LECTURE 1 - THE HEAT EQUATION

The present lecture mainly addresses one of the simplest evolution equations which is the heat equation for which we present some simple but efficient tools for solving it. We next present some semigroup/perturbation arguments for establishing the existence of solutions to more general parabolic equations.

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Content: just resolve the heat equation in the Fourier side

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1. Topic 1. The heat equation and the Fourier transform

We consider the heat equation

$$\partial_t f = \Delta f \quad \text{on} \quad \mathscr{U} := (0,T) \times \mathbb{R}^d, \tag{1.1}$$

$T \in (0,\infty]$, on the function $f = f_t = f(t,x)$, $t \in [0,T)$ the *time variable*, $x \in \mathbb{R}^d$ the *position variable*, where Δ is the Laplace operator

$$\Delta f := \sum_{j=1}^d \partial^2_{jj} f,$$

and we use the shorthands $\partial_t := \frac{\partial}{\partial t}$, $\partial_j := \frac{\partial}{\partial x_j}$ and $\partial^2_{jk} := \frac{\partial^2}{\partial x_j \partial x_k}$. We complement this time evolution equation with an initial condition

$$f(0,\cdot) = f_0 \quad \text{on} \quad \mathbb{R}^d. \tag{1.2}$$

We define the Fourier transform (for functions defined on $\mathbb{R}^d$)

$$(\mathcal{F}f)(\xi) = \hat f(\xi) := \int_{\mathbb{R}^d} f(x) e^{-ix\cdot\xi} d\xi, \quad \forall\, \xi \in \mathbb{R}^d.$$

On the Fourier side, the heat equation (1.1)-(1.2) writes

$$\partial_t \hat f = -|\xi|^2 \hat f, \quad \hat f(0,\cdot) = \hat f_0, \tag{1.3}$$

by observing that $\mathcal{F}(\partial_j f) = i\xi_j \hat f$. We readily solve that equation and we get

$$\hat f(t,\xi) = \Gamma_t(\xi) \hat f_0(\xi), \quad \forall\, t \geqslant 0,\ \xi \in \mathbb{R}^d, \tag{1.4}$$

where we have defined the Gaussian function $\Gamma_t(\xi) := e^{-t|\xi|^2}$. In order to come back to the initial PDE side, we recall that, defining

$$(\check{\mathcal{F}} g)(x) := \int_{\mathbb{R}^d} g(\xi) e^{ix\cdot\xi} d\xi,$$

we have $\mathcal{F}^{-1} = (2\pi)^d \check{\mathcal{F}}$, what means

$$\mathcal{F}^{-1} \circ \mathcal{F} g = \mathcal{F} \circ \mathcal{F}^{-1} g = g$$

for any reasonable (that is smooth enough and decaying fast enough) function g, that $\hat\Gamma_{1/2} = (2\pi)^{d/2}\Gamma_{1/2}$, that $(\mathcal{F} f_\lambda) = \lambda^d (\mathcal{F} f)_{\lambda^{-1}}$, where $g_s(y) := g(y/s)$, and that $\mathcal{F}(fg) = \hat f * \hat g$, where $*$ stands for the convolution operator

$$(f * g)(x) := \int_{\mathbb{R}^d} f(x-y) g(y) dy.$$

In particular, defining the heat kernel

$$\gamma_t(x) := \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x|^2}{4t}},$$

we find $\widehat{\gamma_t} = \Gamma_t$. Taking the inverse Fourier transform of formula (1.4), we thus obtain

$$f(t,\cdot) = \mathcal{F}^{-1}(\Gamma_t \hat f_0) = \gamma_t * f_0, \quad \forall\, t > 0. \tag{1.5}$$

So far, all the discussions have been conducted without much mathematical justification. We explain how to fix it now. For $f_0 \in L^2(\mathbb{R}^d)$, the Plancherel identity tells us that $\hat f_0 \in L^2(\mathbb{R}^d)$, and more precisely $\|\hat f_0\|_{L^2} = (2\pi)^{d/2} \|f_0\|_{L^2}$. As a consequence, $F_t := \Gamma_t \hat f_0 \in L^2(\mathbb{R}^d)$ for any $t \geqslant 0$. More precisely, we have $F \in C([0,T]; L^2(\mathbb{R}^d))$ thanks to the continuity theorem about parameter depending integrals (what is a mere application of the dominated convergence theorem of Lebesgue). We also have $\xi F \in L^2(\mathscr{U})$, because

$$\int_0^T |\xi|^2 e^{-2|\xi|^2 t} dt = \frac{1}{2}\int_0^{2|\xi|^2 T} e^{-u} du \leqslant \frac{1}{2}, \tag{1.6}$$

and then

$$\int_0^T\int_{\mathbb{R}^d} |\xi F|^2 d\xi dt = \int_{\mathbb{R}^d}\int_0^T |\xi|^2 e^{-2|\xi|^2 t} dt |\hat f_0|^2 d\xi \leqslant \frac{1}{2}\|\hat f_0\|^2_{L^2}.$$

From the above discussion, we have thus

$$f \in \mathcal{X} = \mathcal{X}_T := C([0,T]; L^2(\mathbb{R}^d)) \cap L^2(0,T; H^1(\mathbb{R}^d)), \tag{1.7}$$

the last space is just as a notation for telling that $f, \nabla f \in L^2(\mathscr{U})$, $\nabla := (\partial_1, \dots, \partial_d)$. More precisely, we define the Sobolev (type) space

$$\mathscr{H} = \mathscr{H}_T := L^2(0,T;H^1) := \{f \in L^2(\mathscr{U});\ \nabla f \in L^2(\mathscr{U})\}$$

that we endowed with the Hilbert norm defined, for any $f \in \mathscr{H}$, by

$$\|f\|^2_{\mathscr{H}} = \|f\|^2_{L^2(0,T;H^1)} := \int_0^T \|f(s)\|^2_{H^1}\, ds = \int_{\mathscr{U}} (|f|^2 + |\nabla f|^2)\, dxdt.$$

Next, for any $k \geqslant 0$, we have

$$t^k \int_{\mathbb{R}^d} |\xi|^k |F_t|^2 d\xi \leqslant \sup_{\xi\in\mathbb{R}^d} ((t|\xi|^2)^k \Gamma_t(\xi)) \int_{\mathbb{R}^d} |\hat f_0|^2 d\xi \lesssim \|\hat f_0\|^2_{L^2},$$

or in other words

$$\|f(t,\cdot)\|_{H^k(\mathbb{R}^d)} \leqslant \frac{C_k}{t^{k/2}}\|f_0\|_{L^2}, \tag{1.8}$$

and thus in particular $f \in L^\infty(\tau, T; H^k(\mathbb{R}^d))$, for any $\tau \in (0,T)$. Moreover, differentiating in time (1.4), we find

$$\partial_t^\ell \hat f(t,\xi) = (\partial_t^\ell \Gamma_t(\xi))\hat f_0(\xi) = (-|\xi|^2)^\ell \Gamma_t(\xi) \hat f_0(\xi),$$

so that, proceeding similarly as for the last estimate, we have

$$\sup_{t\in[\tau,T]} \int_{\mathbb{R}^d} |\xi^\alpha \partial_t^\ell \hat f(t,\xi)|^2 d\xi \lesssim \|\hat f_0\|^2_{L^2},$$

for any $\tau \in (0,T)$, $\ell, k \geqslant 0$, from what we get $f \in H^s((0,T)\times\mathbb{R}^d))$, for any $s \geqslant 0$, and thus $f \in C^\infty((0,T)\times\mathbb{R}^d))$. Because $F(0,\cdot) = \hat f_0$, we clearly have (1.2) in the a.e. sense (for L^2 functions). Because of (1.3), we clearly have (1.1) in the classical sense. When $f_0 \in H^k(\mathbb{R}^d) \subset C_0(\mathbb{R}^d)$, with $k > d/2$, we may show exactly as above that $f \in C([0,T]; H^k(\mathbb{R}^d)) \subset C([0,T]\times\mathbb{R}^d)$, so that (1.2) holds in the classical (everywhere) sense.

Exercise 1.1. *Apply the same procedure to the heat equation with a source term*

$$\partial_t f = \Delta f + G \quad \text{on} \quad \mathscr{U}, \quad f(0,\cdot) = f_0 \quad \text{on} \quad \mathbb{R}^d, \tag{1.9}$$

with $f_0 \in L^2(\mathbb{R}^d)$ and $G \in L^2(\mathscr{U})$. Build a solution f which
(1) satisfies $f \in C([0,T]; L^2(\mathbb{R}^d))$;
(2) satisfies $f \in L^2(0,T;H^1(\mathbb{R}^d))$ and more precisely

$$\|f\|^2_{L^2(0,T;H^1)} \leqslant \frac{1}{2}\|f_0\|^2_{L^2(\mathbb{R}^d)} + (\frac{2}{3}T^{3/2} + \frac{1}{2}T)\|G\|^2_{L^2(\mathscr{U})}. \tag{1.10}$$

2. Topic 2. The heat equation and the heat kernel

From the very definition of γ_t, we may compute (in the classical sense)

$$\partial_t \gamma_t = -\frac{d}{2t}\gamma_t - \frac{|x|^2}{4t^2}\gamma_t$$

and

$$\Delta\gamma_t = \mathrm{div}(-\frac{x}{2t}\gamma_t) = -\frac{d}{2t}\gamma_t + \frac{|x|^2}{4t^2}\gamma_t,$$

which imply that γ_t satisfies the heat equation

$$\partial_t\gamma_t = \Delta\gamma_t \quad \text{on} \quad (0,\infty)\times\mathbb{R}^d.$$

On the other hand, for $f_0 \in C_0(\mathbb{R}^d)$ or $f_0 \in L^q(\mathbb{R}^d)$, $1 \leqslant q \leqslant \infty$, we define

$$f(t,x) := (\gamma_t * f_0)(x). \tag{2.1}$$

We may then compute (formally, in the sense of distribution or classically)

$$\partial_t f - \Delta f = (\partial_t\gamma_t - \Delta\gamma_t) * f_0 = 0$$

and, because $(\gamma_t)_{t>0}$ is a approximation of the identity (Dirac mass in $x = 0$), we have

$$\gamma_t * f_0 \to f_0 \quad \text{as} \quad t \to 0.$$

We have thus recovered by a direct approach (alternative to the Fourier transform approach presented in the previous section) that (2.1) provides a (weak) solution to the heat equation (1.1)-(1.2).

Recalling the Young inequality for convolution products

$$\|g * h\|_{L^p} \leqslant \|g\|_{L^q}\|h\|_{L^r}, \quad \frac{1}{p} = \frac{1}{q} + \frac{1}{r} - 1, \tag{2.2}$$

for any functions f, g and any (compatible) exponents $p, q, r \in [1,\infty]$, and observing that

$$\|\gamma_t\|_{L^r} = t^{(d/2)(\frac{1}{r}-1)}\|\gamma_1\|_{L^r},$$

we deduce the ultracontractivity estimate

$$\|f(t,\cdot)\|_{L^p} \leqslant \frac{C_{d,r}}{t^{\frac{d}{2}(\frac{1}{q}-\frac{1}{p})}}\|f_0\|_{L^q}, \quad \forall\, 1 \leqslant q \leqslant p \leqslant \infty, \tag{2.3}$$

which may be compared to the regularization estimate (1.8). Two other properties can be readily verified from the integral representation formula (2.1). On the one hand, the following weak maximum principle holds

$$f(t,\cdot) \geqslant 0 \text{ a.e. on } \mathbb{R}^d,\ \forall\, t \geqslant 0, \ \text{ if } \ f_0 \geqslant 0 \text{ a.e. on } \mathbb{R}^d,$$

as a consequence of the fact that $\gamma_t \geqslant 0$ a.e. on $\mathbb{R}^d$. On the other hand, the following strong maximum principle holds

$$f(t,\cdot) > 0 \text{ on } \mathbb{R}^d,\ \forall\, t \geqslant 0, \ \text{ if } \ f_0 \geqslant 0 \text{ and } f_0 \not\equiv 0 \text{ a.e. on } \mathbb{R}^d,$$

as a consequence of the fact that $\gamma_t > 0$ on $\mathbb{R}^d$. This second property also reveals the infinite propagation speed of the heat equation.

Exercise 2.1.
(1) Prove (2.2).
(2) Show that $\gamma_{t+s} = \gamma_t * \gamma_s$ *for any* $t, s > 0$.
(3) Show that

$$\|\nabla_x \gamma_t\|_{L^r} = \frac{C_{d,r}}{t^{\frac{d}{2}(1-\frac{1}{r})+\frac{1}{2}}} \tag{2.4}$$

and recover the regularization estimate (1.8) *when* $k = 1$.

Exercise 2.2 (variation of parameters formula)**.** *Consider the heat equation with a source term* (1.9) *with* $f_0 \in L^2(\mathbb{R}^d)$ *and* $G \in L^2(\mathscr{U})$. *Established (directly) that the function*

$$f(t,\cdot) := \gamma_t * f_0 + \int_0^t \gamma_{t-s} * G(s,\cdot) ds \tag{2.5}$$

(1) is a solution to the heat equation with source term (1.9)*;*
(2) satisfies $f \in C([0,T]; L^2(\mathbb{R}^d))$.
(3) Why this solution is nothing but the one provided by Exercise 1.1?
(4) When furthermore $f_0 = 0$, *establish (directly) that* $f \in L^2(0,T; H^1(\mathbb{R}^d))$ *and more precisely* (1.10).

Exercise 2.3. *For* $G \in L^1(\mathscr{U})$ *establish that the solution* f *to the heat equation with source term given by* (2.5) *satisfies* $f \in L^p(\mathscr{U})$ *for any* $1 < p < 1 + 2/d$. *More generally and more precisely, establish that*

$$\|f\|_{L^p(\mathscr{U})} \lesssim C T^{1-(1+\frac{d}{2})(\frac{1}{q}-\frac{1}{p}-1)} \|G\|_{L^q(\mathscr{U})}, \quad C := \frac{C_{r,d}}{(1-\frac{d}{2}(\frac{1}{q}-\frac{1}{p})r)^{1/r}},$$

under the condition $1 \leqslant q < p$, $(1+\frac{d}{2})(\frac{1}{q}-\frac{1}{p}) < 1$ *and where* $C_{r,d}$ *and* r *are defined in* (2.3).

3. Topic 3. The heat equation and the energy method (a priori estimates)

We present some arguments which make possible to recover at least partially the properties of the solutions to the heat equation presented before. The good thing is that these arguments are very simple and very general (they are useful for general parabolic equations). The negative point is that they are only partial and not completely rigorous. Using these arguments in a completely rigorous way will be the subject of future lectures.

We consider f a solution to the heat equation (1.1)-(1.2). Multiplying the equation by f and integrating in the $x \in \mathbb{R}^d$ variable, we have

$$\frac{1}{2}\frac{d}{dt}\int f^2 = \int (\Delta f) f = -\int |\nabla f|^2,$$

where we have used the Green formula

$$\int_{\mathbb{R}^d} \nabla f \cdot G = -\int_{\mathbb{R}^d} f \mathrm{div} G,$$

for any real function f and vector field G. Integrating in the time variable, we deduce the *energy identity*

$$\int f_t^2 dx + 2\int_0^t \int |\nabla f_s|^2 dx ds = \int f_0^2 dx.$$

In particular, for $f_0 \in L^2$, we have $f \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1)$. Here and below, we note $f \in L^\infty(0,T;L^2)$ if $f \in L^2(\mathscr{U})$ is such that there exists $C \in [0,\infty)$ satisfying

$$\|f(t,\cdot)\|_{L^2(\mathbb{R}^d)} \leqslant C, \text{ for a.e. } t \in (0,T), \tag{3.1}$$

and we define

$$\|f\|_{L^\infty(0,T;L^2)} := \inf\{C \in [0,\infty) \text{ such that (3.1) holds}\}.$$

We now recover (at least part of) the smoothing effect and the ultracontractivity estimates in a very simple way. We indeed observe that

$$\frac{1}{2}\frac{d}{dt}\int |\nabla f|^2 = \int (\Delta \nabla f)\nabla f = -\int |D^2 f|^2$$

and next

$$\frac{d}{dt}\mathcal{H}(t) := \frac{d}{dt}\Big(\int f^2 + 2t\int |\nabla f|^2\Big) = -4\int |D^2 f|^2 \leqslant 0,$$

so that in particular

$$2t\int |\nabla f|^2 \leqslant \mathcal{H}(t) \leqslant \mathcal{H}(0) = \int f_0^2.$$

That is nothing but the smoothing estimate (6.1) in the case $p = q = 2$. In dimension $d \geqslant 3$, from the Sobolev inequality, we deduce that

$$\|f(t,\cdot)\|_{L^p} \lesssim t^{-1/2}\|f_0\|_{L^2}, \quad \forall\, t > 0, \quad \frac{1}{p} = \frac{1}{2} - \frac{1}{d},$$

and we thus recover a part of the ultracontractivity estimate (2.3).

Exercise 3.1. *Apply the same procedure to the heat equation with a source term* (1.9) *with* $f_0 \in L^2(\mathbb{R}^d)$ *and* $G \in L^2(\mathscr{U})$.

Exercise 3.2. *Consider the parabolic equation*

$$\partial_t f = \mathrm{div}(A\nabla f) + \mathrm{div}(af) + b\cdot\nabla f + c\,f \quad \textit{in} \quad \mathscr{U},$$

for some coefficients $A, a, b, c \in L^\infty(\mathbb{R}^d)$ *with* $A \geqslant A_0 I$, $A_0 > 0$. *We complement that equation with the initial condition* (1.2) *for an initial datum* $f_0 \in L^2(\mathbb{R}^d)$.

(1) Establish formally the energy estimate which implies that $f \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1)$.

(2) Same thing when we only assume $a, b \in L^d(\mathbb{R}^d)$ *and* $c \in L^{d/2}(\mathbb{R}^d)$.

4. Topic 4. Duhamel formula and perturbation argument

In this section, we explain how the previous analysis and a perturbation argument make possible to tackle more general parabolic equations. We consider the parabolic equation

$$\partial_t f = \Delta f + b \cdot \nabla f + c\, f \quad \text{in} \quad \mathscr{U}, \tag{4.1}$$

for some coefficients $b, c \in L^\infty(\mathbb{R}^d)$. We complement that equation with the initial condition (1.2) for an initial datum $f_0 \in L^2(\mathbb{R}^d)$.

Because of the variation of parameters formula (2.5), we may look for a function $f \in \mathcal{X}$, $T > 0$, which satisfies the equation in the mild sense

$$f_t = \gamma_t * f_0 + \int_0^t \gamma_{t-s} * [b \cdot \nabla f_s + c\, f_s] ds. \tag{4.2}$$

That identity is named as *the Duhamel formula*. Such a function f will automatically satisfies (2.5) as a consequence of Exercise 2.2-(1). For a given function $g \in \mathcal{X}$ (or just $g \in \mathcal{H} := L^2(0, T; H^1)$), we define

$$h_t := \gamma_t * f_0 + \int_0^t \gamma_{t-s} * [b \cdot \nabla g_s + c\, g_s] ds, \quad \forall\, t \in (0, T),$$

and we denote $g \mapsto \Upsilon g := h$ this mapping. We aim to prove that $\Upsilon : \mathcal{H} \to \mathcal{X} \subset \mathcal{H}$ and that, for $T > 0$ small enough, there exists a unique fixed point $f \in \mathcal{H}$, so that $f = \Upsilon f \in \mathcal{X}$, what is nothing but (4.2).

Because $G := b \cdot \nabla g + c\, g \in L^2(\mathscr{U})$ and of Exercise 1.1, we have $h \in \mathcal{X}$. Now, for $g_1, g_2 \in \mathcal{H}$ and denoting $h_1 := \Upsilon g_1$, $h_2 := \Upsilon g_2$, $h := h_2 - h_1$, $g := g_2 - g_1$, we have

$$h_t \quad = \quad \int_0^t \gamma_{t-s} * [b \cdot \nabla g_s + c\, g_s] ds.$$

From (1.10), we have

$$\|h\|^2_{\mathcal{H}} \leqslant (\frac{2}{3} T^{3/2} + \frac{1}{2} T) \|b \cdot \nabla g + c\, g\|^2_{L^2} \leqslant \alpha_T \|g\|_{\mathcal{H}},$$

with $\alpha_T := (\frac{2}{3} T^{3/2} + \frac{1}{2} T)(\|b\|_{L^\infty} + \|c\|_{L^\infty})$. Choosing $T > 0$ small enough in such a way that $\alpha_T < 1$, we see that Υ is a contraction in $\mathcal{H}$, and the Banach-Picard fixed point theorem for contraction mapping provides a unique fixed point $f \in \mathcal{H}$. We repeat the same procedure in order to build a global (in time) solution to (4.1).

5. Complements to Topic 1.

About Exercise 1.1. By linearity we may consider the equation

$$\partial_t f = \Delta f + G \;\text{ on }\; (0, \infty) \times \mathbb{R}^d, \quad f(0, \cdot) = 0 \;\text{ on }\; \mathbb{R}^d, \tag{5.1}$$

and next add the contribution due to equation (1.1)-(1.2). On the Fourier side, we have

$$\partial_t \hat f = -|\xi|^2 \hat f + \hat G \;\text{ on }\; (0, \infty) \times \mathbb{R}^d, \quad \hat f(0, \cdot) = 0 \;\text{ on }\; \mathbb{R}^d,$$

so that

$$\hat f(t, \cdot) = \int_0^t \Gamma_{t-s} \hat G_s ds.$$

On the one hand, we have

$$\begin{aligned}\int_{\mathbb{R}^d} |\hat f(t,\xi)|^2 d\xi &= \int_{\mathbb{R}^d} \Big(\int_0^t \Gamma_{t-s} \hat G_s ds \Big)^2 d\xi \\ &\leqslant \int_{\mathbb{R}^d} \Big(\int_0^t |\hat G_s| ds \Big)^2 d\xi \\ &\leqslant t^{1/2} \int_0^T \int_{\mathbb{R}^d} |\hat G_s|^2 d\xi ds,\end{aligned}$$

from what we deduce $f \in L^\infty(0,T;L^2(\mathbb{R}^d))$ (thanks to Plancherel identity). We next write

$$\hat f(t,\cdot) - \hat f(t',\cdot) = \int_0^t (\Gamma_{t-s} - \Gamma_{t'-s}) \hat G_s ds + \int_{t'}^t \Gamma_{t'-s} \hat G_s ds =: (\mathrm{I}) + (\mathrm{II}).$$

For the second term, we obviously have (that is the same computation as above)

$$\|(\mathrm{II})\|^2_{L^2} \leqslant |t-t'|^{1/2} \|G\|_{L^2(\mathscr{U}))}.$$

For the first term, for any fixed $t \in (0,T)$ and as $t' \to t$, we have

$$(\Gamma_{t-s} - \Gamma_{t'-s})\hat G_s \to 0 \text{ a.e. and } |(\Gamma_{t-s} - \Gamma_{t'-s})\hat G_s| \leqslant |\hat G_s| \in L^2(\mathscr{U}),$$

so that (I) $\to 0$ in $L^2(\mathbb{R}^d)$ thanks to the dominated convergence theorem of Lebesgue. Both information together imply that $\hat f \in C([0,T];L^2(\mathbb{R}^d))$, so that $f \in C([0,T];L^2(\mathbb{R}^d))$ (thanks to Plancherel identity again).
On the other hand, we compute

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^d} |\xi \hat f|^2 &= \int_0^T \int_{\mathbb{R}^d} |\xi|^2 \Big(\int_0^t \Gamma_{t-s} \hat G_s ds \Big)^2 d\xi dt \\ &\leqslant T \int_0^T \int_0^t \int_{\mathbb{R}^d} |\xi|^2 \Gamma^2_{t-s} |\hat G_s|^2 ds d\xi dt \\ &\leqslant T \int_0^T \int_{\mathbb{R}^d} \int_0^T |\xi|^2 \Gamma^2_\tau d\tau |\hat G_s|^2 d\xi dt \\ &\leqslant \frac{T}{2} \int_0^T \int_{\mathbb{R}^d} |\hat G_s|^2 d\xi dt,\end{aligned}$$

where we have used the Cauchy-Schwarz inequality in the second line and (1.6) in the last line. We immediately deduce that (1.10) holds.

Additional material. Use the Fourier transform method in order to solve

(1) The wave equation

$$\partial^2_{tt} f - c^2 \partial^2_{xx} f = 0 \ \text{ on } \ (0,T) \times \mathbb{R}, \ \ f(0,\cdot) = f_0, \ \partial_x f(0,\cdot) = g_0 \ \text{ on } \ \mathbb{R},$$

with $f = f(t,x)$ and $c > 0$. *Hint. One has to find*

$$f(t,x) = \frac{1}{2}(f_0(x+ct) + f_0(x-ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} g_0(y) dy.$$

(2) The Shrödinger equation on $f = f(t,x)$

$$i\partial_t f = \Delta f \ \text{ on } \ (0,T) \times \mathbb{R}^d, \ \ f(0,\cdot) = f_0 \ \text{ on } \ \mathbb{R}^d.$$

(3) The Kolmogorov equation on $f = f(t,x,v)$

$$\partial_t f + v \cdot \nabla_x f = \Delta_v f \ \text{ on } \ (0,T) \times \mathbb{R}^d \times \mathbb{R}^d, \ \ f(0,\cdot) = f_0 \ \text{ on } \ \mathbb{R}^d \times \mathbb{R}^d.$$

6. Complements to Topic 2 (and to Topic 1 again).

About Exercise 2.1. (1) *Hint: use the Holder inequality.*

(2) We may establish the formula directly from the definition or we may just use the series of identities

$$\gamma_t * \gamma_s = \mathcal{F}^{-1}(\mathcal{F}(\gamma_t * \gamma_s)) = \mathcal{F}^{-1}(\Gamma_t \Gamma_s) = \mathcal{F}^{-1}(\Gamma_{t+s}) = \gamma_{t+s}.$$

(3) We observe that

$$\nabla_x f(t,\cdot) = (\nabla_x \gamma_t) * f_0,$$

so that

$$\|\nabla_x f\|_{L^p} \leqslant \|\nabla_x \gamma_t\|_{L^r} \|f_0\|_{L^q}, \tag{6.1}$$

with

$$\|\nabla_x \gamma_t\|_{L^r} = \frac{C_{p,q}}{t^{1/2+(d/2)(1/q-1/p)}}. \tag{6.2}$$

About Exercise 2.2.

(1) We define

$$\mathcal{G}(t,\cdot) := \int_0^t \gamma_{t-s} * G(s,\cdot) ds \tag{6.3}$$

and we compute

$$\begin{aligned} \partial_t \mathcal{G}(t,\cdot) &= G(t,\cdot) + \int_0^t (\partial_t \gamma_{t-s}) * G(s,\cdot) ds \\ \Delta \mathcal{G}(t,\cdot) &= \int_0^t (\Delta \gamma_{t-s}) * G(s,\cdot) ds, \end{aligned}$$

so that

$$\partial_t \mathcal{G} - \Delta \mathcal{G} = G \ \text{ on } \ (0,\infty)\times\mathbb{R}^d, \quad \mathcal{G}(0,\cdot) = 0 \ \text{ on } \ \mathbb{R}^d.$$

Putting together this result with the first calculus in Section 2, we have established that (2.5) provides a solution to the heat equation with source term (1.9).

(2) For $f_0 \in C_c(\mathbb{R}^d)$, we may establish that $t \mapsto \mathcal{F}_t := \gamma_t * f_0$ belongs to $C([0,T];L^2(\mathbb{R}^d))$ by a mere application of the dominated convergence theorem of Lebesgue. We deduce the same continuity property for $f_0 \in L^2(\mathbb{R}^d)$ thanks to the density $C_c(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$. We may indeed build $(f_{0\varepsilon})$ a sequence of $C_c(\mathbb{R}^d)$ such that $f_{0\varepsilon} \to f_0$ in L^2 as $\varepsilon \to 0$, so that

$$\begin{aligned} \|\mathcal{F}_{t'} - \mathcal{F}_t\|_{L^2} &\leqslant \|\gamma_{t'} * (f_0 - f_{0\varepsilon})\|_{L^2} + \|\gamma_{t'} * f_{0\varepsilon} - \gamma_t * f_{0\varepsilon}\|_{L^2} + \|\gamma_{t'} * (f_0 - f_{0\varepsilon})\|_{L^2} \\ &\leqslant 2\|f_0 - f_{0\varepsilon}\|_{L^2} + \|\gamma_{t'} * f_{0\varepsilon} - \gamma_t * f_{0\varepsilon}\|_{L^2} \to 0, \end{aligned}$$

as $t' \to t$ (and $\varepsilon \to 0$ in an appropriate way). On the other hand, we define

$$\mathcal{G}(t,\cdot) := \int_0^t \gamma_{t-s} * G(s,\cdot) ds$$

and we write

$$\mathcal{G}(t,\cdot) - \mathcal{G}(t',\cdot) = \int_{t'}^t \gamma_{t-s} * G(s,\cdot) ds + \int_0^t (\gamma_{t-s} * G(s,\cdot) - \gamma_{t'-s} * G(s,\cdot)) ds =: (\mathrm{I}) + (\mathrm{II}).$$

For the first term, we obviously have (when $t' < t$ for instance)

$$\begin{aligned} \|(\mathrm{I})\|_{L^2(\mathbb{R}^d)} &\leqslant \int_{t'}^t \|\gamma_{t-s} * G(s,\cdot)\|_{L^2(\mathbb{R}^d)} ds \\ &\leqslant \int_{t'}^t \|G(s,\cdot)\|_{L^2(\mathbb{R}^d)} ds \\ &\leqslant (t-t')^{1/2} \|G(s,\cdot)\|_{L^2(\mathcal{U})} \to 0, \end{aligned}$$

as $t' \to t$. For the second term (II), we argue thanks to a regularization argument as for $\mathcal{F}_t$.

(3) Taking the Fourier transform of the function f defined by (2.5), we get

$$\hat{f}(t,\cdot) = \Gamma_t \hat{f}_0 + \int_0^t \Gamma_{t-s} \hat{G}_s ds,$$

what is nothing but the solution built in Exercise 1.1.

(4) We assume $f_0 = 0$, so that the solution to (1.9) is given by (6.3). On the one hand, for any $t \in (0,T)$, we have

$$\|\mathcal{G}_t\|_{L^2} \leqslant \int_0^t \|\gamma_{t-s} * G_s\|_{L^2} ds \leqslant \int_0^t \|G_s\|_{L^2} ds$$

Using the the Cauchy-Schwarz inequality, we deduce

$$\begin{aligned} \|\mathcal{G}\|^2_{L^2(\mathscr{U})} &\leqslant \int_0^T \Big(\int_0^t \|G_s\|_{L^2} ds\Big)^2 dt \\ &\leqslant \int_0^T t^{1/2} \int_0^t \|G_s\|^2_{L^2} ds dt \leqslant \frac{2}{3} T^{3/2} \|G\|^2_{L^2(\mathscr{U})}. \end{aligned}$$

On the other hand, we have

$$\nabla \mathcal{G}_t = \int_0^t \nabla \gamma_{t-s} * G_s ds,$$

from what we deduce

$$\begin{aligned} \|\nabla G_t\|_{L^2} &\leqslant \int_0^t \|(\nabla \gamma_{t-s}) * G_s\|_{L^2} ds \\ &\leqslant \int_0^t \frac{C}{(t-s)^{1/2}} \|G_s\|_{L^2} ds. \end{aligned}$$

For $\alpha, \beta \geqslant 0$ measurable functions, we observe that

$$\begin{aligned} \int_0^T \Big(\int_0^t \alpha(t-s)\beta(s) ds\Big)^2 dt &\leqslant \int_0^T \int_0^t \alpha(t-s) ds \int_0^t \alpha(t-s)\beta^2(s) ds dt \\ &\leqslant \int_0^T \alpha(\tau) d\tau \int_0^T \int_0^t \alpha(t-s)\beta^2(s) ds dt \\ &\leqslant \Big(\int_0^T \alpha(\tau) d\tau\Big)^2 \int_0^T \beta^2(s) ds, \end{aligned}$$

where we have used the Cauchy-Schwarz inequality in the first line. We deduce that

$$\|\nabla \mathcal{G}\|^2_{L^2(\mathscr{U})} \leqslant \Big(\int_0^T \frac{C}{\tau^{1/2}} d\tau\Big)^2 \int_0^T \|G_s\|^2_{L^2} ds \leqslant 4CT \|G\|^2_{L^2(\mathscr{U})}.$$

The both estimates together, we have recovered (1.10) (in the case $f_0 = 0$).

Additional material. Consider the heat equation with source term

$$\frac{\partial f}{\partial t} = \Delta f + G \quad \text{in } \mathbb{R} \times \mathbb{R}^d, \tag{6.4}$$

with $f, G \in L^2(\mathbb{R}^{d+1})$. We define the Fourier transform (in both variables)

$$\hat{h}(\tau, \xi) := \int_{\mathbb{R}^{d+1}} h(t,x) e^{-i(\tau t + x\cdot\xi)} dt dx.$$

On the Fourier side, the above heat equation is

$$i\tau \hat{f} + |\xi|^2 \hat{f} = \hat{G},$$

from what we immediately compute

$$\int_{\mathbb{R}^{d+1}} (1 + \tau^2 + |\xi|^4) |\hat{f}|^2 = \int_{\mathbb{R}^{d+1}} |\hat{f}|^2 + \int_{\mathbb{R}^{d+1}} \frac{\tau^2 + |\xi|^4}{|i\tau + |\xi|^2|^2} |\hat{g}|^2 = \int_{\mathbb{R}^{d+1}} |\hat{f}|^2 + |\hat{G}|^2.$$

We deduce

$$\|f\|^2_{L^p} \lesssim \|f\|^2_{H^1} \lesssim \|f\|^2_{L^2} + \|G\|^2_{L^2},$$

with $p := 2(d+1)/(d-1) > 2$, from the Sobolev embedding, the Fourier definition of the Sobolev space in $\mathbb{R}^{d+1}$ and the Plancherel identity. This estimate also reveals some gain of integrability of the solution to the heat equation and can be seen as a variant of (2.3).

7. Complements to topic 3.

We consider f a solution of the heat equation (5.1) with source term $G \in L^2(\mathscr{U})$ and vanishing initial datum. Multiplying the equation by $-\Delta f$ and integrating in the x variable, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int |\nabla f|^2 &= \int \partial_t \nabla f \cdot \nabla f \\ &= \int \partial_t f(-\Delta f) \\ &= -\int (\Delta f)^2 + \int G(-\Delta f) \leqslant \frac{1}{4}\int G^2,\end{aligned}$$

where we have used the Young inequality in the last line. Integrating in the time variable, we get

$$\int |\nabla f_t|^2 dx \leqslant \frac{1}{2}\int_0^t\int G_s^2 dx ds, \quad \forall\, t > 0.$$

Integrating once more in the time variable, we get

$$\|f\|^2_{L^2(0,T;\dot H^1)} \leqslant \frac{1}{2} T \|G\|^2_{L^2(0,T;L^2)},$$

and thus we (partially) recover the estimate (1.10).