ON THE INITIAL BOUNDARY VALUE PROBLEM FOR THE VLASOV-POISSON-BOLTZMANN SYSTEM.

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Abstract. We prove existence of *DiPerna-Lions* renormalized solutions to the Boltzmann equation and to the Vlasov-Poisson-Boltzmann system for the initial boundary value problem.

1. Introduction and main results.

This paper deals with the initial boundary value problem for the Boltzmann equation and for the Vlasov-Poisson-Boltzmann system (VPB in short) with general boundary conditions. We establish a stability result for sequences of *DiPerna-Lions* renormalized solutions which enables us to prove the global existence of such a solution.

Let Ω be a smooth, open and bounded set of $\mathbb{R}^3$ and set $\mathcal{O} = \Omega \times \mathbb{R}^3$. We consider a gas confined in $\Omega \subset \mathbb{R}^3$. The state of the gas is given by the distribution function $f(t,x,\xi) \geq 0$ of particles, which at time $t \geq 0$ and at the position $x \in \Omega$, move with the velocity $\xi \in \mathbb{R}^3$. In this model, the evolution of f is governed by the following Boltzmann equation

$$\frac{\partial f}{\partial t} + \xi \cdot \nabla_x f + E \cdot \nabla_\xi f = Q(f,f) \quad \text{in } (0,\infty) \times \mathcal{O}, \tag{1.1}$$

where $Q(f,f)$ is the quadratic Boltzmann collision operator describing the collision interactions (binary elastic shock). For the Boltzmann equation, $E = 0$, and for the VPB system, E is a self-induced force (or mean field) which describes the fact that particles interact by the way of the two-body long range Coulomb force. In this case E is given by $E = E_f(t,x) = -\nabla_x V_f(t,x)$ where V_f is the solution of the following Poisson equation

$$-\Delta V_f = \rho_f = \int_{\mathbb{R}^3} f(t,x,\xi)\, d\xi \qquad \text{on } (0,\infty) \times \Omega, \tag{1.2}$$

$$\text{and} \qquad V_f = 0 \qquad \text{on } (0,\infty) \times \partial\Omega, \tag{1.3}$$

$$\text{or} \qquad \frac{\partial V_f}{\partial n} = \eta \qquad \text{on } (0,\infty) \times \partial\Omega. \tag{1.4}$$

These equations have to be complemented with boundary conditions which take into account how the particles are reflected by the wall. We assume that the boundary $\partial\Omega$ is sufficiently smooth (say a C^2 manifold). We denote by $n(x)$ the outward unit normal vector at $x \in \partial\Omega$ and by $d\sigma_x$ the Lebesgue surface measure on $\partial\Omega$. We define the incoming/outgoing sets by

$$\Sigma_\pm = \{(x,\xi) \in \Sigma; \pm n(x) \cdot \xi > 0\} \qquad \text{where} \qquad \Sigma = \partial\Omega \times \mathbb{R}^3.$$

The boundary conditions take the form of a balance between the values of the traces $\gamma_\pm f$ of f on these sets. Precisely, we assume that the following linear boundary condition holds

$$\gamma_- f = (1-\alpha)\, K\, \gamma_+ f + \alpha\, \phi_- \qquad \text{on } (0,\infty) \times \Sigma_-. \tag{1.5}$$

Here $\alpha \in [0,1]$ is a fixed parameter and $\phi_- \geq 0$ is a given function such that, for all $T < \infty$

$$\int_0^T \!\! \int\!\!\int_{\Sigma_-} \phi_- \left(1 + |\xi|^2 + |\log \phi_-|\right) |n(x) \cdot \xi| \; d\xi d\sigma_x ds < \infty. \tag{1.6}$$

The reflection operator K splits into $K = \lambda L + (1-\lambda) D$, where the accommodation coefficient λ belongs to $[0,1]$, L is a local reflection operator defined by

$$L\,\gamma_+ f(t,x,\xi) = \gamma_+ f(t,x,R_x\,\xi), \tag{1.7}$$

with $R_x\,\xi = -\xi$ (inverse reflection) or $R_x\,\xi = \xi - 2\,(\xi\cdot n(x))\,n(x)$ (specular reflection) and D is a diffuse reflection operator. The precise assumptions we make on D will be detailed later on. A typical example is the Maxwell diffuse reflection

$$D\,\gamma_+ f(t,x,\xi) = M_w(t,x,\xi)\int_{\xi'\cdot n(x)>0}\gamma_+ f(t,x,\xi')\,\xi'\cdot n(x)\,d\xi', \tag{1.8}$$

where M_w is the wall Maxwellian defined by

$$M_w(t,x,\xi) = \frac{1}{2\,\pi\,\Theta^2}\exp(-\frac{|\xi|^2}{2\Theta}), \tag{1.9}$$

with the prescribed temperature $\Theta(t,x)$ which may be constant $\Theta(t,x) = \Theta_w \in (0,\infty)$ or may satisfy

$$0 < \Theta_0 \le \Theta(t,x) \le \Theta_1 < \infty. \tag{1.10}$$

Last, we require on initial condition, so we prescribe f at time $t=0$, i.e.

$$f(t,.) = f_0 \quad \text{on } \mathcal{O}, \tag{1.11}$$

where $f_0 \ge 0$ satisfies

$$\iint_{\mathcal{O}} f_0\,\big(1+|\xi|^2+|\log\, f_0|\big)\,d\xi dx + \nu\int_\Omega |\nabla V_{f_0}|^2\,dx < \infty, \tag{1.12}$$

with the value $\nu = 0$ for the Boltzmann equation and $\nu = 1$ for the VPB system.

In the following we will distinguish three cases according to the values of α and λ. In each case, different *a priori* estimates can be established which lead to three different natural definitions of the solution to the boundary value problem for the Boltzmann equation and the VPB system. These different cases are

- Case 1 : $\alpha \neq 0$ (partially absorbing condition).
- Case 2 : $\alpha = 0$ and $\lambda \neq 1$ (total reflection condition with diffusion).
- Case 3 : $\alpha = 0$ and $\lambda = 1$ (purely local reflection condition).

For the Boltzmann equation we prove in each case a corresponding sequential stability result and we deduce the following existence theorem. The precise meaning of Theorem 1, as well as the meaning of Theorem 2 and 3, is given in section 3 and 4.

Theorem 1. *Let $f_0 \ge 0$ satisfy (1.12) and $\phi_- \ge 0$ satisfy (1.6). Then there exists a renormalized solution $f \in C([0,\infty); L^1(\mathcal{O}))$ of the Boltzmann equation (1.1) corresponding to the initial data f_0 and such that in Case 1 or in Case 3 the trace γf of f satisfies (1.5), and in Case 2 the trace γf satisfies the relaxed boundary condition*

$$\gamma_- f \ge K\,\gamma_+ f \qquad \text{on } (0,\infty)\times\Sigma_-. \tag{1.13}$$

This is a slight generalization to the previous existence results due to K. Hamdache [23] (Case 1 and 3) and L. Arkeryd, N. Maslova [2] (Case 2), since [23] only deals with constant wall temperature and in [2] only the pure diffuse boundary condition ($\alpha = \lambda = 0$) is considered. But, the key point here is that our sequential stability results can be extended to solutions to the VPB system. In order to state the resulting existence theorem, we have to explain which boundary condition is prescribed for the Poisson equation (1.2), since different ıa priori estimates can be obtained. When the Dirichlet condition (1.3) is prescribed the system is noted the VPdB system and when the Neumann condition (1.4) holds, we shall call it the VPnB system.

Theorem 2 (VPdB system). *Assume that Θ is a constant and that Dirichlet condition (1.3) holds. Let $f_0 \geq 0$ satisfy (1.12) and $\phi_- \geq 0$ satisfy (1.6). Then in Case 1 and 3 there exists a renormalized solution $f \in C([0,\infty); L^1(\mathcal{O}))$ to the VPdB system (1.1)-(1.2) corresponding to the initial data f_0 and such that the trace γf of f is well defined and satisfies (1.5).*

Theorem 3 (VPnB system). *Assume that Θ satisfies (1.10) and that Neumann condition (1.4) holds, with η satisfying the compatibility condition*

$$\eta = \frac{1}{meas(\partial\Omega)} \int_\Omega \rho_0(x)\, dx. \tag{1.14}$$

Let $f_0 \geq 0$ satisfy (1.12). Then in Case 2 and 3 there exists a renormalized solution $f \in C([0,\infty); L^1(\mathcal{O}))$ to the VPnB system (1.1)-(1.2) corresponding to the initial data f_0 and such that the trace γf of f is well defined and satisfies the boundary condition (1.5) in Case 3, and the relaxed boundary condition (1.13) in Case 2.

The existence of a weak global solution to the Boltzmann equation for initial data satisfying the natural bound (1.12) was first considered by R.J. DiPerna and P.L. Lions [15,17] who introduce the so-called renormalized solution and the equivalent formulation of mild and exponential solutions. Their proof of existence and all the next ones are based on a sequantial stability or sequantial compactness result: considering a sequence of renormalized solutions to the Boltzmann equation (or to a modified and regularized version of the Boltzmann equation) one shows that there is a subsequence which converges and that the resulting limit is still a renormalized solution to the Boltzmann equation. Next, P.L Lions defined in [27] a more accurate notion of solution, the so-called dissipative solution, using the regularity property of the gain term established in [26]. In [28] he proved the existence of the renormalized solution (in fact a dissipative solution) to the VPB system thanks to a new method of proof which only uses techniques of renormalization of PDEs but does not refer anymore to characteristics which are involved in the definition of a mild solution. This proof, which is even new for the Boltzmann equation, can be seen as a simplification of the initial DiPerna-Lions proof, and its robustness permits it to be adapted in order to prove convergence of discretization schemes for the Boltzmann equation, (see [14] and [31]).

The boundary value problem for the Boltzmann equation has been treated by many authors [23], [11], [8], [1], [2], [20], [24] in the framework of mild and exponential solution. In these works, the trace γf is defined as the limit at the boundary along characteristics (which are lines) of the solution f.

With regard to existence results for the initial value problem for the Vlasov-Poisson system set in the whole space, we refer to Arsenev [3], C. Bardos, P. Degond [5], E. Horst [25], R.J. DiPerna, P.L. Lions [16]. Uniqueness and propagation of moments have been investigated by F. Castella [9], P.L. Lions, B. Perthame [30], B. Perthame [33], K. Pfaffermoser [35], R. Robert [38], G. Rein [37], J. Schaeffer [39]. The initial boundary value problem has been addressed by Y. Guo [21,22], J. Weckler [40], N. Ben Abdallah [6] and the stationary problem by F. Poupaud [36].

In the present work the main difficulty is to define the trace of a solution since the characteristics are no longer lines because of the presence of the field E. The difficulty is overcome thanks to the trace theory developed in [32] and especially the possibility of renormalizing the trace. The trace is then defined by a Green formula written on the renormalized equation. Our sequential stability and existence results are obtained by adapting Lions' proof [28]. It can be seen both as a generalization and as a simplification of the previous existence result of a solution to the Boltzmann boundary value problem.

The paper is organized as follow. In Section 2 we establish some *a priori* estimates for a solution to the VPB system which are available under the natural bounds (1.6) and (1.12). Section 3 is dedicated to make precise the sense of trace we shall use. We prove, in the context of renormalized solutions of the Vlasov equation, a general trace theorem. In Section 4, we present the notion of a weak solution we deal with and state the sequential stability result of renormalized solutions. The proof of the sequential stability is given in Section 5.

2. Reflection operators and a priori estimates.

The a priori estimates that we derive in this section are intimately linked with the assumptions we make on the boundary conditions and that we explain now. We assume that the diffuse operator D can be written

$$D\,\phi(t,x,\xi) = \frac{1}{|n(x)\cdot\xi|}\int_{\xi'\cdot n(x)>0} k(t,x,\xi,\xi')\,\phi(t,x,\xi')\,\xi'\cdot n(x)\,d\xi', \tag{2.1}$$

where the kernel k is a measurable function which satisfies the following assumptions introduced in [2].

(H0) Positivity, i.e. $k\geq 0$ a.e. .

(H1) Normalization, i.e. $\displaystyle\int_{\xi\cdot n(x)<0} k(t,x,\xi,\xi')\,d\xi = 1 \quad$ a.e. on $(0,\infty)\times\Sigma_+$.

(H2) Spreading condition, i.e. there is a constant $\kappa_0>0$ such that

$$\int_{\xi\cdot n(x)<0} k(t,x,\xi,\xi')\,|\xi\cdot n(x)|\,d\xi \geq \kappa_0 \qquad \text{a.e. on } (0,\infty)\times\Sigma_+.$$

(H3) Energy condition, i.e. there is a constant $\kappa_1<\infty$ such that

$$\int_{\xi\cdot n(x)<0} k(t,x,\xi,\xi')\,|\xi|^2\,d\xi \leq \kappa_1 \qquad \text{a.e. on } (0,\infty)\times\Sigma_+.$$

(H4) Reciprocity principle, i.e. there is a wall Maxwellian M_w defined by (1.9) such that

$$\int_{\xi'\cdot n(x)>0} k(t,x,\xi,\xi')\,M_w(t,x,\xi')\,\xi'\cdot n(x)\,d\xi' = |\xi\cdot n(x)|\,M_w(t,x,\xi) \qquad \text{a.e. on } (0,\infty)\times\Sigma_-,$$

with constant temperature or temperature satisfying (1.10).

We refer to [10] or [12] for a physical analysis of the boundary conditions. Let us remark that the Maxwell diffuse reflection (1.8)-(1.10) satisfies the assumptions (H0)–(H4), see [2]. The properties of the reflections operators L and D are collected in the following lemma.

Lemma 1. *Let ϕ satisfy (1.6). For the local operator (1.7) we have for a.e. $(t,x)\in(0,T)\times\partial\Omega$ the following identities*

$$\int_{\xi\cdot n(x)<0} L\,\phi\, p(|\xi|)\ |\xi\cdot n(x)|\,d\xi = \int_{\xi\cdot n(x)>0} \phi\, p(|\xi|)\ \xi\cdot n(x)\,d\xi, \tag{2.2}$$

with $p(y) = a + b\,y^2$, *and*

$$\int_{\xi\cdot n(x)<0} H(L\,\phi)\ |\xi\cdot n(x)|\,d\xi = \int_{\xi\cdot n(x)>0} H(\phi)\ \xi\cdot n(x)\,d\xi, \tag{2.3}$$

where $H(s) = s\,\log s$. *For the diffuse operator the following bounds hold a.e.* $(t,x)\in(0,T)\times\partial\Omega$

$$\int_{\xi\cdot n(x)<0} D\,\phi\ |\xi\cdot n(x)|\,d\xi = \int_{\xi\cdot n(x)>0} \phi\ \xi\cdot n(x)\,d\xi, \tag{2.4}$$

$$\int_{\xi\cdot n(x)<0} D\,\phi\,|\xi|^2\,|\xi\cdot n(x)|\,d\xi \leq \frac{\kappa_1}{\kappa_0}\int_{\xi\cdot n(x)<0} D\,\phi\ (\xi\cdot n(x))^2\,d\xi, \tag{2.5}$$

and the so called Darrozès-Guiraud inequality [13]

$$\int_{\xi\cdot n(x)<0} \big(H(D\,\phi) + \frac{|\xi|^2}{2\,\Theta}\,D\,\phi\big)\,|\xi\cdot n(x)|\,d\xi \leq \int_{\xi\cdot n(x)>0} \big(H(\phi) + \frac{|\xi|^2}{2\,\Theta}\,\phi\big)\,\xi\cdot n(x)\,d\xi, \tag{2.6}$$

where Θ *is the temperature of the wall Maxwellian defined in (H4).*

Proof. The identities (2.2) and (2.3) are immediately deduced by changing variable $V(x,\xi)\to\xi$. The assumption (H1) implies (2.4). From (H2) and (H3) we deduce

$$\int_{\xi\cdot n(x)<0} D\,\phi\,(\xi\cdot n(x))^2\,d\xi \geq \kappa_0 \int_{\xi'\cdot n(x)>0} \phi'\,\xi'\cdot n(x)\ d\xi' \tag{2.7}$$

and

$$\int_{\xi\cdot n(x)<0} D\,\phi\,|\xi|^2\,|\xi\cdot n(x)|\,d\xi \leq \kappa_1 \int_{\xi'\cdot n(x)>0} \phi'\,\xi'\cdot n(x)\ d\xi',$$

from which (2.5) follows.

Last, Jensen's inequality, (H0) and (H4) imply

$$H\big(\frac{D\,\phi}{M_w}\big) \leq \int_{\xi'\cdot n(x)>0} H\Big(\frac{\phi'}{M_w'}\Big)\,k\,\frac{M_w'\,\xi'\cdot n(x)}{M_w\,|n(x)\cdot\xi|}\,d\xi'.$$

Therefore, thanks to (H1) we get

$$\int_{\xi\cdot n(x)<0} M_w\,H\big(\frac{D\,\phi}{M_w}\big)\,|\xi\cdot n(x)|\,d\xi \leq \int_{\xi'\cdot n(x)>0} H\Big(\frac{\phi'}{M_w'}\Big)\,M_w'\ \xi'\cdot n(x)\,d\xi', \tag{2.8}$$

and we deduce (2.6) using (2.4). □

We do not give the explicit expression of the Boltzmann collision operator that we find in [10] or [15] for example. The precise assumptions we make on the cross section are those introduced in [15]. We just recall that the collision operator has the following remarkable properties

$$\int_{\mathbb{R}^3} Q(f,f)\begin{pmatrix}1\\ \xi\\ |\xi|^2\end{pmatrix} d\xi = 0, \tag{2.9}$$

and there is an entropy production term $e(f)\geq 0$ which satisfies

$$\int_{\mathbb{R}^3} e(f)\,d\xi = -\int_{\mathbb{R}^3} Q(f,f)\,\log f\,d\xi. \tag{2.10}$$

We are now able to derive some a priori estimates that satisfy a given solution to the initial boundary value problem of the Boltzmann equation or of the VPB system. In what follows, we assume that f is regular enough and have sufficient decay at infinity such that all the manipulations we perform are justified. We begin with the Boltzmann equation.

Proposition 1 (Boltzmann equation). *For all $T \in (0,\infty)$ there exists a constant $C_T < \infty$ which may depend on f_0 and ϕ_- by the way of bounds (1.6) and (1.12) and such that a solution f to the Boltzmann equation (1.1)-(1.5)-(1.11) satisfies*

$$\sup_{[0,T]}\Big\{\iint_{\mathcal{O}} f\big(1+|\xi|^2+|\log f|\big)\,d\xi dx+\nu\int_\Omega |V_f|^2\,dx\Big\}+\int_0^T\iint_{\mathcal{O}} e(f)\,d\xi dx dt\le C_T, \tag{2.11}$$

and the trace γf satisfies

$$\int_0^T\iint_\Sigma \gamma f\,(\xi\cdot n(x))^2\,d\xi d\sigma_x ds\le C_T. \tag{2.12}$$

Furthermore, in Case 2 one has the additional estimate

$$\int_0^T\iint_\Sigma \gamma f\,(1+|\xi|^2)\,|\xi\cdot n(x)|\,d\xi d\sigma_x ds\le C_T, \tag{2.13}$$

and in Case 1

$$\int_0^T\iint_\Sigma \gamma f\,(1+|\xi|^2+|\log\gamma f|)\,|\xi\cdot n(x)|\,d\xi d\sigma_x ds\le C_T. \tag{2.14}$$

These estimates can be generalized to the VPB system in the following way.

Proposition 2 (VPdB system). *We assume that the wall temperature is constant and we consider Case 1 and 3. For all T there is a constant C_T such that a solution f to the VPdB system (1.1)-(1.2)-(1.3)-(1.5)-(1.11) satisfies (2.11). Furthermore, in Case 1 one has the additional boundary estimate (2.14).*

Proposition 3 (VPnB system). *We assume that the wall temperature satisfies (1.10) and we consider Case 2 and 3. For all T there is a constant C_T such that a solution f to the VPnB system (1.1)-(1.2)-(1.4)-(1.5)-(1.11) satisfies (2.11) and the boundary estimate (2.12). Furthermore, in Case 2 the bound (2.13) holds.*

Proof of Proposition 1 and 3. Let f denote indifferently a solution to the Boltzmann equation or to the VPnB system, with in this last case $\alpha=0$. By a simple integration of equation (1.1) and using (2.9), (2.2) and (2.4) we clearly have

$$\iint_{\mathcal{O}} f_t\,d\xi dx+\alpha\int_0^t\iint_{\Sigma_+}\gamma_+ f\,\xi\cdot n(x)\,d\xi d\sigma_x ds=\iint_{\mathcal{O}} f_0\,d\xi dx+\alpha\int_0^t\iint_{\Sigma_-}\phi_-\,|\xi\cdot n(x)|\,d\xi d\sigma_x ds. \tag{2.15}$$

We remark that when $\alpha=0$ the total mass is conserved

$$\int_\Omega \rho(t,x)\,dx=\int_\Omega \rho_0\,dx,$$

and in particular for the VPnB system, the Poisson equation (1.2) and the Neumann condition (1.4) are compatible since η is given by (1.14).

We denote the total energy by $\mathcal{E}(t)=\iint_{\mathcal{O}} f_t\,|\xi|^2\,d\xi dx+\nu\int_\Omega|\nabla V_f(t,x)|^2\,dx$. Multiplying (1.1) by $|\xi|^2$ and integrating by parts we get

$$\frac{d}{dt}\iint_{\mathcal{O}} f\,|\xi|^2\,d\xi dx+\iint_\Sigma \gamma f\,|\xi|^2\,\xi\cdot n(x)\,d\xi d\sigma_x-2\,\nu\iint_{\mathcal{O}} E\cdot\xi\,f\,d\xi dx=0.$$

For the Boltzmann equation the last term vanishes. For the VPnB system, this is

$$-2\iint_{\mathcal{O}} \nabla_x V_f \cdot \xi\, f\, d\xi dx = -2\int_{\partial\Omega} 2\, V_f(t,x)\big[\int_{\mathbb{R}^3} \gamma f\, n(x)\cdot\xi\, d\xi\big]\, d\sigma_x + 2\iint_{\mathcal{O}} V\,(\xi\cdot\nabla_x f)\, dx d\xi.$$

The boundary term vanishes since (1.5), (2.2) and (2.4) imply that the mass flux is equal to zero when $\alpha = 0$. In order to deal with the last term we use equation (1.1) and (1.2), and we find

$$-2\iint_{\mathcal{O}} E\cdot\xi\, f\, d\xi dx = 2\iint_{\mathcal{O}} V\,(\frac{\partial}{\partial t} f)\, dx d\xi = -2\int_{\Omega} \nabla_x V_f\cdot\nabla_x(\frac{\partial}{\partial t} V_f)\, dx + 2\int_{\partial\Omega} V_f \frac{\partial}{\partial t}(\frac{\partial V_f}{\partial n}) d\sigma_x,$$

and once again the boundary term vanishes thanks to (1.4). We have then proved that the energy $\mathcal{E}$ satisfies

(2.16)

$$\mathcal{E}(t) + \big(1-(1-\alpha)\lambda\big)\int_0^t\iint_{\Sigma_+} \gamma_+ f\, |\xi|^2\, \xi\cdot n(x)\, d\xi d\sigma_x ds =$$
$$= \mathcal{E}(0) + (1-\alpha)(1-\lambda)\int_0^t\iint_{\Sigma_-} D\,\gamma_+ f\, |\xi|^2\ |\xi\cdot n(x)|\, d\xi d\sigma_x ds + \alpha\int_0^t\iint_{\Sigma_-} \phi_-\, |\xi|^2\ |\xi\cdot n(x)|\, d\xi d\sigma_x ds.$$

The smoothness assumption made on the boundary implies the existence of a vector field n which belongs to $\big(W^{1,\infty}(\Omega)\big)^3$ and coincides with the outward unit normal vector at the boundary. The Boltzmann operator is orthogonal to $n(x)\cdot\xi$ thanks to (2.9). Therefore, multiplying equation (1.1) by $n(x)\cdot\xi$ and integrating by parts we get

$$\begin{aligned}&\iint_{\mathcal{O}} f_t\, n(x)\cdot\xi\, d\xi dx + \int_0^t\iint_{\Sigma} \gamma f\,(\xi\cdot n(x))^2\, d\xi d\sigma_x ds =\\ &= \nu\int_0^t\int_{\Omega} \rho\, E\cdot n(x)\, dx ds + \int_0^t\iint_{\mathcal{O}} f\,\xi\cdot\nabla_x n(x)\xi\, d\xi dx ds + \iint_{\mathcal{O}} f_0\ n(x)\cdot\xi\, d\xi dx.\end{aligned} \tag{2.17}$$

To estimate the term whit E we do the following easy computation using the Pohozaev method

$$\begin{aligned}\int_{\Omega} \rho\, E\cdot n(x)\, dx &= \int_{\Omega} \Delta V_f\, \nabla V_f\cdot n(x)\, dx = \int_{\Omega} \partial^2_{ii} V_f\, \partial_j V_f\, n_j\, dx =\\ &= \int_{\partial\Omega} \partial_i V_f\, n_i\ \partial_j V_f\, n_j\ d\sigma_x - \int_{\Omega} \partial_i V_f\, \partial_j V_f\, \partial_i n_j\, dx - \int_{\Omega} \partial_i V_f\, n_j\, \partial^2_{ij} V_f\, dx,\end{aligned} \tag{2.18}$$

with the convention of summation over repeated indices. Integrating by parts the last term once again, we find

$$-\int_{\Omega} \partial_i V_f\, n_j\, \partial^2_{ij} V_f\, dx = -\int_{\partial\Omega} \partial_i V_f\, n_j\ \partial_i V_f\, n_j\ d\sigma_x + \int_{\Omega} \partial_i V_f\, n_j\, \partial^2_{ij} V_f\, dx + \int_{\Omega} \partial_i V_f\, \partial_i V_f\, \partial_j n_j\, dx$$

and we deduce

$$-\int_{\Omega} \partial_i V_f\, n_j\, \partial^2_{ij} V_f\, dx = \frac{1}{2}\int_{\Omega} |\nabla V_f(t,x)|^2\,(div\, n)\, dx - \frac{1}{2}\int_{\partial\Omega} \Big|\frac{\partial V_f}{\partial n}\Big|^2\ d\sigma_x, \tag{2.19}$$

since, thanks to (1.4), one has $|\nabla V_f|^2 = \Big|\dfrac{\partial V_f}{\partial n}\Big|^2$ on $\partial\Omega$.

From (2.17), (2.18) and (2.19) we obtain

$$\int_0^t\iint_{\Sigma} \gamma f\,(\xi\cdot n(x))^2\, d\xi d\sigma_x ds \leq \iint_{\mathcal{O}} (f_t + f_0)\,|\xi|\, d\xi dx + C_{\Omega}\int_0^t \mathcal{E}(s)\, ds + \frac{\nu}{2}\int_0^t\int_{\partial\Omega} \Big|\frac{\partial V_f}{\partial n}\Big|^2\ d\sigma_x ds. \tag{2.20}$$

On the other hand, one deduces from (2.5) and (1.6)

$$\iint_{\Sigma_-} D\gamma_+ f\, |\xi|^2\, |\xi \cdot n(x)|\, d\xi d\sigma_x \le \frac{\kappa_1}{\kappa_0\,(1-\alpha)\,(1-\lambda)} \iint_{\Sigma_-} \gamma_- f\, (\xi \cdot n(x))^2\, d\xi d\sigma_x. \tag{2.21}$$

Here, we have to distinguish two situations. If there is no diffuse reflection at the boundary, i.e. $\alpha = 1$ or $\lambda = 1$, one deduces directly from (2.15) and (2.16) the mass and energy bound

$$\sup_{[0,T]} \Big\{ \iint_{\mathcal{O}} f\big(1 + |\xi|^2\big)\, d\xi dx + \nu \int_\Omega |\nabla_x V_f|^2\, dx \Big\} \le C_T. \tag{2.22}$$

In the case where there is diffuse reflection we have $(1-\alpha)\,(1-\lambda) \neq 0$ and we deduce from (2.16), (2.20), (2.21) and (1.4) the following inequality

$$\mathcal{E}(t) \le C_1 + C_2 \int_0^t \mathcal{E}(s) ds \;\text{ on } (0,T),$$

which implies again (2.22), thanks to the Gronwall lemma.

The first bound (2.12) on the boundary is an easy consequence of (2.20) and (2.22). The outgoing mass flux is estimated by (2.15) in Case 1 and by (2.7) and (2.12) in Case 2. Then the incoming mass flux is controlled thanks to (1.5), (2.2) and (2.4). If diffuse reflection occurs at the boundary, (2.21) and (2.12) imply

$$\int_0^T \iint_{\Sigma_-} D\gamma_+ f\, |\xi|^2\, |\xi \cdot n(x)|\, d\xi d\sigma_x \le C_T,$$

and then (2.16) gives a control of the outgoing energy flux in both cases 1 and 2. The incoming energy flux is then estimated once again thanks to (1.5). To sum up, we have proved that γf satisfies the bound (2.13) in both Case 1 and Case 2.

Last, we come to the entropy estimate. Integrating the equation satisfied by $H(f)$ and using (2.10), we get

$$\begin{aligned} &\iint_{\mathcal{O}} H(f_t)\, d\xi dx + \big(1-(1-\alpha)\lambda\big) \int_0^t \iint_{\Sigma_+} H(\gamma_+ f)\, \xi \cdot n(x)\, d\xi d\sigma_x ds + \int_0^t \iint_{\mathcal{O}} e(f)\, d\xi dx ds \le \\ &\le \iint_{\mathcal{O}} H(f_0)\, d\xi dx + \int_0^T \iint_{\Sigma_-} \big\{ \alpha\, H(\phi_-) + (1-\alpha)(1-\lambda)\, H(D\,\gamma_+ f) \big\}\, |\xi \cdot n(x)|\, d\xi d\sigma_x ds. \end{aligned} \tag{2.23}$$

In Case 3 the boundary terms vanish and in Case 2 the difference between the outgoing and the incoming entropy flux can be controlled by the energy flux thanks to (2.6), (1.10) and (2.13). Therefore in both cases we obtain

$$\sup_{t\in[0,T]} \iint_{\mathcal{O}} H(f_t)\, d\xi dx + \int_0^T \iint_{\mathcal{O}} e(f)\, d\xi dx dt \le C_T. \tag{2.24}$$

In Case 1, we also get (2.24). Then (2.23) leads to the additional estimate of the outgoing entropy flux

$$\int_0^T \iint_{\Sigma_+} H(\gamma_+ f)\, \xi \cdot n(x)\, d\xi d\sigma_x ds \le C_T.$$

We prove that the same bound holds for the incoming entropy flux using (1.5), (2.3), (2.6), (1.10) and (2.13). The proof is ended by the use of the following classical and elementary lemma, that one can find in [15] for instance.

Lemma 2. *There exists an universal constant C_3 such that for all $\phi \geq 0$ we have*

$$\int_{\mathbb{R}^3} \phi\, |\log \phi|\, d\xi \leq \int_{\mathbb{R}^3} \phi\, (\log \phi + |\xi|^2)\, d\xi + C_3.$$

□

Proof of Proposition 2. Since the wall Maxwellian M_w is an absolute Maxwellian we can introduce the relative entropy $H_{M_w}(f) = f\, \log(f/M_w) + M_w - f$ and compute

$$\Big(\frac{\partial}{\partial t} + \xi \cdot \nabla_x + E \cdot \nabla_\xi\Big) H_{M_w}(f) = (\log f - \log M_w)\, Q(f,f) + E \cdot \nabla_\xi M_w + f\, E \cdot \frac{\xi}{\Theta_w}.$$

We integrate this equation using collision invariants (2.9) and entropy production (2.10), and we obtain

$$\begin{aligned}\frac{d}{dt} \iint_{\mathcal{O}} H_{M_w}(f)\, d\xi dx &+ \iint_{\Sigma} H_{M_w}(\gamma f)\, \xi \cdot n(x)\, d\xi d\sigma_x \\ &+ \iint_{\mathcal{O}} e(f)\, d\xi dx + \frac{\nu}{\Theta_w} \int_{\Omega} \nabla V_f \cdot j\, dx = \iint_{\mathcal{O}} E \cdot \nabla_\xi M_w\, d\xi dx,\end{aligned} \tag{2.25}$$

where

$$j(t,x) = \int_{\mathbb{R}^3} \xi\, f(t,x,\xi)\, d\xi.$$

We first remark that integrating equation (1.1) in velocity we have

$$\frac{\partial}{\partial t}\rho + div_x\, j = 0 \quad \text{ on } \ (0,\infty) \times \Omega,$$

and therefore

$$\frac{1}{\Theta_w} \int_{\Omega} \nabla V_f \cdot j\, dx = \frac{1}{\Theta_w} \int_{\Omega} V_f \frac{\partial}{\partial t}\rho\, dx = \frac{1}{2\,\Theta_w} \frac{d}{dt} \int_{\Omega} |\nabla_x V_f|^2\, dx. \tag{2.26}$$

Next, we use (2.2), (2.3), (2.4) and (2.8) to estimate by bellow the boundary term by

$$\begin{aligned}\iint_{\Sigma} H_{M_w}(\gamma f)\, \xi \cdot n(x)\, d\xi d\sigma_x &\geq \iint_{\Sigma_+} H_{M_w}(\gamma_+ f)\, \xi \cdot n(x)\, d\xi d\sigma_x \\ &- \iint_{\Sigma_-} \big\{(1-\alpha)\, \lambda\, H_{M_w}(L\gamma_+ f) + (1-\alpha)\,(1-\lambda)\, H_{M_w}(D\gamma_+ f) + \alpha\, H_{M_w}(\phi_-)\big\}\, |\xi \cdot n(x)|\, d\xi d\sigma_x \\ &\geq \alpha \iint_{\Sigma_+} H_{M_w}(\gamma_+ f)\, \xi \cdot n(x)\, d\xi d\sigma_x - \alpha \iint_{\Sigma_-} H_{M_w}(\phi_-)\, |\xi \cdot n(x)|\, d\xi d\sigma_x.\end{aligned} \tag{2.27}$$

From (2.25), (2.26) and (2.27) we obtain

$$\begin{aligned}&\frac{d}{dt}\Big\{\iint_{\mathcal{O}} H_{M_w}(f)\, d\xi dx + \nu \int_{\Omega} \frac{|\nabla_x V_f|^2}{2\,\Theta_w}\, dx\Big\} + \iint_{\mathcal{O}} e(f)\, d\xi dx + \alpha \iint_{\Sigma_+} H_{M_w}(\gamma_+ f)\, \xi \cdot n(x)\, d\xi d\sigma_x \leq \\ &\leq \nu\, C_{\Theta_w} \int_{\Omega} |\nabla_x V_f|^2\, dx + \alpha \iint_{\Sigma_-} H_{M_w}(\phi_-)\, |\xi \cdot n(x)|\, d\xi d\sigma_x.\end{aligned}$$

From this inequality and using tGronwall lemma and lemma 3 bellow, we can easily deduce that (2.11) holds. When $\alpha \neq 0$, we also obtain the additional boundary estimate (2.14) as we have done in the proof of Proposition 1 and 3.

Lemma 3 [29]. *There is a nonnegative constant $C = C(\Theta_w)$ such that*

$$\iint_{\mathcal{O}} f\left(1 + |\xi|^2 + |\log f|\right) dx d\xi \leq C\Big(meas(\Omega) + \iint_{\mathcal{O}} H_{M_w}(f)\, dx d\xi\Big).$$

A proof of lemma 3 is given in [29]. □

Remark 1. When $\alpha = 0$ we always have the estimate (2.11) but, for the VPdB system, we do not obtain any boundary estimate, in particular, we do not know if estimate (2.12) holds. About Proposition 2, we also note that the diffuse type operator D does not have to satisfy (H2) and (H3) anymore.

3. A trace Theorem for renormalized solutions to the Vlasov equation.

In Theorem 5 we give a general trace result for a renormalized solution to a Vlasov equation which make clear the meaning of the trace we deal with in Theorem 1. In fact, we do not need Theorem 5 in the proof of existence or stability of renormalized solutions that we present in the next sections (at least for case 1 and 2) and we only use in these proofs the fact that one can renormalize a distributional solution to the Vlasov equation as we have shown in [32] and that we recall in Theorem 4 below. By the way, it seems to us, that Theorem 5 is interesting in itself since it states that a renormalized solution f to the VPB equation admits a trace because it solves a Vlasov equation and not because of the way it has been built.

First, we recall the trace result obtained in [32] for a distributional solution $g = g(t, x, \xi) \in L^\infty((0,T) \times \mathcal{O})$ to the Vlasov equation

$$\frac{\partial g}{\partial t} + \xi \cdot \nabla_x g + E \cdot \nabla_\xi g = G \ \text{ in } \mathcal{D}'((0,T) \times \mathcal{O}), \tag{3.1}$$

where $G = G(t, x, \xi)$ is a source term satisfying $G \in L^1((0,T) \times \Omega \times B_R)$ for all $R \in (0, \infty)$, with the notation $B_R = \{\xi \in \mathbb{R}^3,\ |\xi| < R\}$, and $E = E(t, x)$ is a fixed vector field such that

$$E \in L^1(0, T; W^{1,1}(\Omega)). \tag{3.2}$$

Theorem 4 [32]. *Under the above assumptions, there exists a trace function γg such that*

$$\gamma\, g \in L^\infty\big((0,T) \times \Sigma,\ d\xi d\sigma_x dt\big), \tag{3.3}$$

and which is uniquely defined thanks to the Green formula

$$\int_0^T\!\!\iint_{\mathcal{O}} \{g\,(\frac{\partial \phi}{\partial t} + \xi \cdot \nabla_x \phi + E \cdot \nabla_\xi \phi) + G\, \phi\}\, d\xi dx dt = \int_0^T\!\!\iint_{\Sigma} \gamma\, g\, \phi\, n(x) \cdot \xi\, d\xi d\sigma_x dt, \tag{3.4}$$

for every test function $\phi \in \mathcal{D}((0,T) \times \bar{\mathcal{O}})$. Furthermore, for all $\beta \in C^1(\mathbb{R})$ the function $\beta(g)$ solves

$$\frac{\partial}{\partial t}\beta(g) + \xi \cdot \nabla_x \beta(g) + E \cdot \nabla_\xi \beta(g) = \beta'(g)\, G \ \text{ in } \mathcal{D}'((0,T) \times \mathcal{O}), \tag{3.5}$$

and the trace $\gamma\, \beta(g)$ of $\beta(g)$ satisfies

$$\gamma\, \beta(g) = \beta(\gamma g). \tag{3.6}$$

We come now to the renormalized solutions to the Vlasov equation and, to do it, we introduce some notation. Let $(X, \mathcal{B}, \mu)$ be a measured space, and let $L(X)$ denote the space of measurable functions $u : X \to \bar{\mathbb{R}}$. We define $\mathcal{A}$ as the class of all functions $\beta \in C^1(\mathbb{R})$ such that β' has compact support. For every $u \in L(X)$ and $\beta \in \mathcal{A}$ we have $\beta(u) \in L^\infty(X)$.

We assume that E satisfies (3.2) and $G \in L((0,T) \times \mathcal{O})$. We say that $g \in L(D)$ is a renormalized solution of Vlasov equation (3.2) if, for all $\beta \in \mathcal{A}$, we have $\beta'(g)\, G \in L^1((0,T) \times \Omega \times B_R)$ for all $R \in (0, \infty)$ and $\beta(g)$ solves (3.5).

Theorem 5. *Let $g \in L(D)$ be a renormalized solution of equation (3.1)-(3.2). Then g admits a unique trace function $\gamma g \in L((0,T) \times \Sigma)$ in the sense of the Green formula*

$$(3.7) \quad \int_0^T \iint_{\mathcal{O}} \{\beta(g)\,(\frac{\partial \phi}{\partial t} + \xi \cdot \nabla_x \phi + E \cdot \nabla_\xi \phi) + \beta'(g)\, G\, \phi\}\, d\xi dx dt = \int_0^T \iint_\Sigma \beta(\gamma\, g)\, \phi\, n(x) \cdot \xi\, d\xi d\sigma_x ds,$$

for every test function $\phi \in \mathcal{D}((0,T) \times \bar{\mathcal{O}})$ and $\beta \in \mathcal{A}$.

Proof. Let $(\beta_M)_{M \geq 1}$ be a sequence of odd functions which belong to $\mathcal{A}$ and satisfies

$$\beta_M(s) = \begin{cases} s & \text{if } s \in [0, M] \\ M + 1/2 & \text{if } s \geq M+1, \end{cases}$$

and $|\beta_M(s)| \leq |s|$ for all $s \in \mathbb{R}$. The function $\alpha_M(\sigma) := \beta_M(\beta_{M+1}^{-1}(\sigma))$ belongs to $\mathcal{A}$ and satisfies $\alpha_M(s) \leq s$ for all $s \geq 0$ and $\alpha_M(s) \geq s$ for all $s \leq 0$. We will construct the trace γg from the sequence of traces $\gamma \beta_M(g)$ whose existence is given by Theorem 2. Let us define $\Gamma_M^{(\pm)} = \{(t,x,\xi) \in \Gamma, \pm \gamma \beta_M(g)(t,x,\xi) > 0\}$ and $\Gamma_M^{(0)} = \{(t,x,\xi) \in \Gamma, \gamma \beta_M(g)(t,x,\xi) = 0\}$. Thanks to the definition of α_M and the renormalization property of the trace we have $\gamma\, \beta_M(g) = \gamma\, \alpha_M(\beta_{M+1}(g)) = \alpha_M(\gamma\, \beta_{M+1}(g))$, and we deduce the following equality, up to a negligible set

$$\Gamma_M^{(+)} = \Gamma_1^{(+)}, \quad \Gamma_M^{(-)} = \Gamma_1^{(-)} \quad \text{and} \quad \Gamma_M^{(0)} = \Gamma_1^{(0)} \quad \text{for all } M \geq 1.$$

Therefore the sequence $(\gamma\, \beta_M(g))_{M \geq 1}$ is increasing on $\Gamma_1^{(+)}$ and decreasing on $\Gamma_1^{(-)}$. This implies that $\gamma\, \beta_M(g)$ converge a.e. to a limit denoted by γg and which belongs to $L(\Gamma)$. In order to establish the Green formula (3.5) we fix $\beta \in \mathcal{A}$ and $\phi \in \mathcal{D}((0,T] \times \bar{\mathcal{O}})$, we write the Green formula for the function $\beta(\beta_M(g))$ and using the fact that $\gamma\big[\beta \circ \beta_M(g)\big] = \beta(\gamma \beta_M(g))$ we find

$$\int_0^T \iint_{\mathcal{O}} \big(\beta \circ \beta_M(g)\,(\frac{\partial \phi}{\partial t} + \xi \cdot \nabla_x \phi + E \cdot \nabla_\xi \phi) + (\beta \circ \beta_M)'(g)\, G\, \phi\big)\, d\xi dx dt =$$
$$\int_0^T \iint_\Sigma \beta(\gamma\, \beta_M(g))\, \phi\, n(x) \cdot \xi\, d\xi d\sigma_x ds.$$

We prove (3.5) letting M go to ∞ and noting that $\beta \circ \beta_M(s) \to \beta(s)$ for all $s \in \mathbb{R}$. □

4. Renormalized solution to the initial value problem for the VPB system and stability result.

Let now introduce the definition of a renormalized solution to the initial boundary value problem for the VPB system. Therefore, we shall be able to state the corresponding stability results.

With R.J. DiPerna and P.-L. Lions [15,17], [26,27,28] we say that $f \in C([0,\infty); L^1(\mathcal{O}))$ is a renormalized solution of (1.1)-(1.2)-(1.5)-(1.11) if first, f satisfies the bound (2.11) and $\beta(f)$ solves

$$(4.1) \qquad \frac{\partial}{\partial t}\beta(f) + \xi \cdot \nabla_x \beta(f) + E \cdot \nabla_\xi \beta(f) = \beta'(f)\, Q(f,f) \quad \text{in} \quad \mathcal{D}'((0,T) \times \mathcal{O}),$$

for all time $T > 0$, and all $\beta \in \mathcal{B}$, the class of all functions $\beta \in C^1(\mathbb{R})$ such that $\beta(0) = 0$ and $|\beta'(s)| \leq C/(1+s)$ $\forall s \geq 0$. In equation (4.1) the vector field $E = E_f$ is defined thanks to the Poisson equation (1.2) for the VPB system and $E = 0$ for the Boltzmann equation. Remark that thanks to (2.11), the renormalized collision term $Q(f,f)/(1+f)$ belongs to $L^1((0,T) \times \Omega \times B_R)$ for all

$R \in (0,\infty)$ and $E \in L^\infty(0,T;W^{1,1}(\Omega))$ (see [15] and [28] for a proof of these claims), and thus each terms in equation (4.1) makes sense.

Secondly, f must correspond to the initial datum f_0; this means that (1.11) holds in $L^1(\mathcal{O})$, or equivalently that (4.1) holds in $\mathcal{D}'([0,T)\times\mathcal{O})$:

$$(4.2) \qquad \int_0^T\iint_{\mathcal{O}} \{\beta(f)\,(\frac{\partial\phi}{\partial t}+\xi\cdot\nabla_x\phi+E\cdot\nabla_\xi\phi)+\beta'(f)\,Q(f,f)\,\phi\}\,d\xi dx dt+\iint_{\mathcal{O}}\beta(f_0)\,\phi\,d\xi dx=0,$$

for every test function $\phi\in\mathcal{D}([0,T)\times\mathcal{O})$ and $\beta\in\mathcal{B}$.

Last, the trace $\gamma f\geq 0$, defined by Theorem 5, satisfies (2.14) in Case 1, (2.13) in Case 2, no estimate in Case 3, the boundary condition (1.5) and

$$(4.2) \quad \int_0^T\iint_{\mathcal{O}} \{\beta(f)\,(\frac{\partial\phi}{\partial t}+\xi\cdot\nabla_x\phi+E\cdot\nabla_\xi\phi)+\beta'(f)\,Q(f,f)\,\phi\}\,d\xi dx dt=\int_0^T\iint_{\Sigma}\beta(\gamma\,f)\,\phi\,n(x)\cdot\xi\,d\xi d\sigma_x ds,$$

for every test function $\phi\in\mathcal{D}((0,T)\times\bar{\mathcal{O}})$ and $\beta\in\mathcal{B}$. Equation (4.2) is a little more accurate than (3.7).

We are now concerned by stability results for a sequence f^n of renormalized solutions to the initial boundary value problem for the VPB system which satisfies for all $T\in(0,\infty)$

(4.4)

$$\sup_{n\geq 0}\sup_{[0,T]}\Big\{\iint_{\mathcal{O}} f^n\big(1+|\xi|^2+|\log f^n|\big)\,d\xi dx+\nu\int_\Omega|\nabla_x V_{f^n}|^2\,dx\Big\}+\sup_{n\geq 0}\int_0^T\iint_{\mathcal{O}} e(f^n)\,d\xi dx dt\leq C_T<\infty.$$

In the three following propositions we give the stability result corresponding to the a priori estimates that we have obtained in each case in section 2. These results are of course also true for the Boltzmann equation.

Proposition 4 (Case 1: partial absorption). Let f^n be a sequence of renormalized solutions to the VPB system (1.1)-(1.2) which satisfies the bounds (4.4) such that the trace $\gamma\,f^n$ satisfies the boundary condition (1.5) and

$$(4.5) \qquad \int_0^T\iint_{\Sigma}\gamma f^n\,(1+|\xi|^2+|\log\gamma f^n|)\,|\xi\cdot n(x)|\,d\xi d\sigma_x ds\leq C_T.$$

Assume that $f^n(0,.)$ converges to f_0 in $L^1(\mathcal{O})$ weak. Then, up to the extraction of a subsequence, f^n converges weakly in $L^1([0,T]\times\mathcal{O})$ for all $T\in(0,\infty)$ to a renormalized solution f to the initial value problem for the VPB system (1.1)-(1.2)-(1.11) corresponding to the initial datum f_0 with trace satisfying the boundary condition (1.5).

Proposition 5 (Case 2: total reflection with diffusion). Let f^n be a sequence of renormalized solutions to the VPB system (1.1)-(1.2) which satisfies the bounds (4.4) such that the trace $\gamma\,f^n$ satisfies the boundary condition (1.5), or the relaxed boundary condition (1.13), and

$$(4.6) \qquad \int_0^T\iint_{\Sigma}\gamma f^n\,(1+|\xi|^2)\,|\xi\cdot n(x)|\,d\xi d\sigma_x ds\leq C_T.$$

Assume that $f^n(0,.)$ converges to f_0 in $L^1(\mathcal{O})$ weak. Then, up to the extraction of a subsequence, f^n converges weakly in $L^1([0,T]\times\mathcal{O})$ for all $T\in(0,\infty)$ to a renormalized solution f to the initial value problem for the VPB system (1.1)-(1.2)-(1.11) corresponding to the initial datum f_0 and with trace γf satisfying the relaxed boundary condition (1.13).

Proposition 6 (Case 3: total pure local reflection). Let f^n be a sequence of renormalized solutions to the VPB system (1.1)-(1.2) which satisfies the bounds (4.4) such that the trace $\gamma\,f^n$

satisfies the boundary condition (1.5) with $\alpha = 0$ and $\lambda = 1$. Assume that $f^n(0,.)$ converges to f_0 in $L^1(\mathcal{O})$ weak. Then there is a subsequence $f^{n'}$ which converges weakly in $L^1([0,T]\times\mathcal{O})$ for all $T \in (0,\infty)$ to a renormalized solution f to the initial value problem for the VPB system (1.1)-(1.2)-(1.11) corresponding to the initial datum f_0 and with trace γf satisfying the boundary condition (1.5). If furthermore, $f^{n'}(0,.)$ converges strongly in $L^1(\mathcal{O})$ to f_0 then $f^{n'}$ converges strongly in $C([0,T];L^1(\mathcal{O}))$ to f for all $T \in (0,\infty)$.

Remark 2. In Case 3, for the Boltzmann equation and for the VPnB system, we have shown that the a priori estimate (2.12) holds. Therefore, we can give a slightly different version of Proposition 3: if we assume moreover that the sequence $(\gamma\, f^n)$ satisfies

$$\int_0^T\iint_\Sigma \gamma f^n\,(\xi\cdot n(x))^2\,d\xi d\sigma_x ds \le C_T, \tag{4.7}$$

then the trace γf of f also satisfies (2.12).

We conclude this section by alluding briefly to the proof of Theorem 1, 2 and 3. This uses rather standard (and tedious) approximation arguments that are exposed in [23], [2] and [40]. The idea of the proof is to regularise the VPB system. Take a sequence of smooth approximations f_0^n and ϕ_-^n of f_0 and ϕ_-. Consider a sequence of operators Q^n which "approximate" Q, map $L^1\cap L^p$ into itself for all $p \in (1,\infty]$, satisfy the remarkable properties (2.9), (2.10). Regularise E, by convolution for instance. Then prove by Banach fixed point Theorem the existence of a solution f^n to the modified VPB system, for which all the manipulation in section 2 are correct and then which satisfies the bound (4.4). Last, use the Proposition 2, 3 and 4 to conclude.

5. Proof of the stability result.

First we remark that the bound (4.4) and the Dunford-Pettis lemma imply that f^n is weakly compact in $L^p(0,T;L^1(\mathcal{O}))$ for all $p\in[1,\infty)$ and $T\in(0,\infty)$ and then there is a subsequence, not relabeled, such that

$$f^n \underset{n\to\infty}{\rightharpoonup} f \qquad \text{weakly in } L^p(0,T;L^1(\mathcal{O})). \tag{5.1}$$

One can show essentially by a convexity argument, see [17], that f still satisfies the bound (2.11).

We aim to prove that f is a renormalized solution of the VPB system. We have thus to prove that f is a solution of the renormalized equation (4.1) and that its trace γf, which is uniquely defined thanks to the trace Theorem 3 and the Green formula (4.3), satisfies the boundary conditions, possibility relaxed.

We prove the propositions in two steps and the strategy of the proof is based on the one that was introduced in [28]. In step 1, we consider $\beta_\delta(f^n)$ for $\delta\in(0,1]$ and weakly pass to the limit as n goes to ∞ in the equation satisfied by $\beta_\delta(f)$. Then, we renormalize the resulting limit equation and let δ go to 0 to recover (4.1). The same strategy is performed at the boundary in the second step. We consider the sequence $\beta_\delta(\gamma f_n)$, first pass to the limit $n\to\infty$, renormalize the obtained limit and then let $\delta\to 0$.

Step 1. In this step we recall the main idea used in [28] to prove that f solves (4.1).

Extracting a subsequence if necessary, we may assume that for all $\delta > 0$

$$\beta_\delta(f^n) \underset{n\to\infty}{\rightharpoonup} \bar\beta_\delta \qquad \text{weakly} \star \text{ in } L^\infty((0,T)\times\mathcal{O}). \tag{5.2}$$

Furthermore, one can show that estimate (4.4) implies that $Q(f^n,f^n)/(1+f^n)$ is weakly compact in $L^1((0,T)\times\Omega\times B_R)$ for all $R\in(0,\infty)$, see [15], and thus we can also assume

$$\frac{Q(f^n,f^n)}{(1+\delta\, f^n)^2} \underset{n\to\infty}{\rightharpoonup} \bar Q_\delta \qquad \text{weakly in } L^1((0,T)\times\Omega\times B_R). \tag{5.3}$$

Last, P.L. Lions has proved in [28] that (4.4) and the averaging lemma of [19], [18] imply that $\rho^n = \rho_{f^n}$ satisfies

$$\sup_{n\geq 0}\sup_{[0,T]}\int_\Omega \rho^n\,(1+|\log\rho^n|)\,dx \leq C_T \quad \text{ and } \quad \rho^n \underset{n\to\infty}{\longrightarrow} \rho_f \text{ in } L^1((0,T)\times\Omega)).$$

Thus, using (4.4) and a standard property of Poisson equation we obtain

$$E_{f^n} \underset{n\to\infty}{\longrightarrow} E = E_f \quad \text{ in } \quad L^p(0,T;L^a(\Omega)) \tag{5.4}$$

for all $T\in(0,\infty)$, $p\in[1,\infty)$ and $a\in[1,2)$. We pass to the limit in the renormalized equation (4.1) satisfied by f^n with $\beta=\beta_\delta$ and using (5.2), (5.3) and (5.4) we get

$$\frac{\partial\bar\beta_\delta}{\partial t} + \xi\cdot\nabla_x\,\bar\beta_\delta + E\cdot\nabla_\xi\,\bar\beta_\delta = \bar Q_\delta \ \text{ in } (0,T)\times\mathcal{O}. \tag{5.5}$$

Let consider $\beta\in\mathcal{B}$. Renormalizing equation (5.5) by β we find

$$\frac{\partial}{\partial t}\,\beta(\bar\beta_\delta) + \xi\cdot\nabla_x\,\beta(\bar\beta_\delta) + E\cdot\nabla_\xi\,\beta(\bar\beta_\delta) = \beta'(\bar\beta_\delta)\,\bar Q_\delta \ \text{ in } (0,T)\times\mathcal{O}. \tag{5.6}$$

Since we have $0\leq s-\beta_\delta(s) = \delta\,s^2/(1+\delta\,s)\leq \delta\,s\,M + s\,\mathbf{1}_{\{s>M\}}$ for all $M\in(0,\infty)$ we deduce thanks to the bound (4.4) that for all $M\in(0,\infty)$ and $\delta>0$

$$0\leq f-\bar\beta_\delta \leq \delta\,M\,f + g_M \tag{5.7}$$

where g_M is the weak limit of $f^n\,\mathbf{1}_{\{f^n>M\}}$ and thus g_M tends towards 0 in $L^1((0,T)\times\mathcal{O})$ when M goes to ∞. One deduces

$$\bar\beta_\delta \underset{\delta\to 0}{\longrightarrow} f \qquad \text{strongly in } L^1((0,T)\times\mathcal{O}). \tag{5.8}$$

Furthermore, using the average lemma [19] and [15] (see also B. Perthame, P.E. Souganidis [34] for a more general and recent result) one can prove the following lemma

Lemma (P.-L. Lions [28]). *Under the previous assumption one has*

$$\frac{\bar Q_\delta}{1+\bar\beta_\delta} \underset{\delta\to 0}{\longrightarrow} \frac{Q(f,f)}{1+f} \qquad \textit{strongly in } L^1((0,T)\times\Omega\times B_R), \quad \forall R\in(0,\infty). \tag{5.9}$$

It is obvious from (5.8) and (5.9) that passing to the limit in (5.6) we find that f solves (4.1). We also deduce $f\in C([0,\infty);L^1(\mathcal{O}))$.

In order to shorten notation we set $d\mu_1 = |n(x)\cdot\xi|\,d\xi d\sigma_x dt$.

Step 2 of Proposition 4. Without loss of generality, extracting a subsequence if necessary, we can assume

$$\gamma_\pm f_n \underset{n\to\infty}{\rightharpoonup} f_\pm \qquad \text{weakly in } L^1((0,T)\times\Sigma, d\mu_1) \tag{5.10}$$

with $f_\pm$ satisfying the bound (2.14) and

$$\beta_\delta(\gamma_\pm f_n) \underset{n\to\infty}{\rightharpoonup} \bar\beta_{\delta,\pm} \qquad \text{in } L^1((0,T)\times\Sigma_\pm, d\mu_1) \text{ weak and } L^\infty((0,T)\times\Sigma_\pm) \text{ weak } \star. \tag{5.11}$$

Of course, one can pass to the limit in the boundary conditions satisfied by $\gamma_\pm f_n$. One finds

$$f_- = (1-\alpha)\left(\lambda\,L\,f_+ + (1-\lambda)\,D\,f_+\right) + \alpha\,\phi_- \quad \text{ on } (0,\infty)\times\Sigma_-. \tag{5.12}$$

We have only to prove that $\gamma_\pm f = f_\pm$ to conclude. But, on one hand, by the same equiintegrability argument used in the proof of (5.7) we can prove that $\bar\beta_{\delta,\pm} \to f_\pm$ strongly in $L^1((0,T)\times\mathcal{O}, d\mu_1)$ when $\delta \to 0$ and therefore we also have

$$\beta(\bar\beta_{\delta,\pm}) \underset{\delta\to 0}{\longrightarrow} \beta(f_\pm) \qquad \text{strongly in } L^1((0,T)\times\Sigma_\pm, d\mu_1). \tag{5.12}$$

On the other hand, $\bar\beta_{\delta,\pm} = \gamma_\pm \bar\beta_\delta$ as one can see easily passing to the limit $n\to\infty$ in the Green formula (4.3) written for $\beta_\delta(f_n)$. But, by Theorem 2, $\beta_1(\bar\beta_{\delta,\pm}) = \gamma_\pm\beta_1(\bar\beta_\delta)$ which converges to $\gamma_\pm\beta_1(f)$ passing to the limit in the Green formula (4.3) and using (5.7). Combining with (5.12) we have thus proved that $\beta_1(\gamma_\pm f) := \gamma_\pm\beta_1(f) = \beta_1(f_\pm)$ a.e. and, since β_1 is strictly increasing, that $\gamma_\pm f = f_\pm$.

Step 2 of Proposition 5. We now want to prove that γf satisfies the relaxed boundary condition (1.13). The main difficulty is that (5.10) does not hold anymore. As noticed by T. Goudon in [20], one can use the bit lemma [7] and prove that γf_n converges in the sense of Chacon to a limit $f_\pm$ which is the trace of f and satisfies the relaxed boundary condition. We give here a slightly different and simpler proof which does not use the bitting lemma. This one is in fact related to the Chacon's convergence, see [4].

As in the previous case, considering the sequence $\beta_\delta(\gamma f_n)$, we have

$$\beta_\delta(\gamma_\pm f_n) \underset{n\to\infty}{\rightharpoonup} \bar\beta_{\delta,\pm} = \gamma_\pm\bar\beta_\delta \qquad \text{weakly } \star \text{ in } L^\infty((0,T)\times\Sigma_\pm) \text{ and weakly in } L^1((0,T)\times\Sigma_\pm, d\mu_1). \tag{5.13}$$

Furthermore, the sequences $\bar\beta_{\delta,\pm}$ are increasing when $\delta\to 0$ and are uniformly bounded in $L^1((0,T)\times\Sigma_\pm, (1+|\xi|^2)\, d\mu_1)$ thanks to (4.6). By Fatou lemma there exists $f_\pm$ such that

$$\bar\beta_{\delta,\pm} \underset{\delta\to 0}{\longrightarrow} f_\pm \qquad \text{strongly in } L^1((0,T)\times\Sigma_\pm, d\mu_1). \tag{5.14}$$

We show as in Step 2 that $\gamma_\pm\bar\beta_\delta \to \gamma_\pm f$ a.e. and that $\gamma_\pm f = f_\pm$.

We now have to pass to the limit in the relaxed boundary condition (1.13). Since β_δ is concave and $\beta_\varepsilon(s)\le s$ we have

$$\beta_\delta(\gamma_- f_n) \ge \lambda\,\beta_\delta(L\,\gamma_+ f_n) + (1-\lambda)\,\beta_\delta(D\,\beta_\varepsilon(\gamma_+ f_n)) \quad \text{on } (0,T)\times\Sigma_-, \tag{5.15}$$

for all $\varepsilon,\delta\in(0,1]$. Up to extracting a subsequence we can assume

$$\beta_\delta(D\,\beta_\varepsilon(\gamma_+ f_n)) \underset{n\to\infty}{\rightharpoonup} \bar D_{\delta,\varepsilon} \qquad \text{weakly } \star \text{ in } L^\infty((0,T)\times\Sigma_-).$$

Furthermore, $\beta_\delta(L\,\gamma_+ f_n) = L\,\beta_\delta(\gamma_+ f_n)$ and then letting n go to ∞ in (5.15) we get

$$\bar\beta_{\delta,-} \ge \lambda\, L\,\bar\beta_{\delta,+} + (1-\lambda)\,\bar D_{\delta,\varepsilon} \quad \text{on } (0,T)\times\Sigma_-. \tag{5.16}$$

The L^1 continuity of D implies that for every fixed $\varepsilon>0$ we have

$$D\beta_\varepsilon(\gamma_+ f_n) \underset{n\to\infty}{\rightharpoonup} D\bar\beta_{\varepsilon,+} \qquad \text{weakly in } L^1((0,T)\times\Sigma_-, d\mu_1) \tag{5.17}$$

and therefore using the same argument as the one used to prove (5.8) we have that $D_{\delta,\varepsilon}$ is increasing when $\delta\to 0$ and

$$D_{\delta,\varepsilon} \underset{\delta\to 0}{\longrightarrow} D\bar\beta_{\varepsilon,+} \qquad \text{strongly in } L^1((0,T)\times\Sigma_-, d\mu_1). \tag{5.18}$$

Passing first to the limit $\delta\to 0$ in (5.16) we obtain

$$f_- \ge \lambda\, L\, f_+ + (1-\lambda)\, D\bar\beta_\varepsilon \quad \text{on } (0,T)\times\Sigma_-, \tag{5.19}$$

and then passing to the limit $\varepsilon \to 0$ in (5.19) we get

$$f_- \ge \lambda\, L\, f_+ + (1-\lambda)\, D\, f_+ \quad \text{on } (0,T)\times\Sigma_-, \tag{5.20}$$

which is precisely (1.13).

Step 2 of Proposition 6. Here the proof is simplified by the fact that renormalization and pure local reflection commute. Indeed, the trace $\gamma\,\beta_\delta(f^n) = \beta_\delta(\gamma\, f^n)$ satisfies the boundary condition (1.5) and passing to the limit $n \to \infty$ we find that $\gamma\bar\beta_\delta$ satisfies (1.5). Then renormalizing by β and passing to the limit $\delta \to 0$ we see that $\beta(\gamma\, f) = \gamma\,\beta(f)$ also satisfies (1.5) for all $\beta \in \mathcal{B}$. Therefore, the trace of f satisfies the boundary condition. In order to prove strong convergence we follow Lions' proof [27] and [28]. By a convexity argument one proves that $\beta(f^n) = \log(1+f^n)$ converges weakly in $L^1((0,T)\times\mathcal{O})$ to $\bar\beta \le \beta(f)$ and that $\bar\beta$ satisfies the boundary condition (1.5) and also satisfies the inequation

$$\frac{\partial}{\partial t}\,\bar\beta + \xi\cdot\nabla_x\,\bar\beta + E\cdot\nabla_\xi\,\bar\beta \ge \frac{Q(f,f)}{1+f} \quad \text{in } \mathcal{D}'((0,T)\times\mathcal{O}).$$

But since $\beta(f) = \log(1+f)$ is a solution to the renormalized VPB system (4.1) we have

$$\frac{\partial}{\partial t}\,(\bar\beta-\beta(f)) + \xi\cdot\nabla_x\,(\bar\beta-\beta(f)) + E\cdot\nabla_\xi\,(\bar\beta-\beta(f)) \ge 0 \quad \text{in } \mathcal{D}'((0,T)\times\mathcal{O}), \tag{5.21}$$

with $\gamma\,(\bar\beta-\beta(f))$ satisfying the boundary condition (1.5). Then, just as in [27], we integrate (5.21) and find $\displaystyle\iint_{\mathcal{O}}\big\{\bar\beta_t - \beta(f_t)\big\}\,d\xi dx \ge 0$ for all $t \in [0,T]$. Therefore, we have proved that $\log(1+f^n)$ weakly converges to $\log(1+f)$ and by standard convexity argument we find that $f^n \to f$ in $L^p(0,T;L^1(\mathcal{O}))$ for all $T \in (0,\infty)$ and $p \in [1,\infty)$. We refer to [28] for the uniform convergence in the t variable. $\square$

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