A CRASH COURSE ON EVOLUTION (LINEAR) PDEs

This course aims to present several approaches and techniques for dealing with the well-posedness issue for evolution (linear) PDEs.

Lecture 1 - The heat equation (3h)

• Topic 1. The heat equation and the Fourier transform

Content: just resolve the heat equation in the Fourier side

Exercises: The FT and other equations (Wave, Schrödinger, Kolmogorov)

• Topic 2. The heat equation and the heat kernel

Content: the heat kernel using Fourier and/or a direct computation, ultracontractivity and other $W^{1,p}$ estimates

Exercises: The heat equation with source term

• Topic 3. The heat equation and the energy method (a priori estimates)

Content: evolution of the L^2 norm, of the $\dot{H}^1$ norm, recover (at least partially) the ultracontractivity

Exercises: A priori estimates for general parabolic equations

• Topic 4. Duhamel formula and perturbation argument

Content: Parabolic equation as a perturbation of the heat equation.

Lecture 2 - Transport equations (3h)

• Topic 5. The Gronwall lemma

Exercises: Several other versions of the Gronwall lemma

• Topic 6. Transport equations and smooth data

Content: the characteristics method for smooth data

Exercises: The free transport equation, the renewal equation

• Topic 7. Ttransport equations and non-smooth data

Content: a priori L^p estimates, the characteristic method for non-smooth data

• Topic 8. Duhamel formula and perturbation argument (bis)

Content: The relaxation equation as a perturbation of the free transport equation

Exercises: The renewal equation

Lecture 3 - Parabolic equations (3h)

• Introduction to the parabolic equations framework

Content: Parabolic equations, a priori estimates, weak solutions and main result

• Topic 9. Existence of solutions - an implicit Euler scheme approach

Content: The implicit Euler scheme and its convergence

• Topic 10. Existence of solutions - the variational approach of J.-L. Lions

Content: A variant of the Lax-Milgram theorem, an alternative proof in the time independent case, a time dependent variant

• Topic 11. Generalities about evolution equations and semigroups

Content: Evolution equation and semigroup, explicit semigroup, the spectral analysis approach, perturbation of semigroup, the variational approach, the Hille-Yosida approach

Lecture 4 - Uniqueness and qualitative properties (3h)

• Topic 13. Uniqueness - the Gronwall lemma approach

• Topic 14. Uniqueness - the duality approach

• Topic 15. Weak maximum principle and strong maximum principle

• Topic 16. Nash approach to ultracontractivity for parabolic equations