mathbb{R})$ such that $\beta(0) = \beta'(0) = 0$ and $\beta'' \in L^\infty$. Prove that any variational solution $f \in X_T$ to the above linear parabolic equation satisfies

$$\int_{\mathbb{R}^d} \beta(f_t)\, dx \le \int_{\mathbb{R}^d} \beta(f_0)\, dx + \int_0^t \int_{\mathbb{R}^d} \{c\, f\, \beta'(f) - (div b)\, \beta(f)\}\, dx ds,$$

for any $t \ge 0$.

2) Assuming moreover that $\beta \ge 0$ and there exists a constant $K \in (0, \infty)$ such that $0 \le s\, \beta'(s) \le K \beta(s)$ for any $s \in \mathbb{R}$, deduce that for some constant $C := C(b, c, K)$, there holds

$$\int_{\mathbb{R}^d} \beta(f_t)\, dx \le e^{Ct} \int_{\mathbb{R}^d} \beta(f_0)\, dx, \quad \forall\, t \ge 0.$$

3) Prove that for any $p \in [1, 2]$, for some constant $C := C(b, c)$ and for any $f_0 \in L^2 \cap L^p$, there holds

$$\|f(t)\|_{L^p} \le e^{Ct} \|f_0\|_{L^p}, \quad \forall\, t \ge 0.$$

[Hint. For $p \in (1, 2]$, define $\beta \simeq s^p$ on $\mathbb{R}_+$ and extend it to $\mathbb{R}$ by symmetry. More precisely, define $\beta''_\alpha(s) = 2\theta \mathbf{1}_{s \le \alpha} + p(p-1)s^{p-2}\mathbf{1}_{s>\alpha}$, with $2\theta = p(p-1)\alpha^{p-2}$, and then the primitives which vanish at the origin, which are thus defined by $\beta'_\alpha(s) = 2\theta s \mathbf{1}_{s \le \alpha} + (ps^{p-1} + p(p-2)\alpha^{p-1})\mathbf{1}_{s>\alpha}$, $\beta_\alpha(s) = \theta s^2 \mathbf{1}_{s\le\alpha} + (s^p + p(p-2)\alpha^{p-1}s + A\alpha^p)\mathbf{1}_{s>\alpha}$, $A := p(p-1)/2 - 1 - p(p-2)$. Observe that $s\beta'_\alpha(s) \le 2\beta_\alpha(s)$ because $s\beta''_\alpha(s) \le \beta'_\alpha(s)$ and $\beta_\alpha(s) \le \beta(s)$ because $\beta''_\alpha(s) \le \beta''(s)$. Pass to the limit $p \to 1$ in order to deal with the case $p = 1$.]

4) Prove that for any $p \in [2, \infty]$ and for some constant $C := C(a, c, p)$ there holds

$$\|f(t)\|_{L^p} \le e^{Ct} \|f_0\|_{L^p}, \quad \forall\, t \ge 0.$$

[Hint. Define $\beta''_R(s) = p(p-1)s^{p-2}\mathbf{1}_{s\le R} + 2\theta \mathbf{1}_{s>R}$, with $2\theta = p(p-1)R^{p-2}$, and then the primitives which vanish in the origin and which are thus defined by $\beta'_R(s) = ps^{p-1}\mathbf{1}_{s\le R} + (pR^{p-1} + 2\theta(s-R))\mathbf{1}_{s>R}$, $\beta_R(s) = s^p \mathbf{1}_{s\le R} + (R^p + pR^{p-1}(s-R) + \theta(s-R)^2)\mathbf{1}_{s>R}$. Observe that $s\beta'_R(s) \le p\beta_R(s)$ because $s\beta''_R(s) \le (p-1)\beta'_R(s)$ and $\beta_R(s) \le \beta(s)$ because $\beta''_R(s) \le \beta''(s)$. Pass to the limit $p \to \infty$ in order to deal with the case $p = \infty$.]

5) Prove that for any $f_0 \in L^p(\mathbb{R}^d)$, $1 \le p \le \infty$, there exists at least one weak (in the sense of distributions) solution to the linear parabolic equation (2.3). [Hint: Consider $f_{0,n} \in L^1 \cap L^\infty$ such that $f_{0,n} \to f_0$ in L^p, $1 \le p < \infty$, and prove that the associate variational solution $f_n \in X_T$ is a Cauchy sequence in $C([0, T]; L^p)$. Conclude the proof by passing to the limit $p \to \infty$.] Prove that $f \ge 0$ if furthermore $f_0 \ge 0$.

3. Nash argument

Exercise 3.1. (Poincaré Wirtinger inequality) *Consider* $f \in L^1_{\rm loc}(\mathbb{R}^d)$ *such that* $\nabla f \in L^2(\mathbb{R}^d)$ *and* $0 \le \rho \in L^1_2(\mathbb{R}^d)$ *with unit integral. Prove that*

$$\|f - f * \rho\|_{L^2(\mathbb{R}^d)} \le C\Big(\int_{\mathbb{R}^d} \rho(z)|z|^2 dz\Big)^{1/2} \|\nabla f\|_{L^2(\mathbb{R}^d)}.$$

Exercise 3.2. (Nash inequality) *Establish the Nash inequality with the help of the above Poincaré Wirtinger inequality.*

Exercise 3.3. (Variant proofs of Nash inequality using the Sobolev inequality)
1. Give another proof of the Nash inequality by using the Sobolev inequality in dimension $d \ge 3$*. (Hint. Write the interpolation estimate*

$$\|f\|_{L^2} \le \|f\|^\theta_{L^1} \, \|f\|^{1-\theta}_{L^{2^*}}$$

and then use the Sobolev inequality associated to the Lebesgue exponent $p = 2$*).*
2. Give another proof of the Nash inequality by using the Sobolev inequality in dimension $d = 2$*. (Hint. Prove the interpolation estimate*

$$\|f\|_{L^2} \le \|f\|^{1/4}_{L^1} \, \|f^{3/2}\|^{1/2}_{L^2},$$

then use the Sobolev inequality associated to the Lebesgue exponent $p = 1$ *and* $p^* := 2$ *and finally the Cauchy-Schwartz inequality in order to bound the second term).*
3. Give another proof of the Nash inequality by using the Sobolev inequality in dimension $d = 1$*. (Hint. Prove the interpolation estimate*

$$\|f\|_{L^2} \le \|f\|^{1/2}_{L^1} \, \|f^{3/2}\|^{1/3}_{L^\infty},$$

then use the Sobolev inequality associated to the Lebesgue exponent $p = 1$ *and* $p^* := \infty$ *and finally the Cauchy-Schwartz inequality in order to bound the second term).*

Exercise 3.4. (A regularity estimate) *We consider the parabolic equation*

$$\partial_t f = \Delta f + \mathrm{div}(af), \quad f(0) = f_0,$$

with $a \in W^{1,\infty}(\mathbb{R}^d)$*. Establish that*

$$\|\nabla f(t)\|_{L^p} \le \frac{C}{t^\alpha} \|f_0\|_{L^1}, \quad \forall\, t > 0,$$

for any $p \in [2, \infty]$ *and for some associated constant* $C, \alpha > 0$*.*

Exercise 3.5. (A gain of integrability estimate)
(1) Using the Fourier transform technique, establish that the solution to the heat equation with source term We consider the heat equation with source term

$$\partial_t f = \Delta f + \mathrm{div} G,$$

with $f, G \in L^2(\mathbb{R}^{d+1})$*, satisfies* $f \in L^p(\mathbb{R}^{d+1})$*, with* $p > 2$*.*
(2) Deduce that any weak solution to the parabolic equation

$$\partial_t f = \mathrm{div}(A \nabla f),$$

with $0 < \nu I \le A \in L^\infty$*, satisfies* $f \in L^p((T_0, T_1) \times \mathbb{R}^d)$*, with* $p > 2$*, for any* $0 < T_0 < T_1 < \infty$*.*

4. The McKean equation

Exercise 4.1. *Consider a sequence (f_n) such that $f_n \to f$ in $L^2(\mathcal{U})$ and, for some $k > d/2$,*

$$f_n \in \mathcal{Z} := \{g \in L^2(\mathcal{U});\ g \geq 0,\ \|g(t)\|_{L^1} \leq A(t),\ \|g(t)\|_{L^2_k} \leq B(t)\}.$$

(1) Prove that $f \geq 0$. [Hint. Prove that for $g \in L^1(\mathcal{U})$, we have $g \geq 0$ if and only if $\langle g, \varphi \rangle \geq 0$ for any $\varphi \in L^\infty(\mathcal{U})$.]

(2) Prove that $\|f\|_{L^1(\mathbb{R}^d)} \leq A$ a.e. on $(0,T)$. [Hint. Prove that $\|f_n(t,\cdot)\|_{L^1} \to \|f(t,\cdot)\|_{L^1}$ in $L^1(0,T)$ by using the Cauchy-Schwartz inequality and conclude by using the reverse sense of the dominated convergence Lebesgue theorem.]

(3) Prove that $\|f\|_{L^2_k(\mathbb{R}^d)} \leq B$ a.e. on $(0,T)$. [Hint. For any $k' \in [0,k)$, prove that $f_n \langle x \rangle^{k'} \to f \langle x \rangle^{k'}$ strongly $L^2(\mathcal{U})$ and that $\|f_n\|_{L^2_{k'}} \to \|f_n\|_{L^2_{k'}}$ a.e. on $(0,T)$. Next deduce that $\|f(t,\cdot)\|_{L^2_{k'}} \leq B$ a.e. on $(0,T)$ for any $k' \in (0,k)$ and conclude.]

Exercise 4.2. *Consider a sequence (f_n) such that $f_n \rightharpoonup f$ in $L^2(\mathcal{U})$ and $f_n \in \mathcal{Z}$, for some $k > d/2$.*

(1) Prove that $f \geq 0$. [Hint. Prove that for $g \in L^1(\mathcal{U})$, we have $g \geq 0$ if and only if $\langle g, \varphi \rangle \geq 0$ for any $\varphi \in \mathcal{D}(\mathcal{U})$.]

(2) Prove that $\|f\|_{L^1(\mathbb{R}^d))} \leq A$ a.e. on $(0,T)$. [Hint. Prove that $f_n \rightharpoonup f$ in $L^2(0,T;L^2_{k'})$ for any $k' \in [0,k)$ and there exists a sequence (g_n) such that g_n is a convex combination of $f_1, \dots, f_n$ and $g_n \to f$ in $L^2(0,T;L^2_{k'})$. Conclude with the help of Exercise 4.1.]

(3) - Prove that $f_n^2 \rightharpoonup g$ weakly and $g \geq f^2$. [Hint. Consider the family $\mathscr{A}$ of real affine functions such that $\ell \in \mathscr{A}$ iff $\ell(s) \leq s^2$ for any $s \in \mathbb{R}$ and observe that $\ell(f_n) \rightharpoonup \ell(f)$ weakly.]

- We define $G_n(t) := \|f_n(t,\cdot)\|^2_{L^2_k}$. Prove that, up to the extraction of a subsequence, $G_n \rightharpoonup G$ weakly and $G(t) \geq \langle g(t,\cdot) \rangle$ a.e. on $(0,T)$. [Hint. Take $\psi(t)\chi_R(x)$ as a test function].

- Conclude that $\|f\|_{L^2_k(\mathbb{R}^d)} \leq B$ a.e. on $(0,T)$.

(4) - For $0 \leq F \in L^2(\mathcal{U})$ such that

$$\int F\varphi \leq C\|\varphi\|_{L^1(0,T;L^2)}, \quad \forall\, \varphi,$$

establish that $\|F\|_{L^\infty(0,T;L^2)} \leq C$ by proving first

$$\int (F \wedge n)^2 \psi \leq C\|\psi\|_{L^1(0,T)}, \quad \forall\, \psi, \forall\, n.$$

- Establish that $\|f(t,\cdot)\|_{L^2_k} \leq C = B(T)$ a.e. on $(0,T)$.

(5) For $0 \leq F \in L^1(\mathcal{U})$ such that

$$\int F\psi \leq C\|\psi\|_{L^1(0,T))}, \quad \forall\, \psi,$$

establish that $\|F\|_{L^\infty(0,T;L^1)} \leq C$. [Hint. Consider $\psi := \mathbf{1}_{F \geq C+\varepsilon}$, $\varepsilon > 0$]. Recover (2).

Exercise 4.3. (McKean-Vlasov equation) *Consider the linear parabolic equation*

$$\partial_t f = \mathcal{L}_g f := \Delta f + \mathrm{div}(a_g f), \quad f(0) = f_0, \tag{4.1}$$

with

$$a_g := a * g, \quad a \in L^\infty(\mathbb{R}^d)^d, \tag{4.2}$$

associated to the nonlinear McKean-Vlasov equation. We prove the existence and uniqueness of the solution to this equation by using directly the J.-L. Lions theorem in the flat L^2 *and associated Sobolev spaces.*

1) Defining $F := f\langle x\rangle^{2k}$*, establish that* F *is a solution to the linear parabolic equation*

$$\partial_t F = \mathcal{M}_g F := \Delta F + div(a_g F) + b\cdot\nabla F + c_g F, \tag{4.3}$$

with b *and* c_g *to be determined. [Hint.* $b := -4kx/\langle x\rangle^2$, $c_g := \langle x\rangle^{-2k}(8|\nabla\langle x\rangle^k|^2 - \Delta\langle x\rangle^{2k}) + \frac{1}{2}a_g\cdot b$*.]*

2) Establish that for any $F_0 \in L^2$ *and* $g \in L^\infty(0,T;L^1)$*, there exists a unique variational solution* $F \in X_T$ *to the parabolic equation* (4.3)*.*

3) Establish that for $f_0 \in L^2_k$ *and* $g \in L^\infty(0,T;L^1)$*, there exists a unique variational solution* $f \in Y_T$ *to the parabolic equation* (4.1) *with* $Y_T = C([0,T];H)\cap L^2(0,T;V)\cap H^1(0,T;V')$, $H := L^2_k$, $V := H^1_k$*.*

Exercise 4.4. (McKean-Vlasov equation again) *We consider the same linear parabolic equation as in Exercise 4.3 and the associated nonlinear McKean-Vlasov equation. We extend the existence of solutions to a larger class of initial data.*

1) Prove that for $f_0 \in L^2_k$, $k > d/2$*, and* $g \in L^1(\mathcal{U})$*, the solution* $f \in X_T$ *to the linear parabolic equation satisfies*

$$\|f(t,\cdot)\|_{L^1} \le \|f_0\|_{L^1}, \quad \forall\, t \ge 0. \tag{4.4}$$

[Hint. Define $f^\pm$ the solutions associated to the initial data $f_{0\pm} \ge 0$. Prove that $f = f^+ - f^-$ and conclude.*]*

2) When $\mathrm{div} a \in L^\infty$*, recover* (4.4) *by using a convenient family of renormalizing functions.*

3) Prove the existence and uniqueness of a solution to the nonlinear McKean-Vlasov equation for any $f_0 \in L^2_k$, $k > d/2$*.*

4) Prove the existence of a weak solution to the nonlinear McKean-Vlasov equation (4.1) *for any initial datum* $f_0 \in L^1 \cap L^2_k$, $k > 0$*.*