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On a Quantum Boltzmann equation for a gas of photons

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On a Quantum Boltzmann equation for a gas of photons

Abstract - We prove existence and uniqueness of the solution of a homogeneous quantum Boltzman equation describing the photon-electron interaction. We study the asymptotic behaviour of the solutions, and show in particular, that the photon density distribution condensates at the origin asymptotically in time when the total number of photons is larger than a given positive constant. We also recover the Kompaneets equation as a Fokker-Planck type limit of this Boltzman model.

Équation de Boltzmann Quantique pour un gaz de photons

Résumé - Nous démontrons l'existence et l'unicité de la solution d'une équation de Boltzmann quantique homogène décrivant l'interaction photons-électrons. Nous étudions le comportement asymptotique des solutions, et nous montrons, en particulier, que la densité de photons se condense à l'origine en temps infini lorsque le nombre de photons est suffisamment grand. Nous retrouvons aussi l'équation de Kompaneets comme une limite de type Fokker-Planck à partir de ce modèle d'équation de Boltzmann.

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On a Quantum Boltzmann equation for a gas of photons

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Abstract - We prove existence and uniqueness of the solution of a homogeneous quantum Boltzman equation describing the photon-electron interaction. We study the asymptotic behaviour of the solutions, and show in particular, that the photon density distribution condensates at the origin asymptotically in time when the total number of photons is larger than a given positive constant. We also recover the Kompaneets equation as a Fokker-Planck type limit of this Boltzman model.

Key words - Boltzmann equation, Quantum particles, Bose-Einstein condensation phenomena, long time behavior, Kompaneets equation

1. Introduction.

We are concerned in this paper with the Boltzmann-Compton equation

$$k^2 \frac{\partial f}{\partial t} = \int_0^\infty \big(f'\,(1+f)\,B(k',k;\theta) - f\,(1+f')\,B(k,k';\theta)\big)\,dk'. \tag{1.1}$$

Following A.S. Kompaneets and others (see [24], [16], [8]), this equation describes the dynamics of a low energy, homogeneous, isotropic photon gas that interacts via Compton scattering with a low energy electron gas, at low temperature $\theta > 0$ and with a Maxwellian distribution of velocities $e^{-k/\theta}$. The scalar quantity $f(t,k) \geq 0$ represents the density of photons which at time $t \geq 0$ have energy $k \geq 0$. In equation (1.1) we have adopted the usual notations $f = f(t,k)$ and $f' = f(t,k')$. The cross section $B(k,k';\theta)/k^2$ is the probability for a given particle at energy state k to be scattered to the energy state k'. This one must satisfy the detailed balance law

$$e^{k/\theta}\,B(k',k;\theta) = e^{k'/\theta}\,B(k,k';\theta). \tag{1.2}$$

In all the sequel we take $\theta = 1$ without any loss of generality.

For a given state $f \geq 0$ we introduce the two following "macroscopic" quantities: the total number of photons $N(f)$ and the entropy $S(f)$ defined by

$$N(f) = \int_0^\infty f(k)\,k^2\,dk \quad \text{and} \quad S(f) = \int_0^\infty s(f(t,k),k)\,k^2\,dk, \tag{1.3}$$

where $s(x,k) = (1+x)\,\ln(1+x) - x\,\ln x - k\,x$ is the entropy density. The fundamental physical properties of a solution f to (1.1) is that, formally at least,

$$\frac{d}{dt}N(f(t,.)) = 0 \quad \text{and} \quad \frac{d}{dt}S(f(t,.)) \geq 0 \qquad \forall t \geq 0, \tag{1.4}$$

so that the total number of photons is preserved and the entropy is increasing along the trajectory of (1.1). A large part of the physic described by this model is contained in these only two properties in (1.4).

The Boltzmann-Compton equation (1.1) is a spatially homogeneous Boltzmann equation and its study is therefore simplified by the absence of a transport term. But on the other hand, as a quantum kinetic equation, it has received much less attention in the mathematical literature than the classical (which means non quantic) equations. The classical Boltzmann equation in an spatially homogeneous framework has been extensively studied since the precursor work by L. Arkeryd [3]. For recent development in this direction and further references, we refer to [38] for existence results, to [32] for uniqueness results and to [37] for the asymptotic trend to the equilibrium. Let us just emphasize that the Quantum Boltzmann equation for a gas of Fermi particles has been addressed by J. Dolbeault [15] and P.-L. Lions [26], and also linear version arising in semi-conductor theory have been studied, see [31] for more references. But, concerning the Quantum Boltzmann equation for Bose gases (remember that photons are a particular type of Bose particles) we only know the very recent work of X. Lu [30]. As we show in this work, classic and quantum Boltzmann equations may exhibit solutions with quite different behaviors. This can already be seen in the expression of the collision kernel appearing in (1.1), $f'\,(1+f)\,B(k',k;\theta) - f\,(1+f')\,B(k,k';\theta)$, while in the classical equations the kernel takes the form $f'\,f\,B(k',k;\theta) - f\,f'\,B(k,k';\theta)$. The reason for that difference comes from the following. The particles whose density is to be described by the function f, i.e. the photons, are quantum particles. They obey Bose statistics and thus tend to be all at the same energy level. Therefore, if there is already a particle at energy level k, this enhances the probability for another particle, at an energy level k', to jump to the same energy level k. This accounts for the terms in $1+f$ and $1+f'$. One interesting mathematical consequence, which has also been observed by X. Lu in [30], is that an uniform bound of the entropy $S(f_j)$, for a family of suitable functions f_j, does not provide weak convergence of that family in L^1 as it does for the classical homogeneous Boltzmann equation. The fact that the entropy is not superlinear makes more difficult the statement of existence theory, but it is strongly related to the condensation phenomena that we will introduce below.

The purposes of this paper is, first, to study the existence of solutions for the Cauchy problem associated to (1.1). We show that under "reasonable" conditions on the cross section B and for a large class of initial datum f_{in} there exists a global (in time) solution to (1.1) associated to f_{in}, which furthermore is unique. Moreover, if f_{in} is a measurable function (not a singular measure) then $f(t,.)$ is also a measurable function. Next, we can consider the long time behavior, as $t \to +\infty$, of these solutions. Thanks to (1.4) it is expected that $f(t,.)$ converges, as $t \to +\infty$, to an equilibrium state which is uniquely associated to the number of photons $N = N(f(t,.))$ $\forall t \geq 0$. Heuristically, the equilibrium state must be the maxima of the entropy $S(f)$ for all the densities f with prescribed total number of photons $N(f) = N$.

This is the first main question we are interested in, and we would like now to concentrate us on this maximum entropy problem which is simple and very enlightening both in a physic point of view and for the mathematical analysis of equation (1.1). Moreover the maximisation entropy problem is physically relevant, since the statistical physics says that the solution of this problem is the most probably state of the gas: it is the thermodynamical equilibrium. Let us first briefly see why the entropy $S(f)$ is well defined. To this end we remark that

$$\frac{\partial s}{\partial x} = \ln(1+x) - \ln x - k, \qquad \frac{\partial^2 s}{\partial x^2} = \frac{1}{1+x} - \frac{1}{x} < 0, \quad \forall x > 0.$$

Then, for every fixed $k > 0$, $s(k,\cdot)$ is a concave function of x with an unique maximum. That maximum obviously depends on k and is usually denoted by $f_0(k)$. It is given by $\frac{\partial s}{\partial x}(k, f_0(k)) = 0$, or equivalently,

$$f_0(k) = \frac{1}{e^k - 1}, \tag{1.5}$$

and is called the Planck distribution. Therefore, for every measurable and non negative function f, we have $s(f(k),k) \leq s(f_0(k),k) = \ln \frac{e^k}{e^k - 1}$ and

$$S(f) \leq S(f_0) \equiv \int_0^\infty k^2 \ln \frac{e^k}{e^k - 1} dk < \infty. \tag{1.6}$$

This shows that $S(f)$ is well defined and $S(f) \in [-\infty, S(f_0)]$. Let us emphasize that (1.6) implies that f_0 is the global maximum of the entropy S; note moreover that $N(f_0) < \infty$.

In order to get a better insight into the maximum entropy problem, we introduce the Bose-Einstein distributions defined by

$$\forall \mu > 0, \qquad f_\mu(k) = \frac{1}{e^{k+\mu} - 1}. \tag{1.7}$$

Observe that the Planck distribution corresponds to $\mu = 0$. We note $N_\mu = N(f_\mu)$. One easily checks that the functions f_μ are ordered ($f_\nu < f_\mu$ if $\nu > \mu$), that for any $N \in (0, N_0]$ there exists an unique $\mu \geq 0$ such that $N_\mu = N$, and that the corresponding distribution f_μ solves the maximisation problem

$$S(f_\mu) = \max_{N(f)=N} S(f). \tag{1.8}$$

One can also remark that f_μ is a stationary solution of the equation (1.1), i.e. $Q(f_\mu, f_\mu) = 0$. More precisely, whenever B satisfies the detailed balance condition (1.2) we have

$$f'_\mu\,(1 + f_\mu)\,B(k', k; 1) - f_\mu\,(1 + f'_\mu)\,B(k, k'; 1) = 0, \qquad \forall\, k > 0,\ k' > 0.$$

Now since the the maximisation problem (1.8) has been solved for $N \in (0, N_0]$, one can then wonder whether it has a solution or not when $N > N_0$. That question was solved by R.E. Caflisch and C.D. Levermore in [6] with the following remark. If φ_n is a regular approximation of δ_a, the Dirac mass at the point $k = a$ with $a \geq 0$, then

$$S(f + \alpha\,\frac{\varphi_n}{k^2}) \underset{n\to\infty}{\longrightarrow} S(f) - \alpha\, a \qquad \text{and} \quad N(f + \alpha\,\frac{\varphi_n}{k^2}) \underset{n\to\infty}{\longrightarrow} N(f) + \alpha. \tag{1.9}$$

In order to be more precise we perform the change of variables: $g = k^2\, f$. Consider now a distribution F of the form $F = g + \alpha\,\delta_a$ where $g \in L^1(\mathbb{R}_+)$, $\alpha \in \mathbb{R}$ and $g,\ \alpha \geq 0$. When $\alpha > 0$, the singular part $\alpha\,\delta_a$ has to be interpreted as a Bose condensate: a macroscopic part of the gas of photons is concentrated in the single energy level $k = a$. We define the "total mass" $M(F)$ of such a distribution F as

$$M(F) := \int_0^\infty dF(k) = M(g) + \alpha = \int_0^\infty g(k)\, dk + \alpha, \tag{1.10}$$

and its entropy

$$H(F) := H(g) - \alpha\, a, \qquad \text{with} \qquad H(g) = \int_0^\infty h(g, k)\, dk, \tag{1.11}$$

where $h(x, k) = (k^2 + x)\, \ln(k^2 + x) - x\, \ln x - k^2\, \ln k^2 - k\, x$. By construction, if g is a measurable and non negative function and $f(k) = k^{-2} g(k)$ we have

$$M(g) = N(f) \quad \text{and} \quad H(g) = S(f).$$

Therefore $M(g)$ and $H(g)$ are well defined for every nonnegative measurable function g and $M(g) \in [0, +\infty]$, $H(g) \in [-\infty, S(f_0)]$. Finally, we define the Bose distributions

$$\mathcal{B}_m = g_\mu + \alpha\, \delta_0 \tag{1.12}$$

with $\alpha = 0$ and $\mu \geq 0$ such that $M(g_\mu) = m$ if $m \leq N_0$; $\mu = 0$ and $\alpha = m - N_0$ if $m > N_0$. Under these notations, the result by R.E. Caflisch and C.D. Levermore may be stated as follows.

Theorem 1 ([6]). *For every $m > 0$, $H(\mathcal{B}_m) = \max_{M(F)=m} H(F)$.*

It is fundamental to emphasize that when $m > N_0$, the "thermodynamical equilibrium" condensates at the origin since in this case $\mathcal{B}_m = g_0 + (m - N_0)\, \delta_0$. From a physical point of view, this is known has a Bose condensation type phenomena. Of course, we will also show that the Bose-Einstein distribution $\mathcal{B}_m$ are the only stationary solution of (1.1).

Coming back to the evolution equation (1.1), we will be able to prove that $k^2 f(t,k)$ converges, when $t \to +\infty$ to the Bose-Einstein state $\mathcal{B}_m$, with $m = N(f_{in})$. As a first conclusion, this leads to our most physical value result: starting from an initial regular state f_{in} such that $N(f_{in}) > N_0$ no Bose condensation appears in finite time as we have already mentioned it (the photon density distribution f does not concentrate in a Dirac mass), but Bose condensation appears at the origin in infinite time. From the point of view of the physical model, this indicates that part of the photons tend to concentrate at the zero energy state and create a condensate as $t \to \infty$. This condensation phenomena for a photons gas bears some similarity with the classical Bose condensation phenomena for gas constitued of true Bose-Einstein particles (we mean no photons). Nevertheless, it must be emphasized that this is an infinite time process while in the Bose condensation, the condensate has been predicated to appear in finite time [25], [33], [35]. In his work [30], X. Lu establish the existence of global L^1-solution to the Boltzmann equation for Bose-Einstein particles under strong troncature assumption (and somewhat not physical) on the cross-section: no more condensation appears in finite time. X. Lu also studies the asymptotic behavior of the solutions in some specific cases. We will come back on the questions in the forthcoming work [19], where we present some general and basic properties of quantum and relativistic Boltzmann equation.

The question of convergence to the equilibrium state or more generally asymptotic behavior of solution when $t \to \infty$ is one of the main question in kinetic theory. It has been treated by many authors for the homogeneous Boltzmann equation [4], [37] and for the unhomogeneous Boltzmann equation [26], [11], [28], as for other models [34], [5], [13], [14].

We introduce now the second main goal of this paper: to justify rigorously the approximation of the Boltzmann equation (1.1) by the Kompaneets equation

$$x^2 \frac{\partial f}{\partial t} = \frac{\partial}{\partial x}\Big(x^4\,(\frac{\partial f}{\partial x} + f + f^2)\Big), \quad \text{for} \quad t > 0, x > 0. \tag{1.13}$$

This equation is the well known Fokker-Planck approximation of the Boltzmann-Compton equation (1.1) introduced by A. S. Kompaneets in [24], under the hypothesis that the energy transferred in each separate act is small in comparison with the energy quantum: $k' - k << k$.

It is a classical device to approximate classic Boltzmann equation with Coulomb interactions by Landau equation. This corresponds physically to the fact that small angle collisions are much more important than collisions resulting in large momentum changes (Chapman Cowling [7], second edition, pages 178-179). This leads to the formal method often used for treating such systems, in which one expands the collision integrand of the Boltzmann equation in powers of the momentum change per collision. With regard to the classical Boltzmann equation, the Fokker-Planck limit, which corresponds to the asymptotic behavior when the collisions become grazing, has been extensively studied in [10], [12], [21], [38] and we refer to [39] for a general presentation of the problem and for more references.

Now, Compton scattering is not a long but a short range interaction. Nevertheless the formal expansion argument in powers of the momentum change may still be performed but for a different reason. It actually corresponds to consider that the main contribution in the collision integral of the equation (1.1) comes from the region where $k' - k$ is small ($|k' - k| << k$). This does not come from the type of interaction, which has been said to be short range, but from the fact that $b(k, k')$ is very peaked arround $k \sim k'$ and the presence of the exponentially decaying terms. Remark that this is due to the fact that the electrons are decoupled and supposed to be at equilibrium. Moreover, thanks to this last assumption, the formal expansion method gives a partial differential equation which is the Kompaneets equation. In general, without the condition of decoupling, the method gives an integro-(partial) differential equation, see [19] for a formal derivation of this model.

In order to derive rigorously (1.13) from (1.1) we will consider a family (B_ε) of cross-section which tends to concentrate the interaction between particles on the pairs of particles of energy k and k' with $k' \simeq k$ (see Theorem 7 for precise assumptions on B_ε). Then, for an initial datum f_{in} such that $0 \le f_{in} \le f_0$, we prove that the family of solutions f_ε of (1.1) (associated to f_{in} and B_ε) converges to a solution f of the Kompaneets equation (1.13).

The associated mixed problem in $\mathbb{R}_+^\star \times \mathbb{R}_+^\star$

$$
\begin{cases}
x^2 \dfrac{\partial f}{\partial t} = \dfrac{\partial}{\partial x}\Big(x^4 \big(\dfrac{\partial f}{\partial x} + f + f^2\big)\Big), \quad \text{for} \quad x > 0, t > 0,\\
f(0,x) = f_{in}(x),\\
x^4 \big(\dfrac{\partial f}{\partial x} + f + f^2\big) \to 0 \qquad \text{as } x \to 0 \text{ and } x \to \infty,
\end{cases}
\tag{1.14}
$$

has received much more attention in the physic and mathematic literature than the Cauchy problem associated to (1.1) and has been widely studied; in particular, by G. Cooper [8], R.E. Caflisch & C. D. Levermore [6], O. Kavian [23] and M. Escobedo, M. A. Herrero & J. J. L. Velazquez [17]. There is a lot of similarity between Boltzmann-Compton equation (1.1) and Kompaneets equation (1.14). The flux condition at $x = 0$ and as $x \to \infty$ is natural from the following point of view. As we have already remarked, the total mass of the solutions is preserved in the Boltzmann-Compton model. As a formal integration by parts shows, the property of mass preserving in the Kompaneets model (1.13) requires the flux condition to be satisfied at both $x = 0$ and $x \to \infty$. Moreover, the function S defined in (1.3) is also an increasing entropy for the solutions of (1.14) and the Planck and Bose-Einstein distributions are still stationary solutions of equation (1.13).

But, on the other hand, it was proved in [17] that problem (1.14) is unstable in the following sense. There are initial data f_{in} (rather general) such that for a finite time $T^* > 0$, there exists an unique function f, defined in $\mathbb{R}_+^\star \times \mathbb{R}_+^\star$, which is the unique classical solution of (1.14) for $t \in [0, T^*)$, which satisfies the Kompaneets equation and the flux condition at $x \to \infty$ for all $t > 0$, and such that

$$
\lim_{x\to 0} x^4 \big(\frac{\partial f}{\partial x}(x,t) + f(x,t) + f^2(x,t)\big) > 0, \qquad \forall t \geq T^*,
$$

i.e. it does not satisfy the flux condition at the origin for $t \geq T^*$. As we shall see, this implies that, for some initial datum, the approximation of the Boltzmann equation by equation (1.14) breaks down in finite time.

2. Main results.

In this section, we present in details the new results that we have obtained and which were announced in [18]. For that purpose we begin specifying the cross section we deal with. We introduce

$$
b(k,k') = B(k',k;1)\, e^k\, k^{-2}\, k'^{-2}. \tag{2.1}
$$

Note that assumption (1.2) on B implies that b is a symmetric function. We always assume that b satisfies

$$
\exists \eta \in [0,1); \quad b(k,k') = e^{\eta\, k}\, e^{\eta\, k'}\, \sigma(k,k'), \tag{2.2}
$$

for some function $0 \leq \sigma \in L^\infty(\mathbb{R}_+^2)$ symmetric. We will also need a more restrictive assumption on b, precisely that for some $\sigma_\star, \sigma^\star, \nu > 0$, $\gamma \in [0,1)$

$$
\sigma(k,k') \equiv \sigma(k'-k) \quad \text{and} \quad 0 < \sigma_\star\, e^{-\nu\, |z|^\gamma} \leq \sigma(z) \leq \sigma^\star \qquad \forall z \in \mathbb{R}. \tag{2.3}
$$

It is difficult to find in the literature the reasonable physic assumption that one has to make on b. The question of physical relevance of the cross-section assumption will be addressed in a next work [19] where we will see that the Compton scattering cross-section has a structure not so far to (2.2) or (2.3). We will also see (in Theorem 7) that the Kompaneets equation (1.13) is a Fokker-Planck limit of the Boltzmann-Compton equation (1.1) for a cross-section B satisfying (2.2) with $\eta = 1/2$.

From a mathematical point of view these assumptions are uniquely used in the proof of existence (and uniqueness) of solutions to the Boltzmann-Compton equation (1.1) with unbounded cross-section b and for a general class of initial data f_{in} (including the case $N(f_{in}) > N_0$).

Let us now introduce the space of distributions where we look for solutions to (1.1). Theorem 1 shows that the natural space for the solutions of equation (1.1) is the set of bounded and not negative measures $M^1(\mathbb{R}_+) = \big(C_b(\mathbb{R}_+)\big)'$. In the sequel, for a given $0 \le F \in M^1$ we note

$$\begin{cases} F = g + G, \ \text{with} \\ g \in L^1(\mathbb{R}_+), G \text{ a singular measure with respect to the Lebesgue measure in } \mathbb{R}_+. \end{cases} \tag{2.4}$$

With these notations and the change of variables $F = g = k^2 f$ (so that $G = 0$) the equation (1.1) writes

$$\frac{\partial F}{\partial t} = Q(F,F) = \int_{\mathbb{R}_+} b(k,k')\big(F'\,(k^2+F)\,e^{-k} - F\,(k'^2+F')\,e^{-k'}\big)\,dk'. \tag{2.5}$$

But in fact, $Q(F,F)$ is also well defined for all nonnegative measures F of $M^1(\mathbb{R}_+)$ (at least when b is bounded); therefore, equation (2.5) makes sense for such general states. Equation (2.5) can also be written as the following system of equations for the regular part g and the singular part G

$$\begin{cases} \dfrac{\partial g}{\partial t} = Q_1(g,G) = Q_1^+(g,G) - Q_1^-(g,G) = (k^2+g)\,e^{-k}\,L(F) - g\,L\big((k^2+F)\,e^{-k}\big), \\ \dfrac{\partial G}{\partial t} = Q_2(g,G) = Q_2^+(g,G) - Q_2^-(g,G) = G\,\big[L(F)\,e^{-k} - L\big((k^2+F)\,e^{-k}\big)\big], \end{cases} \tag{2.6}$$

with $L(\phi) := \displaystyle\int_{\mathbb{R}_+} b(k,k')\,\phi'\,dk'$.

On the other hand, since we are interested in the Cauchy problem, we add an initial datum

$$F(0,.) = g(0,.) + G(0,.) = F_{in} = g_{in} + G_{in}. \tag{2.7}$$

Due to the particular form of equation (2.5), when the cross section b is a bounded function, a natural space to look for solutions is

$$\mathcal{E}_0 = \{F \in M^1(\mathbb{R}_+),\ F \ge 0,\ M((1+k)\,F) < \infty\}.$$

Since we want to consider more general cross sections b of the form (2.2), we also introduce the spaces

$$\mathcal{E}_\eta = \{F \in M^1(\mathbb{R}_+),\ F \ge 0,\ Y_\eta(F) := M(e^{\eta\,k}\,F) < \infty\} \quad \text{if } \eta > 0.$$

Recall that $M(F)$ denote the mass of F defined by $M(F) = \displaystyle\int_{\mathbb{R}_+} dF(k)$. We shall then assume that $F_{in} \in \mathcal{E}_0$ if $\eta = 0$ and $F_{in} \in \mathcal{E}_\theta$ for some $\theta > 0$ if $\eta \in (0,1)$

Two basic properties of the solutions of (2.5) are the conservation of mass and the fact that a suitably defined entropy is increasing. The formal proofs of these facts are simple calculations and so they will be done here. The validity of these calculations under the assumptions of our theorems will be checked in each case.

To show the conservation of mass we integrate equation (2.5) over $\mathbb{R}_+$ with respect to k. Then, by the change of variables $(k,k') \to (k',k)$ we obtain

$$\frac{d}{dt}M(F) = \int_{\mathbb{R}_+} Q(F,F)\,dk = 0,$$

which means that the number of photons is conserved and

$$M(F(t,.)) = M(F_{in}) =: m \quad \text{for all } \ t \ge 0. \tag{2.8}$$

On the other hand, we define the entropy for a general state $F = g + G$ by

$$H(F) = H(g) - M(k\,G), \tag{2.9}$$

where $H(g)$ is defined in (1.11). By (2.9), $H(F)$ is well defined for every distribution F given by (2.4) and $H(F) \in [-\infty, S(f_0)]$.

We now show that the entropy $H(F)$ is not decreasing along the trajectories of (2.5), precisely

$$\frac{d}{dt} H(F) = \frac{1}{2} D(F), \tag{2.10}$$

where $D(F) \geq 0$ is a the so-called dissipation entropy rate that we define below. Let j be the function

$$j(u,v) = \begin{cases} (v-u)(\ln v - \ln u) \quad \text{if} \quad u>0, v>0, \\ 0 \quad \text{if} \quad u = v = 0 \\ +\infty \quad \text{elsewhere.} \end{cases} \tag{2.11}$$

Whenever $Q_1^{\pm}(g,G)\,h'(g,k) \in L^1$ and $Q_2^{\pm}(g,G)\,k \in M^1$, we state that

$$\begin{aligned} &\int_{\mathbb{R}_+} \big\{Q_1(g,G)\,h'(g,k) - Q_2(g,G)\,k\big\}\,dk = \\ &= \int_{\mathbb{R}_+} \big\{(k^2+g)\,e^{-k}\,L(g) - g\,L\big((k^2+g)\,e^{-k}\big\}\,h'(g,k)\,dk \\ &+ \int_{\mathbb{R}_+} \big\{\big[(k^2+g)\,e^{-k}\,L(G) - g\,L\big(G\,e^{-k}\big)\big]\,h'(g,k) - G\,\big[L(g)\,e^{-k} - L\big((k^2+g)\,e^{-k}\big)\big]\,k\big\}\,dk \\ &- \int_{\mathbb{R}_+} G\,\big[L(G)\,e^{-k} - L\big(G\,e^{-k}\big)\big]\,k\,dk =: \frac{1}{2}D_1(g) + D_2(g,G) + \frac{1}{2}D_3(G) =: \frac{1}{2}D(F), \end{aligned} \tag{2.12}$$

where the dissipation of entropy terms D_i are given by

$$\begin{aligned} D_1(g) &= \iint_{\mathbb{R}_+^2} b\,j\left((k^2+g)\,e^{-k}\,g', (k'^2+g')\,e^{-k'}\,g\right)dk'dk, \\ D_2(g,G) &= \iint_{\mathbb{R}_+^2} b\,j\left((k^2+g)\,e^{-k}, g\,e^{-k'}\right)dG(k')dk, \\ D_3(G) &= \iint_{\mathbb{R}_+^2} b\,j\left(e^{-k}, e^{-k'}\right)dG(k')dG(k). \end{aligned} \tag{2.13}$$

Indeed, we just make the following computation:

$$\begin{aligned} &\int_{\mathbb{R}_+} \big\{(k^2+g)\,e^{-k}\,L(g) - g\,L\big((k^2+g)\,e^{-k}\big\}\,h'(g,k)\,dk = \\ &\qquad = \iint_{\mathbb{R}_+^2} b\,\big\{(k^2+g)\,e^{-k}\,g' - g\,(k'^2+g')\,e^{-k'}\,\big[\ln\left((k^2+g)\,e^{-k}\right) - \ln\,g\big]\,dk'dk \\ &\qquad = \iint_{\mathbb{R}_+^2} b\,\big\{(k'^2+g')\,e^{-k'}\,g - g'\,(k^2+g)\,e^{-k}\,\big[\ln\left((k'^2+g')\,e^{-k'}\right) - \ln\,g'\big]\,dkdk' \\ &\qquad = \frac{1}{2}\iint_{\mathbb{R}_+^2} b\,j\big((k^2+g)\,e^{-k}\,g', g\,(k'^2+g')\,e^{-k'}\big)\,dk'dk = \frac{1}{2}D_1(g), \end{aligned}$$

$$\int_{\mathbb{R}_+} \left\{ \left[(k^2+g)\, e^{-k}\, L(G) - g\, L\big(G\, e^{-k}\big) \right] h'(g,k) - G \left[L(g)\, e^{-k} - L\big((k^2+g)\, e^{-k}\big) \right] k \right\} dk =$$

$$= \iint_{\mathbb{R}^2_+} b \left[(k^2+g)\, e^{-k} - g\, e^{-k'} \right] \left[\ln\big((k^2+g)\, e^{-k}\big) - \ln\, g) \right] dG(k')dk$$

$$+ \iint_{\mathbb{R}^2_+} b \left[g\, e^{-k'} - (k^2+g)\, e^{-k} \right] \ln e^{-k'}\, dG(k')dk = D_2(g,G),$$

and

$$- \int_{\mathbb{R}_+} \left[L(G)\, e^{-k} - L\big(G\, e^{-k}\big) \right] k\, dG(k) = \iint_{\mathbb{R}^2_+} b\, (e^{-k} - e^{-k'})\, (-k)\, dG(k)dG(k')$$

$$= \iint_{\mathbb{R}^2_+} b\, (e^{-k} - e^{-k'})\, (-k')\, dG(k)dG(k') = \frac{1}{2} D_3(G).$$

Then, from (2.12), we get, at least formally,

$$\frac{d}{dt} H(F) = \int_{\mathbb{R}_+} \left\{ h'(g,k)\, \frac{\partial g}{\partial t} - k\, \frac{\partial G}{\partial t} \right\} dk = \int_{\mathbb{R}_+} \left\{ Q_1(g,G)\, h'(g,k) - Q_2(g,G)\, k \right\} dk = \frac{1}{2} D(F). \tag{2.14}$$

We may now state our main results. As it is typical in the study of Boltzmann equations, we first consider the set of stationary solutions of (2.5) and give different characterizations of them.

Theorem 2. *Assume (2.2) with $b > 0$. Let F be a bounded non negative measure such that $M(F) = m$. The following assertions are equivalent:*

$$F = \mathcal{B}_m, \tag{2.15}$$

$$F \text{ is the solution of the maximisation problem } H(F) = \max_{M(F')=m} H(F') \tag{2.16}$$

$$D(F) = 0, \tag{2.17}$$

$$Q(F,F) = 0 \quad \text{and} \quad F \in \mathcal{E}_\eta. \tag{2.18}$$

Our next step is to consider the existence of solutions for the evolution problem. We say that a distribution $F \in C([0,\infty); M^1(\mathbb{R}_+))$ is an entropy solution of the Cauchy problem (2.5)$-$(2.7) if

$$\int_{\mathbb{R}_+} F(t,k)\, \phi(t,k)\, dk = \int_{\mathbb{R}_+} F_{in}(k)\, \phi(0,k)\, dk + \int_0^t \int_{\mathbb{R}_+} Q(F,F)\, \phi\, dk ds, \tag{2.19}$$

$\forall \phi \in C_c([0,\infty) \times \mathbb{R}_+)$, and satisfies either the entropy inequality

$$\int_{t_1}^{t_2} D(F(s,.))\, ds \le H(F(t_2,.)) - H(F(t_1,.)) \qquad \text{for all } t_2 \ge t_1 \ge 0, \tag{2.20}$$

or the entropy dissipation bound

$$\int_0^\infty D(F(t,.))\, dt \le H(\mathcal{B}_m) - H(F_{in}); \tag{2.21}$$

this will be specified in each case.

Theorem 3 (First existence result). *Assume that b satisfies (2.2) with $\eta = 0$. Then for any initial datum $F_{in} = g_{in} + G_{in} \in \mathcal{E}_0$ there exists an unique entropy solution to (2.5), (2.7) and (2.20), $F = g + G \in C([0,\infty), \mathcal{E}_0)$. Moreover, F satisfies (2.8) and is such that*

$$supp\, G(t,.) \;\subset\; supp\, G_{in}. \tag{2.22}$$

In particular, if $F_{in} = g_{in} \in L^1(\mathbb{R}_+)$ then $G(t,.) = 0$ for every $t \geq 0$ and thus $F(t,.) = g(t,.) \in L^1(\mathbb{R}_+)$ for every $t \geq 0$.

Theorem 4 (Second existence result). *Assume that b satisfies (2.3). Then, for all initial datum $F_{in} = g_{in} + G_{in} \in \mathcal{E}_{\theta'}$ with $\theta' > 0$ there exists an unique global entropy solution to equation (2.5)-(2.7) and (2.20), $F = g + G \in C([0,T), \mathcal{E}_\theta) \cap L^1(0,T; \mathcal{E}_{\eta+\theta})$ for all $T > 0$ and all $0 < \theta < \min(\theta', \eta, 1-\eta)$. Moreover, it satisfies (2.20) and (2.22).*

Theorem 5 (Third existence result). *Assume that b satisfies (2.2) for some $\eta \in [0,1)$ and that the initial datum has the special shape*

$$F_{in} = g_{in} + \alpha_{in}\,\delta_0, \qquad \text{with} \qquad 0 \leq g_{in} \leq g_0 \quad \text{and} \quad \alpha_{in} \geq 0. \tag{2.23}$$

Then there exists an entropy solution to (2.5), (2.7) and (2.21), $F = g + \alpha\,\delta_0 \in C([0,T), \mathcal{E}_1)$. Moreover, F satisfies (2.8) and

$$0 \leq \alpha(t) \leq \alpha_{in}, \qquad 0 \leq g(t,.) \leq g_0 \qquad \forall t \geq 0. \tag{2.24}$$

Remark 2.1. Theorem 3 has to be seen as a first simple step in the existence theory: we deal with general initial data and bounded cross-section but without the (may be) artificial assumption (2.3). In fact, the assumption of boundedness of the croos-section b seems to be more unphysical that assumption (2.3). Theorem 4 provide a good framework in order to investigate long time behavior for unbounded cross-section and initial data f_{in} such that $N(f_{in}) > N_0$. Theorem 5 allow us to get inside the Kompaneets asymptotic. Observe that the solutions obtained in Theorems 3 and 4, i.e. under the more restrictive conditions on b, are unique and satisfy the entropy inequality (2.20). Under the less restrictive condition (2.2), the solution constructed satisfies the weaker entropy dissipation bound (2.21) and moreover we do not know whether it is unique. We believe that it should be possible to adapt the results of X. Lu [29], [30] in order to prove that equality holds in (2.20) in all the cases. Concerning the uniqueness see also Remarks 5.1 and 5.2.

Remark 2.2. The main difficulty in the proofs of Theorems 3, 4 and 5 with respect to the classic Boltzmann equation is that an uniform bound on the entropy does not provide weak convergence in L^1. In the existence proof we use two different strategies. On one hand, in Theorems 3 and 4, we do restrictive assumption on the cross section but we deal with (quite) general initial data. In this case, we are able to prove that a sequence of solutions to a regularized problem is a Cauchy sequence in some appropriate space. When $\eta = 0$ we just follow the method of Arkeryd [3]. When $\eta > 0$, the collision operator Q does not map M^1 into itself. In this case, we follow the spirit of the moment method developed for the classical Boltzmann with hard potential. Using the specific shape (2.3) we prove that exponential momentum of the solution (or of a sequence of regularized solutions) can be bounded; condition (2.3) is used in order to gain momentum, which is crucial in the proof. A similar method has been already used in [32]. On the other hand, in Theorem 5, we deal with general cross section but we make strong restriction on the initial data. In this case, we are able to prove the maximum principle (2.24) and then we can use a L^∞ compactness argument.

We next consider the asymptotic behavior of our global solutions. Our main result is the following.

Theorem 6 (Asymptotic behavior). *Assume that $0 < b$ and F_{in} satisfy the assumptions of one of the existence Theorems 3, 4 or 5. Let be $m = M(F_{in})$, $\mathcal{B}_m = g_\mu + \alpha\delta_0$ the Bose distribution of mass m defined in (1.12) and $F \in C([0,\infty); M^1)$ the corresponding solution. Then we have*

$$\begin{cases} F(t,.) \underset{t\to\infty}{\rightharpoonup} \mathcal{B}_m \text{ weakly} \star \text{ in } \big(C_c(\mathbb{R}_+)\big)' \\ \lim\limits_{t\to\infty} \|g(t,.) - g_\mu\|_{L^1((k_0,\infty))} = 0 \quad \forall k_0 > 0. \end{cases} \tag{2.25}$$

Moreover if $m \leq N_0$ or $0 \leq g_{in} \leq g_0$ we can take $k_0 = 0$.

Remark 2.3. Let us observe the following consequence of the above results. Assume we start with a regular initial data $F_{in} \equiv g_{in} \in L^1$. Then, the solution F remains regular for all time: $F(t) \equiv g(t) \in L^1$. Moreover, suppose that $M(F_{in}) \equiv M(g_{in}) = m > N_0$. Then $F(t,\cdot) \equiv g(t,\cdot) \rightharpoonup \mathcal{B}_m$ where $\mathcal{B}_m = k^2 f_0 + (m - N_0)\delta_0$. This precisely shows that a regular initial state of total mass greater that N_0 does not condense in finite time (Theorem 3 or 4) but does condense at the origin in infinite time (Theorem 6).

Remark 2.4. Suppose now that we start with an initial datum which already has a condensate, say $F_{in} = g_{in} + \alpha_0\delta$. By Theorem 6, if $m = M(F_{in}) \le N_0$ then $g(t,.) \to g_\mu$ in $L^1(\mathbb{R}_+)$ and $\alpha(t) \to 0$ as $t \to \infty$. It is an interesting question to know what happens if $m > N_0$. We know by Theorem 6 that $g(t) + \alpha(t)\delta \rightharpoonup g_0 + (m - N_0)\delta$ in $\sigma\big(M^1(\mathbb{R}_+), C_c(\mathbb{R}_+)\big)$ weak $\star$ and $g(t,.) \to g_0$ in $L^1([k_0,\infty))$ for all $k_0 > 0$. But this does not tell us anything about the asymptotic behavior of $\alpha(t)$ and of $g(t)$ near $k = 0$. If for instance, $M(g_{in)} > N_0$, then part of the mass of $g(t)$ must be transferred to the condensate. Does this happens continuously at all times $t > 0$ or does it happens only asymptotically as $t \to \infty$? i.e. do we have

$$\overline{\alpha} \equiv \lim_{t\to\infty} \alpha(t) = m - N_0 \quad \text{and} \quad \lim_{t\to\infty} \|g(t) - g_0\|_1 = 0$$

or

$$\overline{\alpha} \equiv \lim_{t\to\infty} \alpha(t) < m - N_0 \quad \text{and} \quad g(t) \rightharpoonup g_0 + (m - N_0 - \overline{\alpha})\,\delta \quad \text{in} \quad \sigma\big(M^1(\mathbb{R}_+), C_c(\mathbb{R}_+)\big)\,\text{weak}\,\star\ ?$$

If $g_{in} \le g_0$ (but nevertheless $m > N_0$), we know by Theorem 6 that $g(t) \le g_0$ for all $t > 0$, and $g(t) \to g_0$ in $L^1(\mathbb{R}_+)$. Then we must have, $\overline{\alpha} = m - N_0$ and we are in the first case. This and related questions are considered in a forthcoming work [20].

We finally turn to the Kompaneets limit. Our result is the following

Theorem 7 (Kompaneets limit). *Assume that* $b(k,k') = e^{k/2} e^{k'/2}$ *and consider* $\sigma \in \mathcal{D}(\mathbb{R})$ *even,* $\mathit{supp}\,\sigma \subset [-2,2]$, $\sigma > 0$ *over* $[-1,1]$ *with* $\int_{\mathbb{R}} \sigma(z)\,dz = 1$, $\int z^2\sigma(z)\,dz = 2$. *We define*

$$b_\varepsilon(k,k') = b(k,k')\,\frac{\sigma_\varepsilon(k'-k)}{\varepsilon^2} \qquad \text{with} \qquad \sigma_\varepsilon(z) = \frac{1}{\varepsilon}\,\sigma\big(\frac{z}{\varepsilon}\big). \tag{2.26}$$

For a given initial datum $0 \le g_{in} \le g_0$ *we denote by* $g_\varepsilon \in C([0,\infty), L^1(\mathbb{R}_+))$ *the solution to the Boltzmann equation (2.6) corresponding to the cross-section* b_ε *and the initial datum* g_{in} *which is given by Theorem 5. Then, for all* $T > 0$,

$$\lim_{\varepsilon\to 0} ||g_\varepsilon - g||_{C([0,T],L^2(\mathbb{R}_+))} = 0, \tag{2.27}$$

where $g \equiv k^2 f$, *and* $f \in C([0,T]; L^1(\mathbb{R}_+))$ *is the unique solution to the Cauchy problem*

$$\begin{cases} k^2\,\dfrac{\partial f}{\partial t} \ = \ Q_0(f,f) \ = \ \dfrac{\partial}{\partial k}\big\{k^4\,\big(f^2 + f + \dfrac{\partial f}{\partial k}\big)\big\} \quad \mathit{in} \quad \mathcal{D}'((0,T)\times\mathbb{R}_+)\ \forall T > 0,\\ f(x,0) = k^{-2} g_{in}(x) \quad \mathit{for} \quad x > 0 \end{cases} \tag{2.28}$$

such that $0 \le f \le f_0$.

Remark 2.5. The existence and uniqueness of such a solution f was proved in [17]. The existence of a solution $f \in C([0,\infty); L^1(\mathbb{R}_+^*))$ for all $T > 0$ also follows from the proof of Theorem 7, but not the uniqueness. More generally, we can consider cross sections b satisfying

$$0 \le b\,e^{-\eta k} e^{-\eta k'} \equiv \sigma(k - k') \quad \text{with} \quad \eta \in [0,1) \tag{2.29}$$

and $\sigma \in \mathcal{D}(\mathbb{R})$ even, $\operatorname{supp}\sigma \subset [-2,2]$, $\sigma > 0$ over $[-1,1]$ with $\int \sigma(z)\,dz = 1$, $\int_{\mathbb{R}} z^2\,\sigma(z)\,dz = \Sigma$. We prove in that case, for every $T > 0$

(i) the existence of a function h such that $h = k^2\varphi$ with $0 \le \varphi \le f_0$, $\varphi \in C([0,\infty); L^1(\mathbb{R}_+^*))$ solution to

$$\begin{cases} k^2\,\dfrac{\partial \varphi}{\partial t} \ = \ \dfrac{\partial}{\partial k}\big\{\alpha(k) k^4\,\big(\varphi^2 + \varphi + \dfrac{\partial \varphi}{\partial k}\big)\big\} \quad \text{in} \quad \mathcal{D}'((0,T)\times\mathbb{R}_+)\ \forall T > 0,\\ \varphi(x,0) = k^{-2} g_{in}(x) \quad \text{for} \quad x > 0 \end{cases} \tag{2.30}$$

where

$$\alpha(k) := \frac{\Sigma}{2}\, b(k,k)\, e^{-k}; \tag{2.31}$$

(ii) the existence of a subfamily $(g_{\varepsilon_\ell})_{\ell\in\mathbb{N}}$ such that

$$\lim_{\ell\to\infty} \|g_{\varepsilon_\ell} - h\|_{L^2((0,T)\times\mathbb{R}_+)} = 0.$$

Since we do not know whether such a solution φ to problem (2.30) is unique or not, the function h may depend on the subfamily $(g_{\varepsilon_\ell})_{\ell\in\mathbb{N}}$ and the time T. In particular we can not be sure that all the family (g_ε) converges to h in $L^2((0,T)\times\mathbb{R}_+)$ for every $T>0$.

Remark 2.6. Theorem 7 shows that, under a suitable hypothesis on b, the Cauchy problem for the equation (1.1) may be approximated by the Cauchy problem for the Kompaneets equation with the same initial datum f_{in}, whenever $0\le f_{in}\le f_0$. In that case, the solution of the Kompaneets equation f_K also satisfies the flux conditions:

$$\lim_{k\to 0} x^4 \left(\frac{\partial f_K}{\partial x} + f_K + f_K^2\right) = \lim_{k\to\infty} x^4 \left(\frac{\partial f_K}{\partial x} + f_K + f_K^2\right) = 0$$

for all time. Notice that the flux condition is already taken into account in the formulation (2.28). The function f_K satisfies therefore the problem (1.10) with the total mass preserved (see [17]).

Remark 2.7. Note that from [17] one can find initial data g_{in} such that the solution f of the Kompaneets equation associated to $f_{in} = g_{in}/k^2$ satisfies $m(t) = N(f(t))$ is decreasing and $m(t) < N(f_{in})$ for $t \ge T^\star > 0$. Of course, g_{in} does not satisfy $0\le g_{in}\le g_0$. Therefore,

$$\|g_\varepsilon(t) - k^2 f(t)\|_{L^1} \ge \|g_\varepsilon(t)\|_{L^1} - \|k^2 f(t)\|_{L^1} = N(f_{in}) - m(t) > 0,$$

for every $t\ge T^\star$, $\varepsilon>0$. As a conclusion, the Kompaneets equation is not an approximation of the Boltzmann Compton equation after $T^\star$. It is an open problem to understand what happens to the sequence (g_ε) built in the statement of Theorem 7 when the initial datum does not satisfy $0\le f_{in}\le f_0$.

In the next Section we study the stationary states and prove Theorem 2. The detailed analysis of the entropy and the entropy dissipation terms is done in Section 4. Section 5 is devoted to the proofs of the existence results stated in Theorem 3, Theorem 4 and Theorem 5. The long time behavior of the solutions is studied in Section 6. Finally the approximation by the Kompaneets equation is studied in Section 7, where we prove Theorem 7.

3. The stationary problem: proof of Theorem 2.

We start studying the stationary states of the equation (2.2). These are particular solutions of the equation which, moreover, are important for the dynamics of the general solutions of the Cauchy problem. We only consider stationary states which are bounded non negative measures. This section is devoted to the proof of Theorem 2. It is divided in three steps.

Step 1. We begin with the equivalence of (2.15) and (2.16) and so give an alternative proof to the proof of Theorem 1 presented in [6]. For that purpose we use the following Lemma.

Lemma 3.1. *Let $F = g + G$ be given as in (2.4) such that $M(F) = m$ and $\mathcal{B}_m = g_\mu + \alpha\,\delta_0$ be the Bose state of mass m. Then,*

$$H(g|g_\mu) \equiv \int_{\mathbb{R}_+} \big[(k^2+g)\,\ln\frac{k^2+g}{k^2+g_\mu} - g\,\ln\frac{g}{g_\mu}\big]\,dk,$$

is well defined, $H(g|g_\mu) \in [-\infty, 0]$ and

$$H(F) - H(\mathcal{B}_m) = H(g|g_\mu) - M(G\,(k+\mu)). \tag{3.1}$$

We accept this Lemma for the moment and end the proof of Step 1. We remark that the function $\psi(x,y) = (k^2+x)\,\ln\dfrac{k^2+x}{k^2+y} - x\,\ln\dfrac{x}{y}$ (with fixed k and y) has an unique maximum which is $x = y$, and $\psi(y,y) = 0$. Therefore, for any non negative measurable function g and all $k > 0$, $\psi(g(k), g_\mu(k)) \le 0$, $H(g|g_\mu)$ is well defined and $H(g|g_\mu) \le 0$ with equality, if, and only if $g = g_\mu$. We deduce that for any $F = g + G$ such that $M(F) = m$ we have $H(F) \le H(\mathcal{B}_m)$. Moreover, if $H(F) = H(\mathcal{B}_m)$ then $g = g_\mu$ and $M(G\,(k+\mu)) = 0$, so that $G = \alpha\,\delta_0$ and $\alpha\,\mu = 0$. This exactly means that $F = \mathcal{B}_m$. It is clear on the other hand that if $F = \mathcal{B}_m$, $H(F) = H(\mathcal{B}_m)$, which shows (2.16).

Proof of Lemma 3.1. We start writing

$$\begin{aligned} H(F) - H(\mathcal{B}_m) &= \int_{\mathbb{R}_+} \big(g\,\ln\frac{(k^2+g)\,e^{-k}}{g} + k^2\,\ln(k^2+g) - k^2\,\ln k^2\big)\,dk - M(k\,G) \\ &\quad - \int_{\mathbb{R}_+} \big(g_\mu\,\ln\frac{(k^2+g_\mu)\,e^{-k}}{g_\mu} - k^2\,\ln(k^2+g_\mu) + k^2\,\ln k^2\big)\,dk. \end{aligned} \tag{3.2}$$

We remark that $(k^2+g_\mu)\,e^{-k}/g_\mu = e^\mu$ and, since by (1.15), $\mu\,\alpha = 0$, we have

$$\int_{\mathbb{R}_+} g_\mu\,\ln e^\mu = \mu\,\big[M(\mathcal{B}_m) - \alpha\big] \equiv \mu\,M(\mathcal{B}_m) = \mu\,M(F) = \int_{\mathbb{R}_+} g\,\ln e^\mu + \mu\,M(G), \tag{3.3}$$

Finally, (3.1) follows from (3.2) and (3.3). □

Step 2. Equivalence of (2.15) and (2.17). From the expression for D given in (2.13), it is clear that $D(\mathcal{B}_m) = 0$. Assume now that $F = g + G$ and $M(F) = m$. Since all the terms $D_i(F)$, $i = 1, 2, 3$ are well defined and non negative, $D(F)$ is also well defined. Assume moreover that

$$D(F) = D_1(g) + 2\,D_2(g,G) + D_3(G) = 0.$$

All the terms are non negative, and so must be zero. In particular

$$D_1(g) = \iint_{\mathbb{R}_+^2} b(k,k')\,j\big(g'\,(k^2+g)\,e^{-k}, g\,(k'^2+g')\,e^{-k'}\big)\,dk'dk = 0.$$

But, since $b > 0$, we have

$$g'\,(k^2+g)\,e^{-k} = g\,(k'^2+g')\,e^{-k'} \quad a.e. \quad k, k' \ge 0,$$

and so

$$\frac{g}{k^2+g}\,e^k = \frac{g'}{k'^2+g'}\,e^{k'} \quad a.e. \quad k,k' \geq 0.$$

This shows that $\dfrac{g}{k^2+g}\,e^k$ is independent of k and is then a constant say, γ. We deduce that either $g \equiv 0$ or $\gamma > 0$. In the last case, we may write $\gamma = e^{-\mu}$, so that $g = g_\mu$ with $\mu \geq 0$, since $g \in L^1(\mathbb{R}_+)$. Moreover, from

$$D_4(G) = \iint_{\mathbb{R}^2_+} b\, j\big(e^{-k}, e^{-k'}\big)\, dG(k)\, dG(k') = 0,$$

we deduce that $G = \alpha\,\delta_a$ for some $a \geq 0$ and $\alpha \geq 0$. Finally, since $(k^2+g_\mu)\,e^{-k} = g_\mu\,e^\mu$, we deduce

$$D_2(g,G) = \iint_{\mathbb{R}^2_+} b\; j\big((k^2+g)\,e^{-k}, g\;e^{-k'}\big)\, dG(k')dk = \alpha\, j\big(e^{\mu}, e^{-a}\big)\, L_a(g_\mu) = 0,$$

which may only happen when $\alpha = 0$ or $\mu = -a = 0$. □

Step 3. Equivalence of (2.15) and (2.14). It is clear that $\mathcal{B}_m \in \mathcal{E}_1 \subset \mathcal{E}_\eta$ and $Q(\mathcal{B}_m, \mathcal{B}_m) = 0$. Let be now $F = g + G \in \mathcal{E}_\eta$ such that $M(F) = m$ and $Q(F,F) = 0$, which implies

$$Q_1(g,G) = (k^2+g)\,e^{-k}\,L(F) - g\,L\big((k^2+F)\,e^{-k}\big) = 0 \tag{3.4}$$

and

$$Q_2(g,G) = G\,\big[L(F)\,e^{-k} - L\big((k^2+F)\,e^{-k}\big)\big] = 0. \tag{3.5}$$

Define the continuous function $\mu : \mathbb{R}_+ \to \mathbb{R}$ by

$$e^{\mu(k)} := \frac{L((k^2+F)\,e^{-k})}{L(F)},$$

so that, from (3.4), we get $(k^2+g)\,e^{-k} = g\,e^\mu$ for almost every $k \geq 0$. Then

$$g(k) = \frac{k^2}{e^{k+\mu}-1} \qquad a.e.\ k \geq 0.$$

Since $g \geq 0$ and $g \in L^1$ we deduce $\mu \geq -k$. We observe that

$$g\,e^\mu \leq g\,e\,\mathbf{1}_{\{\mu\leq 1\}} + \frac{k^2}{e^k - e^{-\mu}}\,\mathbf{1}_{\{\mu\geq 1\}} \text{ belongs to } L^1(\mathbb{R}_+)$$

and therefore that

$$g\,|\mu| \leq g\,k\,\mathbf{1}_{\{\mu\leq 0\}} + g\,e^\mu\,\mathbf{1}_{\{\mu\geq 0\}} \text{ belongs to } L^1(\mathbb{R}_+),$$

since $F \in \mathcal{E}_\eta$, and in particular, $k\,g \in L^1(\mathbb{R}_+)$.

On the other hand, since

$$h'(g,k) = \ln(k^2+g) - \ln g - k = \mu$$

we have

$$\begin{cases} Q_1^-(g,G)h'(g,k) \equiv g\,L((k^2+F)\,e^{-k})\,h'(g,k) = g\,\mu\,L(G\,e^{-k} + g\,e^\mu) \in L^1 \\ Q_2^+(g,G)k \equiv G\,L(F)\,e^{-k}\,k \in M^1. \end{cases} \tag{3.6}$$

Finally, from $Q_1(g,G) = 0$ et $Q_2(g,G) = 0$ we deduce

$$
(3.7)\qquad \begin{cases} Q_1^+(g,G)h'(g,k) \equiv (k^2+g)\, e^{-k}\, L(F)\, h'(g,k) \in L^1 \\ Q_2^-(g,G)k \equiv G\, L\big((k^2+F)\, e^{-k}\big)\, k \in M^1. \end{cases}
$$

Therefore, the formal calculations performed in Section 2 leading to (2.12) are allowed, so that we get

$$
\frac{1}{2} D(F) = \int_{\mathbb{R}_+} \{Q_1(g,G)\, h'(g,k) - Q_2(g,G)\, k\}\, dk = 0.
$$

We conclude using step 2. □

4. Analysis of the entropy term and of the entropy dissipation term.

This section is devoted to a detailed analysis of the entropy (1.14), (2.9) and the entropy dissipation (2.12) defined for a distribution F given by (2.4). For that purpose we need some results about convex functions of measures. These questions have already been studied by R. Temam [36], F. Demangel & R. Temam [9] and T. Hadhri [22]. We briefly show that their results extend to the more general functions that are needed for our purposes.

We start with the following elementary result.

Lemma 4.1. *There exists a constant C_1 such that for every state $F = g + G$ defined in (2.4) and for which $H(F)$ and $M((1+k)F)$ are well defined, the following inequalities hold*

$$
(4.1)\qquad M(k\, F) \leq C_1 (1 + M(g) - H(F))
$$

$$
(4.2)\qquad |H(F)| \leq C_1\, M((1+k)\, F).
$$

Proof of Lemma 4.1. We show only the proof of (4.1); the proof of (4.2) is similar. We first write

$$
(4.3)\qquad \int_{\mathbb{R}_+} k\, g\, dk + \int_{\mathbb{R}_+} k\, dG(k) = H_0(g) - H(F),
$$

where we have defined

$$
(4.4)\qquad H_0(g) = \int_{\mathbb{R}_+} \left[(g+k^2)\, \ln(g+k^2) - g\, \ln g - k^2\, \ln k^2\right] dk.
$$

We use, without proof, the following elementary estimates.

Lemma 4.2.

(i) For every $s \in (0,1)$ and $k > 0$

$$
(4.5)\qquad 0 \leq (s+k^2)\, \ln(s+k^2) - k^2\, \ln k^2 \leq s\, (1 + \ln(1+k^2)).
$$

(ii) There exists a positive constant C_1 such that, for all $s \geq 1$ and $k > 1$,

$$
(4.6)\qquad 0 \leq (s+k^2)\, \ln(s+k^2) - s\, \ln s - k^2\, \ln k^2 \leq 2s\, (C_1 + \ln(1+k^2)).
$$

(iii) For all $\delta \in (0,1)$ there exists a positive constant C_δ such that, for all $s \geq 1$ and $k \in (0,1]$

$$
(4.7)\qquad 0 \leq (s+k^2)\, \ln(s+k^2) - s\, \ln s \leq \delta s + C_\delta.
$$

Thanks to Lemma 4.2, we have:

$$
\begin{aligned}
&\int_{\mathbb{R}_+} ((g+k^2)\,\ln(g+k^2) - k^2\,\ln k^2)\,dk - \int_{\mathbb{R}_+} g\,\ln g\,\mathbf{1}_{\{g\geq 1\}}\,dk = \\
&= \int_{\mathbb{R}_+} ((g+k^2)\,\ln(g+k^2) - k^2\,\ln k^2 - g\,\ln g)\,\mathbf{1}_{\{g\geq 1, k>1\}}\,dk \\
&+ \int_{\mathbb{R}_+} ((g+k^2)\,\ln(g+k^2) - k^2\,\ln k^2 - g\,\ln g)\,\mathbf{1}_{\{g\geq 1, 0<k\leq 1\}}\,dk \\
&+ \int_{\mathbb{R}_+} ((g+k^2)\,\ln(g+k^2) - k^2\,\ln k^2)\,\mathbf{1}_{\{0<g<1\}}\,dk \\
&\leq 2\int_{\mathbb{R}_+} g\,(C_1 + \ln(1+k^2))\,dk + \delta\int_{\mathbb{R}_+} g\,dk + C_\delta \\
&\qquad - \int_0^1 k^2\,\ln k^2\,dk + \int_{\mathbb{R}_+} g\,(1+\ln(1+k^2))\,dk,
\end{aligned}
$$

from where, for some positive constant C,

$$
\int_{\mathbb{R}_+} ((g+k^2)\,\ln(g+k^2) - k^2\,\ln k^2)\,dk - \int_{\mathbb{R}_+} g\,\ln g\,\mathbf{1}_{\{g\geq 1\}}\,dk \leq C\int_{\mathbb{R}_+} g\,(1+\ln(1+k^2))\,dk + C. \tag{4.8}
$$

On the other hand, since $s \mapsto -s\,\ln s$ is increasing over $[0, e^{-1}]$ and $s \mapsto -\ln s$ is decreasing, we have

$$
\begin{aligned}
-\int_{\mathbb{R}_+} g\,\ln g\,\mathbf{1}_{\{0\leq g\leq 1\}}\,dk &= -\int_0^1 g\,\ln g\,\mathbf{1}_{\{0\leq g\leq 1\}}\,dk \\
&\quad - \int_1^\infty g\,\ln g\,\mathbf{1}_{\{0\leq g\leq e^{-\sqrt{k}}\}}\,dk - \int_1^\infty g\,\ln g\,\mathbf{1}_{\{e^{-\sqrt{k}}\leq g\leq 1\}}\,dk \\
&\leq C_1 + \int_1^\infty e^{-\sqrt{k}}\,\sqrt{k}\,dk + \int_1^\infty g\,\sqrt{k}\,dk \\
&\leq C_1 + C_2 + M(g) + \frac{1}{4}\int_{\mathbb{R}_+} g\,k\,dk,
\end{aligned} \tag{4.9}
$$

where $C_2 = \int_1^\infty e^{-\sqrt{k}}\,\sqrt{k}\,dk$. Using (4.3), (4.4), (4.8) and (4.9) we obtain (4.1) □

In order to analyze the different terms of the entropy dissipation defined in (2.13) we first remark that they are all defined by mean of the convex, proper lower semi continuous (l.s.c.) function j defined in (2.11). As usual we denote $j^\star$ its conjugate function, i.e.

$$
j^\star(a) = \sup_{b\in\mathbb{R}^2} (a\cdot b - j(b)).
$$

Since j is homogeneous of degree 1, we have $j^\star = I_K$, with $I_K(a) = 0$ if $a \in K$ and $I_K(a) = +\infty$ if $a \notin K$, where K is a closed, convex subset of $\mathbb{R}^2$. We can also verify, for example, that

$$
(-\infty, 0] \times (-\infty, 0] \subset K \subset \left(\mathbb{R}^2_+ \backslash \{(0,0)\}\right)^c,
$$

but in fact, we do not need in the sequel, the exact description of K. As a consequence, we have

$$
j(b) = j^{\star\star}(b) = \sup_{a\in\mathbb{R}^2} (b\cdot a - j^\star(a)) = \sup_{a\in K} b\cdot a.
$$

We consider now $F \in M^1(\mathbb{R}_+)$, $F = g + G$ satisfying (2.4). Let us define the following measures of $M^1(\mathbb{R}_+^2)$:

$$
\text{(4.10)} \qquad \begin{cases} A = (k^2 + F)\, e^{-k}\, F', & B = (k'^2 + F')\, e^{-k'}\, F \\ A_1 = (k^2 + g)\, e^{-k}\, g', & B_1 = (k'^2 + g')\, e^{-k'}\, g \\ A_2 = G\, e^{-k}\, g', & B_2 = (k'^2 + g')\, e^{-k'}\, G \\ A_3 = (k^2 + g)\, e^{-k}\, G', & B_3 = G'\, e^{-k'}\, g \\ A_4 = G\, G'\, e^{-k}, & B_4 = G\, G'\, e^{-k'}, \end{cases}
$$

in such a way that

$$
\text{(4.11)} \qquad A = A_1 + A_2 + A_3 + A_4, \quad B = B_1 + B_2 + B_3 + B_4.
$$

Finally, for all $\mathcal{A}, \mathcal{B} \in M^1(\mathbb{R}_+^2)$ we define

$$
\text{(4.12)} \qquad J_X(\mathcal{A}, \mathcal{B}) = \sup_{(u,v)\in\mathcal{K}} < b\,\mathcal{A}, u > + < b\,\mathcal{B}, v >,
$$

where

$$
\mathcal{K} := \{(u,v) \in X^2,\ (u(x), v(x)) \in K\ \forall x \in \mathbb{R}_+^2\} \quad \text{and} \quad X = C_b(\mathbb{R}_+^2).
$$

Theorem 4.3. *Let $F \in M^1(\mathbb{R}_+)$ be defined as in (2.4). With the preceding notation we have*

$$
\text{(4.13)} \qquad J_X(A, B) = J_X(A_1, B_1) + J_X(A_2, B_2) + J_X(A_3, B_3) + J_X(A_4, B_4).
$$

Moreover,

$$
\text{(4.14)} \qquad J_X(A_1, B_1) = D_1(g),
$$

$$
\text{(4.15)} \qquad J_X(A_2, B_2) = J_X(A_3, B_3) = D_2(g, G),
$$

$$
\text{(4.16)} \qquad J_X(A_4, B_4) = D_3(G),
$$

and therefore

$$
\text{(4.17)} \qquad J_X(A, B) = D(F).
$$

Remark 4.4. We obtain that Theorem 4.3 also holds with $X = C_c(\mathbb{R}_+^2)$ by standard troncature arguments.

Proof of Theorem 4.3. We divide the proof in three steps.

Step 1: proof of (4.13). By the definition (4.12), for all $(u, v) \in \mathcal{K}$ we have

$$
< b\,A, u > + < b\,B, v > \leq\ J_X(A_1, B_1) + J_X(A_2, B_2) + J_X(A_3, B_3) + J_X(A_4, B_4),
$$

so that

$$
J_X(A, B)\ \leq\ J_X(A_1, B_1) + J_X(A_2, B_2) + J_X(A_3, B_3) + J_X(A_4, B_4).
$$

It is then enough to show the reverse inequality in order to prove (4.13). For any $\varepsilon > 0$ fixed, there exist $(u_i, v_i) \in \mathcal{K}$ such that

$$
\text{(4.18)} \qquad J_X(A_i, B_i) \leq < b\,A_i, u_i > + < b\,B_i, v_i > + \varepsilon \qquad \text{for } i = 1, 2, 3, 4.
$$

Consider now a sequence $(\theta^n) \in X$ such that $0 \leq \theta^n \leq 1$, $\theta^n = 1$ over $\operatorname{supp} G$ and $\theta^n \to 0$ a.e. in $\mathbb{R}_+^2$. Define

$$
w_n := (1 - \theta^n)\,(1 - \theta^{n\prime})\,u_1 + \theta^n\,(1 - \theta^{n\prime})\,u_2 + (1 - \theta^n)\,\theta^{n\prime}\,u_3 + \theta^n\,\theta^{n\prime}\,u_4,
$$

$$
z_n := (1 - \theta^n)\,(1 - \theta^{n\prime})\,v_1 + \theta^n\,(1 - \theta^{n\prime})\,v_2 + (1 - \theta^n)\,\theta^{n\prime}\,v_3 + \theta^n\,\theta^{n\prime}\,v_4.
$$

Observe that, for every n, $(w_n, z_n) \in \mathcal{K}$. Since $(w_n, z_n) \to (u_1, v_1)$ strongly in $L^1(\mathbb{R}_+^2)$, we have

$$< b\,A_1, w_n > + < b\,B_1, z_n > \to < b\,A_1, u_1 > + < b\,B_1, v_1 > . \tag{4.19}$$

Moreover,

$$< b\,A_2, w_n > + < b\,B_2, z_n > = < b\,A_2, (1-\theta^{n\prime})\,u_2 + \theta^{n\prime}\,u_4 > + < b\,B_2, (1-\theta^{n\prime})\,v_2 + \theta^{n\prime}\,v_4 >,$$

because $G\,\theta_n = G$ and $G\,(1-\theta_n) = 0$. Using the fact that $((1-\theta^{n\prime})\,u_2 + \theta^{n\prime}\,u_4, (1-\theta^{n\prime})\,v_2 + \theta^{n\prime}\theta^{n\prime}\,v_4) \to (u_2, v_2)$ strongly in $L^1(\mathbb{R}_+^2)$ we obtain

$$< b\,A_2, w_n > + < b\,B_2, z_n > \to < b\,A_2, u_2 > + < b\,B_2, v_2 > . \tag{4.20}$$

We show in the same way:

$$< b\,A_3, w_n > + < b\,B_3, z_n > \to < b\,A_3, u_3 > + < b\,B_3, v_3 > . \tag{4.21}$$

Finally, for every n

$$< b\,A_4, w_n > + < b\,B_4, z_n > = < b\,A_4, u_4 > + < b\,B_4, v_4 > . \tag{4.22}$$

We deduce from (4.18) and (4.19)–(4.22) that, for n sufficiently large, one has

$$\begin{aligned} < b\,A, w_n > + < b\,B, z_n > &= \sum_{i=1}^{4} < b\,A_i, w_n > + < b\,B_i, z_n > \\ &\geq \sum_{i=1}^{4} < b\,A_i, u_i > + < b\,B_i, v_i > -4\,\varepsilon \geq \sum_{i=1}^{4} J_X(A_i, B_i) - 8\,\varepsilon, \end{aligned}$$

which implies

$$J_X(A, B) \geq \sum_{i=1}^{4} J_X(A_i, B_i) - 8\,\varepsilon.$$

We let $\varepsilon \to 0$ to get the conclusion.

In the following steps we identify the different terms $J_X(A_i, B_i)$. Identity (4.14) is classical and we refer to F. Demengel & R. Temam [9] for the proof which is anyway very similar to those we give here to identify the other terms. We first need the following Lemma that will be proved later.

Lemma 4.5. *Let $\phi : \mathbb{R}^2 \times \mathbb{R}_+^2 \to \mathbb{R} \cup \{+\infty\}$ be such that:*

- for every $\xi \in \mathbb{R}^2$, $x \mapsto \phi(\xi, x)$ is measurable,

- for almost every $x \in \mathbb{R}_+^2$, $\xi \mapsto \phi(\xi, x)$ is a proper, l.s.c., strictly convex function,

- there exists $\phi_0 : \mathbb{R}^2 \to \mathbb{R} \cup \{+\infty\}$ such that $\phi_0(\xi) \to \infty$ when $\xi \to \infty$ and $\phi(\xi, x) \geq \phi_0(\xi)$ for all $\xi \in \mathbb{R}^2$ and almost every $x \in \mathbb{R}_+^2$.
Then, for almost every $x \in \mathbb{R}_+^2$, there exists an unique ξ_x such that $\phi(\xi_x, x) \leq \phi(\xi, x)$ for every $\xi \in \mathbb{R}^2$, and the map $x \mapsto \xi_x$ is measurable.

Moreover, if ϕ is l.s.c. in the two variables ξ and x, and if for every $\xi \in \mathbb{R}^2$, $x \mapsto \phi(\xi, x)$ is continuous, then $x \mapsto \xi_x$ is also continuous.

Finally, if $x = (k, k')$, $k \mapsto \phi(\xi, k, k')$ is continuous for every $\xi \in \mathbb{R}^2$ and almost every $k' \in \mathbb{R}_+$ and $(\xi, k) \mapsto \phi(\xi, k, k')$ is l.s.c. for almost every $k' \in \mathbb{R}_+$ then $k \mapsto \xi_{(k,k')}$ is continuous for almost every $k' \in \mathbb{R}_+$.

Step 2: proof of (4.16). We first remark that

$$J_X(A_4,B_4) = \sup_{(u,v)\in\mathcal{K}} \iint_{\mathbb{R}_+^2} b\,(e^{-k}\,u + e^{-k'}\,v)\,dG(k)dG(k')$$
$$\le \iint_{\mathbb{R}_+^2} b\,j(e^{-k},e^{-k'})\,dG(k)dG(k') = D_4(G),$$

so that we only have to prove the reverse inequality. We denote $\xi = (s,t)$ and $x = (k,k')$. Let us define

$$j_n^\star(\xi) = j^\star(\xi) + \frac{|\xi|^2}{n}\,H_n(\xi) \qquad \text{with} \qquad H_n(\xi) = \begin{cases} 1 & \text{if} \quad |\xi| \le n \\ \infty & \text{elsewhere.} \end{cases}$$

Then, the function $\phi : \mathbb{R}^2 \times \mathbb{R}_+^2 \to \mathbb{R}\cup\{+\infty\}$ defined by

$$\phi(\xi,x) = j_n^\star(\xi) - a\,s - b\,t, \qquad a(x) = e^{-k}, \quad b(x) = e^{-k'},$$

is a l.s.c., proper convex function such that $\phi(\xi,x) = +\infty$ if $|\xi| \ge n$. By Lemma 4.5, there exists an unique $\xi_x = (u_n(x),v_n(x))$ such that

$$\sup_{\xi\in\mathbb{R}^2}\; a\,s + b\,t - j_n^\star(\xi)\,] = [\,a\,u_n(x) + b\,v_n(x) - j_n^\star(\xi_x)\,]. \tag{4.23}$$

Moreover, the map $x \mapsto \xi_x$ is continuous with respect to x, and

$$(u_n(x),v_n(x)) \in K, \quad |(u_n(x),v_n(x))| \le n \qquad \text{for every } x \in \mathbb{R}_+^2.$$

We deduce that

$$j_n(a,b) = a\,u_n + b\,v_n - j_n^\star(u_n,v_n) \qquad \text{for every } x \in \mathbb{R}_+^2,$$

since $j_n^{\star\star} = j_n$. On the other hand, by definition of $j_n^\star$, we have $j_n \nearrow j$ pointwise, and then by monotone convergence

$$D_{4,n}(G) := \iint_{\mathbb{R}_+^2} b\,j_n(a,b)\,dG(k)dG(k') \nearrow D_4(G).$$

Fix now $\varepsilon > 0$, choose n large enough, we have

$$D_4(G) - \varepsilon \le D_{4,n}(G) \le \iint_{\mathbb{R}_+^2} \big[a\,u_n + b\,v_n - j_n^\star(u_n,v_n)\big]\,dG(k)dG(k')$$
$$\le \iint_{\mathbb{R}_+^2} \big[a\,u_n + b\,v_n - j^\star(u_n,v_n)\big]\,dG(k)dG(k') \le J_X(A_4,B_4),$$

and we let $\varepsilon \to 0$ to conclude.

Step 3: proof of (4.15). By the definition of $j^\star$ we have

$$J_X(A_2,B_2) \le J(A_2,B_2) := \iint_{\mathbb{R}_+^2} b(k,k')\,G\,j((k'^2+g')\,e^{-k'}, g'\,e^{-k})\,dkdk'.$$

We only need to show the reverse inequality. We use again the notations of step 2 with $a = (k'^2+g')\,e^{-k'}$ and $b = g'\,e^{-k}$. By Lemma 4.5, there exists an unique $\xi_x = (u_n(x),v_n(x))$ satisfying (4.23). This function is continuous with respect to k for almost every k', measurable in k' for every k and satisfies

$$j_n(a,b) = a\,u_n + b\,v_n - j_n^\star(u_n,v_n) \quad \text{for every} \quad k \in \mathbb{R}_+ \text{ and a.e. } k' \in \mathbb{R}_+.$$

If

$$\mathcal{K}' := \left\{ \begin{array}{l} (u,v) : \mathbb{R}_+^2 \to \mathbb{R}^2, \text{ bounded and measurable ; } (u(x),v(x)) \in K \quad \forall k \in \mathbb{R}_+, \\ \text{a.e. } k' \in \mathbb{R}_+, \ k \mapsto (u(k,k'),v(k,k')) \quad \text{is continous for almost every } k' \in \mathbb{R}_+. \end{array} \right\}$$

we obtain as in step 2

$$J(A_2,B_2) = \sup_{(u,v)\in\mathcal{K}'} < A_2, u > + < B_2, v > .$$

Finally, (4.15) follows by a density argument: if ρ_ε is an approximation of the identity, for every $(u,v) \in \mathcal{K}'$, the pair

$$u_\varepsilon(k,k') = (u(k,\cdot) *_{k'} \rho_\varepsilon)(k'), \quad v_\varepsilon(k,k') = (v(k,\cdot) *_{k'} \rho_\varepsilon)(k'),$$

satisfies, $(u_\varepsilon, v_\varepsilon) \in \mathcal{K}$ (this is trivial for step functions and follows in general by density). Moreover, as $\varepsilon \to 0$, $< A_2, u_\varepsilon > \to < A_2, u >$, $< B_2, v_\varepsilon > \to < B_2, v >$ and therefore

$$J(A_2,B_2) = \sup_{(u,v)\in\mathcal{K}'} < A_2, u > + < B_2, v > = \sup_{(u,v)\in\mathcal{K}} < A_2, u > + < B_2, v > . \qquad \square$$

Proof of Lemma 4.5. For every $n \in \mathbb{N}^\star$, we define the dyadic grid

$$D_n := \Delta_n^2, \qquad \Delta_n = \{\lambda \in \mathbb{Q}, \ |\lambda| \le 2^n \text{ and } 2^n \lambda \in \mathbb{Z}\},$$

and

$$R_x^n = \{s \in D_n, \ \phi(s,x) = \inf_{s \in D_n} \phi(s,x)\}.$$

This is a non empty set with, at most, four elements (in a given horizontal, vertical or diagonal line of D_n only two elements of D_n can belong to R_x^n because of the convexity of $\phi(.,x)$). We finally define r_x^n as the center of mass of the elements of R_x^n, so that we have built a measurable application $r^n : \mathbb{R}_+^2 \to \mathbb{R}^2$ such that

$$\phi(r_x, x) \le \phi(r, x) \quad \forall r \in D_n.$$

Moreover, for almost every $x \in \mathbb{R}_+^2$, the minimum ξ_x of $\phi(.,x)$ exists, is unique and, if n is large enough, satisfies $|\xi_x| \le 2^n$. Therefore, by the construction above we deduce that

$$|\xi_x - r_x^n| \le 2^{-n-1}.$$

This shows that, for almost every $x \in \mathbb{R}_+^2$, $\xi_x = \lim_{n\to\infty} r_x^n$, so that the map $x \mapsto \xi_x$ is measurable application from $\mathbb{R}_+^2$ to $\mathbb{R}^2$.

Assume now that $x \mapsto \phi(x,\xi)$ is continuous and that ϕ is l.s.c. with respect to the two variables $x \in \mathbb{R}_+^2$ and $\xi \in \mathbb{R}^2$. Let $(x_n) \subset \mathbb{R}_+^2$ be a sequence such that $x_n \to x \in \mathbb{R}_+^2$, denote $\xi_n = \xi_{x_n}$. Due to the condition of uniform lower bound at infinity, we know that (ξ_n) is bounded. Therefore, there exists a subsequence, still denoted by (ξ_n), and $\bar\xi \in \mathbb{R}^2$ such that $\xi_n \to \bar\xi$ and $\phi(\bar\xi, x) \ge \phi(\xi_x, x)$. Let us prove that the equality holds. If not, for some $\varepsilon > 0$ we have $\phi(\bar\xi, x) \ge \phi(\xi_x, x) + 3\,\varepsilon$. Then, by lower semicontinuity in both variables we have, for n large enough,

$$\phi(\xi_n, x_n) \ge \phi(\xi_x, x) + 2\,\varepsilon.$$

By continuity in x we have, for n large enough:

$$\phi(\xi_n, x_n) \ge \phi(\xi_x, x_n) + \varepsilon,$$

which contradicts the definition of ξ_n. We have thus proved that $\phi(\bar\xi, x) = \phi(\xi_x, x)$, which implies $\bar\xi = \xi_x$. Therefore, the whole sequence ξ_n converges to $\bar\xi$ and the map $x \mapsto \xi_x$ is continuous.

The last part of the Lemma is proved in a similar way; we have to consider the maps $k \mapsto \phi(\xi, k, k')$ which are continuous for every $\xi \in \mathbb{R}^2$ and $k' \in B$ where B is a Borel set of $\mathbb{R}_+$ such that B^c has measure zero. $\square$

We summarize the results obtained in this section in the following

Theorem 4.6.

(i) Let F be a non negative measure such that $M(F) < \infty$. Then, $M(k\,F) < \infty$ if and only if $-H(F) < \infty$. Moreover, $F \mapsto -H(F)$ is a continuous and convex function from $\mathcal{E}_0$ to $\mathbb{R}$.

(ii) Assume that (F_n) is a bounded sequence of $L^\infty(0,T;M^1(\mathbb{R}_+))$ which satisfies

$$\int_{\mathbb{R}_+} \psi\, dF_n(k) \underset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}_+} \psi\, dF(k) \tag{4.24}$$

strongly in $L^1(0,T)$ for any $\psi \in C_c(\mathbb{R}_+)$. Then we have

$$\int_0^T D(F)\, dt \le \liminf_{n\to\infty} \int_0^T D(F_n)\, dt. \tag{4.25}$$

Proof of Theorem 4.6. The point (i) of the Theorem is an immediate consequence of Lemma 4.1. Let us prove the point (ii) and assume that (4.24) holds. Then, there exists a subsequence $(F_{n'})$ of (F_n) such that (with obvious notation)

$$< b\, A_{n'}(t),\, u > + < b\, B_{n'}(t),\, v > \underset{n'\to\infty}{\longrightarrow} < b\, A(t),\, u > + < b\, B(t),\, v > \tag{4.26}$$

for any $(u,v) \in \mathcal{K}$ and for a.e. $t \in [0,T]$. Moreover, Theorem 4.5 and the Remark 4.4 imply that for a.e. $t \in [0,T]$ and every $\varepsilon > 0$ there exists $(u_\varepsilon, v_\varepsilon) \in \mathcal{K}$ such that

$$D(F(t,.)) = J_X(A(t),B(t)) \ \le < b\, A(t),\, u_\varepsilon > + < b\, B(t),\, v_\varepsilon > + \,\varepsilon.$$

Therefore, from (4.26) we get

$$D(F(t,.)) \ \le \ \liminf_{n'} \ < b\, A_{n'}(t),\, u_\varepsilon > + < b\, B_{n'}(t),\, v_\varepsilon > + \,\varepsilon \ \le \ \liminf_{n'} \ D(F_{n'}(t,.)) + \varepsilon,$$

for any $\varepsilon > 0$, so that $D(F(t,.)) \le \liminf\ D(F_{n'}(t,.))$ for a.e. $t \in [0,T]$. We conclude thanks to Fatou's Lemma. □

5. Existence and Uniqueness: proofs of Theorems 3, 4 and 5.

This section is devoted to the existence and uniqueness of solutions to the Cauchy problem (2.2)−(2.7).

Proof of Theorem 3. We follow the proof of L. Arkeryd [3]. We divide the proof in two steps. In the first we assume that the initial data is "well prepared", i.e. satisfies a technical hypothesis. In the second step we remove this unnecessary condition.

First Step: Assume F_{in} to be well prepared. Assume that the initial data $F_{in} = g_{in} + G_{in} \in M^1$ satisfies

$$\begin{cases} \exists \theta \in (0,1),\ \exists(\gamma_0, \Gamma_0);\ 0 < \gamma_0 \le \Gamma_0 \quad \text{such that} \ \ \gamma_0\, e^{-k} \le g_{in} \le \Gamma_0\, e^{-\theta\, k} \quad \forall k > 0, \\ \operatorname{supp} G_{in} \ \text{is a compact subset of } \mathbb{R}_+. \end{cases} \tag{5.1}$$

Define the auxiliary space

$$\mathcal{E}_T := \left\{ \begin{array}{c} F \in C([0,T], M^1);\ \operatorname{supp} G(t) \subset \operatorname{supp} G_{in},\ M(F(t)) = M(F_{in}), \\ \gamma(t)\, e^{-k} \le g(k,t) \le \Gamma(t)\, e^{-\theta\, k},\ G(t) \ge 0,\, \forall t \in [0,T],\ \forall k > 0 \end{array} \right\} \tag{5.2}$$

where

$$\tag{5.3} \begin{cases} \gamma(t) := \gamma_{in}\, e^{-C_0\, t}, \quad C_0 := b^\star\,(M(k^2\, e^{-k}) + m) \\ \Gamma(t) := \Gamma_{in}\, e^{C_0\, t} + C_1\,(e^{C_0\, t} - 1),\ C_1 := \sup\limits_{k\geq 0} k^2\, e^{(\theta-1)\, k}. \end{cases}$$

It is a closed subset of $C([0,T]; M^1(\mathbb{R}_+))$.

Given $F \in \mathcal{E}_T$ let $\bar{F}$ be the solution to

$$\tag{5.4} \begin{cases} \dfrac{\partial \bar{F}}{\partial t} + C_0\, \bar{F} = (k^2 + F)\, e^{-k}\, L(F) + \big(C_0 - L((k^2+F)\, e^{-k})\big)\, F, \\ \bar{F}(0,.) = F_{in}. \end{cases}$$

It is clear that $\bar{F} = \bar{g} + \bar{G}$ with $\bar{g} \in C([0,T], L^1)$, $\bar{G} \in C([0,T], M^1)$, $\operatorname{supp} \bar{G} \subset \operatorname{supp} G_{in}$. If we integrate the differential equation and the initial condition in (5.4), we obtain

$$\frac{d}{dt} M(\bar{F}) + C_0\, M(\bar{F}) = C_0\, M(F_{in}), \qquad M(\bar{F})(0) = M(F_{in}),$$

and so $M(\bar{F}) = M(F_{in})$ for every $t \in [0,T]$. Moreover, $M(F) = M(F_{in})$ implies $C_0 - L((k^2 + F)\, e^{-k}) \geq 0$ and

$$\frac{\partial \bar{F}}{\partial t} + C_0\, \bar{F} \geq 0.$$

It follows that $\bar{G} \geq 0$ and $\bar{g} \geq e^{-C_0\, t}\, \gamma_{in}\, e^{-k}$. Finally, if $\bar{u} := \sup_{k\geq 0} \bar{g}\, e^{\theta\, k}$ we obtain, by definition of $\Gamma(t)$,

$$\begin{cases} \dfrac{d\bar{u}}{dt} + C_0\, \bar{u} \leq C_1\, C_0 + 2\, C_0\, \Gamma = \dfrac{d\Gamma}{dt} + C_0\, \Gamma, \\ u(0) \leq \Gamma_{in} = \Gamma(0), \end{cases}$$

which implies $\bar{u} \leq \Gamma$ for all $t \in [0,T]$. This shows that $\bar{F} \in \mathcal{E}_T$.

On the other hand, for every $F_1, F_2 \in \mathcal{E}_T$, the corresponding solutions $\bar{F}_1, \bar{F}_2$ to (5.4) satisfy

$$\frac{d}{dt} \|\bar{F}_2 - \bar{F}_1\| + C_0\, \|\bar{F}_2 - \bar{F}_1\| \leq 6\, C_0\, \|F_2 - F_1\|,$$

where here and below $\|\ \| = \|\ \|_{M^1}$ stands for the total variation norm in $M^1(\mathbb{R}_+)$. We get

$$\tag{5.5} \sup_{[0,T]} \|\bar{F}_2 - \bar{F}_1\| \leq 6\, \big(1 - e^{-C_0\, T}\big) \sup_{[0,T]} \|F_2 - F_1\|.$$

This implies that for T small enough so that

$$\tag{5.6} 6\, \big(1 - e^{-C_0\, T}\big) < 1 \iff T < \frac{\ln 6 - \ln 5}{C_0},$$

the map $F \mapsto \bar{F}$ is a contraction from $\mathcal{E}_T$ into itself. This map admits an unique fixed point that we denote by F, which is the unique solution of (2.2) belonging to $\mathcal{E}_T$. Observe moreover that, by (5.6), the time existence interval $[0,T]$ of this solution is such that $T > \dfrac{\ln 6 - \ln 5}{2C_0}$. Therefore, by iteration of this argument we obtain a global solution F of (2.2) satisfying $F(t) \in \mathcal{E}_T$ for all $T \geq 0$. This solution is actually unique in $C([0,\infty), M^1)$. If our solution has the form $F = g + G$, then

$$e^{-k} \leq \frac{g + k^2}{g}\, e^{-k} \leq e^{-k} + \gamma^{-1}\, k^2.$$

Therefore the function $h'(g,k) \equiv \ln(\frac{g+k^2}{g} e^{-k})$ satisfies

$$-1 \le \frac{h'(g,k)}{k} \le \frac{\ln(e^{-k} + \gamma^{-1} k^2)}{k} \in L^\infty(\mathbb{R}_+), \;\; Q^\pm(F,F)\, k \in M^1(\mathbb{R}_+).$$

and so

$$\frac{h'(g,k)}{k} \in L^\infty(\mathbb{R}_+).$$

Using that $b \in L^\infty$, it is then trivial to check, that

$$Q_1^\pm(g,G)\, h'(g,k) \in L^1 \quad \text{and} \quad Q_2^\pm(g,G)\, k \in M^1.$$

Therefore, formulas (2.12), (2.13) and (2.14) hold actually (the calculation in Section 2 makes sense) true for our solution F and

$$H(F)(t_2) - H(F)(t_1) = \frac{1}{2} \int_{t_1}^{t_2} D(F)(\tau)\, d\tau, \tag{5.7}$$

for every $t_1, t_2 \ge 0$.

Second Step: General initial data. Suppose now that $F_{in} \in \mathcal{E}_0$, i.e. $M(F_{in}) < \infty$ and $-H(F_{in}) < \infty$. Define $F_{in}^n := g_{in}^n + G_{in}^n$ where $g_{in}^n := g_{in} \wedge (n\, e^{-k}) + \frac{1}{n}\, e^{-k}$ et $G_{in}^n := G_{in}\, \mathbf{1}_{k \in [0,n]}$. By the first step, there exists a solution F^n with initial datum F_{in}^n and $M(F^n) = M(F_{in}^n)$ with

$$H(F^n)(t_2) - H(F^n)(t_1) = \frac{1}{2} \int_{t_1}^{t_2} D(F^n)(\tau)\, d\tau, \tag{5.8}$$

for every $t_1, t_2 \ge 0$. Since $F_{in}^n \to F_{in}$ in $\mathcal{E}_0$, we have $H(F_{in}^n) \to H(F_{in})$. Therefore, for n large enough,

$$H(F^n) \ge H(F_{in}) - 1 \qquad \text{and} \qquad \int_0^\infty D(F^n)\, ds \le 2\,(S(f_0) + 1 - H(F_{in})).$$

By Lemma 4.1 we deduce that for some positive constant C_3

$$M(k\, F^n) \le C_3, \quad \forall t > 0, \; \forall n.$$

Moreover,

$$\begin{aligned}\frac{\partial}{\partial t}\big((1+k)\,(F^m - F^n)\big) &= (1+k)\,(k^2 + F^m)\, e^{-k}\, L(F^m - F^n) + (1+k)\, e^{-k}\,(F^m - F^n)\, L(F^m) \\ &\quad + L((k^2 + F^m)\, e^{-k})\,(1+k)\,(F^m - F^n) + (1+k)\, F^m\, L((F^m - F^n)\, e^{-k}),\end{aligned}$$

so that

$$\begin{aligned}\frac{d}{dt}\|(1+k)\,(F^m - F^n)\| &\le C_2\,\big(1 + \|(1+k)\,F^m\|\big)\,\|(1+k)\,(F^m - F^n)\| \\ &\le C_2\,\big(1 + C_3\big)\,\|(1+k)\,(F^m - F^n)\|.\end{aligned}$$

Finally

$$\|(1+k)\,(F^m - F^n)(t)\| \le \|(1+k)\,(F_{in}^m - F_{in}^n)\|\, e^{C_2\,(1+C_3)t}.$$

This shows that (F^n) is a Cauchy sequence in $C([0,\infty); \mathcal{E}_0)$ and thus converges to a limit $F \in C([0,\infty); \mathcal{E}_0)$. This function F trivially satisfies $M(F(t)) = M(F_{in})$ for every $t > 0$.

On the other hand, by construction, for every n, the function F^n satisfies

$$\int_{\mathbb{R}_+} F^n(t,k)\, \phi(t,k)\, dk = \int_{\mathbb{R}_+} F_{in}^n(k)\, \phi(0,k)\, dk + \int_0^t \int_{\mathbb{R}_+} Q(F^n, F^n)\, \phi(t,k)\, dk ds, \tag{5.9}$$

$\forall \phi \in C_c([0,\infty) \times \mathbb{R}_+)$ and

$$H(F^n(t_2,.)) - H(F^n(t_1,.)) = \int_{t_1}^{t_2} D(F^n(s,.))\,ds \qquad \text{for all } t_2 \geq t_1 \geq 0. \tag{5.10}$$

Passing to the limit in the equation (5.9) we deduce first that F satisfies (2.19) with initial datum F_{in}. Moreover, by lower semi continuity, we deduce (2.20) from (5.10). As a consequence, F is an entropy solution of (2.2)−(2.7). □

Remark 5.1. When b satisfies (2.2) with $\eta \in [0,1/4)$ and $F_{in} \in \mathcal{E}_{2\,\eta}$ we can prove the existence of a solution $F \in C([0,T];\mathcal{E}_\eta) \cap L^\infty(0,T;\mathcal{E}_{2\,\eta})$ of (2.5). Indeed, performing the same kind of computation that we present in the proof of Theorem 4 we get the (formal) a priori bound

$$\sup_{[0,T]} Y_{2\,\eta}(F) \leq C_T. \tag{5.11}$$

Then, we establish that the sequence we introduce in (5.12) satisfies (5.11) and we prove that (F_n) is a Cauchy sequence in $C([0,T];\mathcal{E}_\eta)$.

Remark 5.2. Using Gronwall Lemma we establish without difficulty that (2.5) has at most one solution in the class $C([0,T];\mathcal{E}_\eta) \cap L^\infty(0,T;\mathcal{E}_{2\,\eta})$ when $\eta \in [0,1/2)$. This provides uniqueness result under general assumption (2.2) on the cross-section b in the following two cases:

- if $\eta \in [0,1/4)$ then there exists an unique solution to (2.5);
- if $\eta \in [0,1/2)$ and $0 \leq g_{in} \leq g_0$ then there exists an unique solution g to (2.5) such that $0 \leq g \leq g_0$.

Proof of Theorem 4. We follow in this demonstration the arguments introducing in [32]. Let $\tau > 0$ satisfy $\theta + 3\tau < \min(\theta', \eta, 1-\eta)$, and assume that

$$Y_{\theta'}(F) := \int_{\mathbb{R}_+} e^{\theta'\,k}\,d|F|(k) < \infty.$$

We define F^n_{in} as in the previous step and consider the cross section $b_n(k,k') := \sigma(k'-k)\,e^{\eta\,k_n}\,e^{\eta\,k'_n}$, $k_n = k \wedge n = \min(k,n)$. Let $F_n \in C([0,\infty), M^1(\mathbb{R}_+))$ be the solution to

$$\begin{cases} \dfrac{\partial F_n}{\partial t} = e^{\eta\,k_n - k}\,(k^2 + F_n)\,\ell(e^{\eta\,k_n}\,F_n) - e^{\eta\,k_n}\,F_n\,\ell(e^{\eta\,k_n - k}\,(F_n + k^2)) \\ F_n(0,.) = F^n_{in}, \end{cases} \tag{5.12}$$

given by Step 1; with the notations $\ell(\phi) = \displaystyle\int_{\mathbb{R}_+} \sigma\phi'\,dk'$. We also may assume that

$$Y_{\theta'}(F^n_{in}) \leq Y_{\theta'}(F_{in}) + 1. \tag{5.13}$$

If we multiply equation (5.12) by $e^{(\theta+3\tau)\,k}$ and integrate with respect to k we obtain

$$\frac{d}{dt} Y_{\theta+3\tau}(F_n) \leq \sigma^\star\,\|k^2\,e^{(\theta+3\tau+\eta-1)\,k} + F_n\|\,\|e^{\eta\,k_n}\,F_n\| - \sigma_\star\|I_n(k)\,e^{(\theta+3\tau)\,k+\eta\,k_n}\,F_n\|,$$

where we have set

$$I_n(k) := \int_{\mathbb{R}_+} k'^2\,e^{\eta\,k'_n - k'}\,e^{-\nu\,|k'-k|^\gamma}\,dk'.$$

It is clear that there exists a positive constant α_τ such that $I_n(k) \geq \alpha_\tau\,e^{-\tau\,k}$, so that

$$\|I_n(k)\,e^{(\theta+3\tau)\,k+\eta\,k_n}\,F_n\| \geq \alpha_\tau\,\|e^{(\theta+2\tau)\,k+\eta\,k_n}\,F_n\|. \tag{5.14}$$

Moreover,

$$\|e^{\eta\, k_n}\, F_n\| = \int_0^R e^{\eta\, k_n}\, F_n\, dk + \int_R^\infty e^{\eta\, k_n+(\theta+2\tau)\, k}\ e^{-(\theta+2\tau)\, k} F_n\, dk$$
$$\le \int_0^R e^{\eta\, k_n}\, F_n\, dk + e^{-(\theta+2\tau)\, R} \int_R^\infty e^{\eta\, k_n+(\theta+2\tau)\, k}\ F_n\, dk,$$

so that, for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\|e^{\eta\, k_n}\, F_n\| \le \varepsilon\, \|e^{\eta\, k_n+(\theta+2\tau)\, k}\, F_n\| + C_\varepsilon\, M(F_n). \tag{5.15}$$

Therefore, for $\varepsilon > 0$ sufficiently small we obtain, for some positive constants C_1 and C_2

$$\frac{d}{dt} Y_{\theta+3\tau}(F_n) + C_1\, \|e^{(\theta+2\tau)\, k+\eta\, k_n}\, F_n\| \le C_2. \tag{5.16}$$

We deduce

$$\sup_{[0,T]} \|e^{(\theta+2\tau)\, k+\tau\, k_n)}\, F_n\| \le C_T \qquad \text{and} \qquad \int_0^T \|e^{(\theta+2\tau)\, k+\eta\, k_n}\, F_n\|\, dt \le \frac{C_T}{C_1}, \tag{5.17}$$

where $C_T := C_2\, T + Y_{\theta+3\tau}(F_{in}) + 1$.

We prove now that (F_n) is a Cauchy sequence in the $Y_{\theta+\tau}$ norm. For every $m > n$

$$\begin{aligned}
\frac{\partial}{\partial t}(F_m - F_n) =\ & (k^2 + F_m)\, e^{-k}\, L_n(F_m) - F_m\, L_n((k^2 + F_m)\, e^{-k}) \\
& - (k^2 + F_n)\, e^{-k}\, L_n(F_n) + F_n\, L_n((k^2 + F_n)\, e^{-k}) \\
& + (k^2 + F_m)\, e^{-k}\, L_m(F_m) - F_m\, L_m((k^2 + F_m)\, e^{-k}) \\
& - (k^2 + F_m)\, e^{-k}\, L_n(F_m) + F_m\, L_n((k^2 + F_m)\, e^{-k}) \\
=\ & (F_m - F_n)\, e^{-k}\, L_n(F_m) + (k^2 + F_n)\, e^{-k}\, L_n(F_m - F_n) \\
& + (F_n - F_m)\, L_n((k^2 + F_m)\, e^{-k}) + F_n\, L_n((F_n - F_m)\, e^{-k}) \\
& + (k^2 + F_m)\, e^{-k}\, (e^{\eta\, k_m} - e^{\eta\, k_n})\, \ell(e^{\eta\, k_m}\, F_m) + (k^2 + F_m)\, e^{\eta\, k_n - k}\, \ell((e^{\eta\, k_m} - e^{\eta\, k_n})\, F_m) \\
& + F_m\, e^{\eta\, k_m}\, \ell((e^{\eta\, k_n} - e^{\eta\, k_m})\, (k^2 + F_m)\, e^{-k}) + F_m\, (e^{\eta\, k_n} - e^{\eta\, k_m})\, \ell(e^{\eta\, k_n}\, (k^2 + F_m)\, e^{-k}),
\end{aligned}$$

with the obvious notations $L_n(\phi) = \int_{\mathbb{R}_+} b_n\, \phi'\, dk'$. Then, we compute

$$\begin{aligned}
\frac{d}{dt} Y_{\theta+\tau}(F_m - F_n) \le\ & \sigma^\star\, \|F_m - F_n\|\, \|e^{\eta\, k_n}\, F_m\| + \sigma^\star\, C_1\, \|e^{\eta\, k_n}\, (F_m - F_n)\| \\
& - \sigma_\star\, \alpha_\tau\, \|e^{\theta\, k+\eta\, k_n}\, (F_m - F_n)\| + \sigma^\star\, \|e^{(\theta+\tau)\, k+\eta\, k_n}\, F_n\|\, \|F_m - F_n\| \\
& + \sigma^\star\, \|(k^2 + F_m)\, b_{m,n}\, e^{(\theta+\tau-1)\, k}\|\, \|e^{\eta\, k_m}\, F_m\| + \sigma^\star\, C_1\, \|b_{m,n}\, F_m\| \\
& + \sigma^\star\, \|b_{m,n}\, (k^2 + F_m)\, e^{-k}\|\, \|F_m\, e^{(\theta+\tau)\, k} e^{\eta k_m}\| + \sigma^\star\, C_1\, \|b_{m,n}\, e^{(\theta+\tau)\, k}\, F_m\|
\end{aligned}$$

where we have set $C_1 := Y_{\theta+\tau+\eta-1}(k^2) + M(F_{in})$, which is finite since $\theta+2\tau < 1-\eta$, and $b_{m,n} := e^{\eta\, k_m} - e^{\eta\, k_n}$. We estimate now each of the terms of the right hand side.

First, it is clear that

$$\begin{cases} \|F_m - F_n\|\, \|e^{\eta\, k_n}\, F_m\| \le \|e^{(\theta+\tau)\, k}(F_m - F_n)\|\, \|e^{\eta\, k_n+(\theta+2\tau)\, k}\, F_m\| \\ \|F_m - F_n\|\, \|e^{(\theta+\tau)\, k+\eta\, k_n}\, F_n\| \le \|e^{(\theta+\tau)\, k}(F_m - F_n)\|\, \|e^{\eta\, k_n+(\theta+2\tau)\, k}\, F_n\|. \end{cases} \tag{5.18}$$

Using the same argument as in the proof of (5.15) it is clear that for every $\varepsilon > 0$ there exists a positive constant C_ε such that

$$\|e^{\eta\, k_n}\, (F_m - F_n)\| \le \varepsilon\, \|e^{\theta\, k+\eta\, k_n}\, (F_m - F_n)\| + C_\varepsilon\, \|(F_m - F_n)\, e^{(\theta+\tau)\, k}\| \tag{5.19}$$

We observe now that $b_{m,n} \le e^{\eta\, k_m + \tau\, k}\, e^{-\tau\, n}$, so that

$$
\begin{cases}
\|b_{m,n}\,(k^2 + F_m)\, e^{(\theta+\tau-1)\,k}\| \le e^{-\tau\,n}\, C_1, \\
\|b_{m,n}\, F_m\| \le e^{-\tau\,n}\, \|e^{\eta\,k_m + (\theta+2\tau)\,k}\, F_m\|, \\
\|b_{m,n}\, F_m\, e^{(\theta+\tau)\,k}\| \le e^{-\tau\,n}\, \|e^{\eta\,k_m + (\theta+2\tau)\,k}\, F_m\|.
\end{cases} \tag{5.20}
$$

From (5.18)$-$(5.20) we deduce

$$
\frac{d}{dt} Y_{\theta+\tau}(F_m - F_n) \le g_{m,n}\,\big(Y_{\theta+\tau}(F_m - F_n) + e^{-\tau\,n}\big),
$$

where $g_{n,m} := C_4\,(1 + \|F_n e^{(\theta+2\tau)k + \eta k_n}\| + \|F_m e^{(\theta+2\tau)\,k + \eta k_m}\|)$. Finally, since $g_{n,m}$ is bounded in $L^1([0,T])$ uniformly with respect to n and m, we obtain, by Gronwall's lemma

$$
\sup_{[0,T]} Y_{\theta+\tau}(F_m - F_n) \underset{n,m\to\infty}{\longrightarrow} 0.
$$

Therefore, (F_n) is a Cauchy sequence for the $Y_{\theta+\tau}$ norm and converges to a limit, say, F. We easily pass to the limit in (5.12) and we get that F is an entropy solution to (2.2)–(2.7). Furthermore, passing to the limit in (5.16), we get

$$
\frac{d}{dt} Y_{\theta+2\tau}(F) + C_1\, Y_{\theta+2\tau+\eta}(F) \le C_2 \qquad \text{on} \quad (0,T),
$$

where C_i do not depend on T. We deduce of this differential inequality that

$$
\sup_{[0,\infty)} Y_{\theta+2\tau}(F) < \infty \quad \text{and} \quad \int_0^\infty Y_{\theta+2\tau+\eta}(F)\, dt < \infty.
$$

□

Remark 5.3. In fact, when $\eta < 1/2$, we can also prove (see [32])

$$
\sup_{[0,\infty)} Y_\eta(F) < \infty \qquad \text{if} \qquad Y_\eta(F_{in}) < \infty.
$$

Proof of Theorem 5. Assume now that

$$
F_{in} = g_{in} + \alpha_{in}\,\delta_0, \qquad \text{with} \qquad 0 \le g_{in} \le g_0 \quad \text{and } \alpha_{in} \ge 0.
$$

First Step. Suppose first that $b \in L^\infty$ and define the space

$$
\mathcal{F}_T := \{F = g + \alpha\,\delta_0,\ g \in C([0,T];L^1),\ 0 \le g \le g_0,\ \alpha \in C([0,T]),\ \alpha \ge 0\}.
$$

For every $F \in \mathcal{F}_T$ consider the solution $\bar F \in C^1([0,T], M^1)$ to the equation

$$
\begin{cases}
\dfrac{\partial \bar F}{\partial t} + \bar F\,\big[L((k^2 + F)\,e^{-k}) - e^{-k}\,L(F)\big] = k^2\,e^{-k}\,L(F), \\
\bar F(0,.) = F_{in}.
\end{cases} \tag{5.21}
$$

We remark that $g \le g_0$ and $F = g + \alpha\,\delta_0$ implies that $F \le (k^2 + F)\,e^{-k}$. Therefore,

$$
L((k^2 + F)\,e^{-k}) - e^{-k}\,L(F) \ge L((k^2 + F)\,e^{-k}) - L(F) \ge 0.
$$

Multiplying the equation by $-\mathbf{1}_{\bar F\le 0}$, we get, after integration,

$$\frac{d}{dt}\int_{\mathbb{R}_+} \bar F^-\,dk \le 0, \qquad \int_{\mathbb{R}_+} \bar F^-(0,k)\,dk = 0.$$

This implies $\bar F^- \equiv 0$ and $\bar F \ge 0$. We then rewrite equation (5.21)

$$\begin{aligned}\frac{\partial \bar F}{\partial t} &= (k^2+\bar F)\,e^{-k}\,L(F) - (k^2+g_0)\,e^{-k}\,L(F) + (k^2+g_0)\,e^{-k}\,L(F)\\ &\quad - g_0\,L((k^2+F)\,e^{-k}) + g_0\,L((k^2+F)\,e^{-k}) - \bar F\,L((k^2+F)\,e^{-k}),\end{aligned}$$

so that

$$\frac{\partial}{\partial t}(\bar F - g_0) + (\bar F - g_0)\left[L((k^2+F)\,e^{-k}) - e^{-k}\,L(F)\right] = g_0\left[L(F) - L((k^2+F)\,e^{-k})\right], \tag{5.22}$$

since $(k^2+g_0)\,e^{-k} = g_0$. Multiplying (5.22) by $\mathbf{1}_{\bar F - g_0 \ge 0}$ we obtain

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}_+}(\bar F - g_0)^+\,dk &\le -\int_{\mathbb{R}_+}(\bar F - g_0)^+\left[L((k^2+F)\,e^{-k}) - e^{-k}\,L(F)\right]dk\\ &\quad + \int_{\mathbb{R}_+} g_0\left[L(F) - L((k^2+F)\,e^{-k})\right]dk \le 0,\end{aligned}$$

which implies $\bar g \le g_0$ and $\bar\alpha \le \alpha_{in}$. This shows that the application $F \mapsto \bar F$ maps $\mathcal{F}_T$ into itself. Moreover, for $F_1, F_2 \in \mathcal{F}_T$ we have

$$\begin{aligned}&\frac{\partial}{\partial t}(\bar F_2 - \bar F_1) + (\bar F_2 - \bar F_1)\left[L((k^2+F_2)\,e^{-k}) - e^{-k}\,L(F_2)\right] \le\\ &\qquad \le \bar F_1\left[L((F_1-F_2)\,e^{-k}) - e^{-k}\,L(F_1-F_2)\right] + k^2\,e^{-k}\,L(F_2-F_1).\end{aligned}$$

Therefore,

$$\frac{d}{dt}\|\bar F_2 - \bar F_1\|_{M^1} \le b^\star\,(2\,M(g_0+\alpha_{in}\,\delta_0) + M(k^2\,e^{-k}))\,\|F_2 - F_1\|_{M^1}.$$

By the Banach contraction Theorem the map $F \mapsto \bar F$ has an unique fixed point in $\mathcal{F}_T$.

Second Step. Assume now that the cross section b satisfies the condition $0 \le b\,e^{-\eta\,k}\,e^{-\eta\,k'} \in L^\infty$ with $\eta \in [0,1)$. Let us define

$$b_n(k) = b(k) \wedge n.$$

From the first step we know that for any $n \ge 0$, there exists a solution $F_n \in \mathcal{F}_T$ to

$$\frac{\partial F_n}{\partial t} = \int_{\mathbb{R}_+} b_n(k,k')\big(F_n'\,(k^2+F_n)\,e^{-k} - F_n\,(k'^2+F_n')\,e^{-k'}\big),$$

with initial datum F_{in}. Moreover F_n satisfies

$$0 \le F_n \le g_0 + \alpha_{in}\,\delta_0 \qquad \text{and} \qquad \int_0^T D(F_n)\,dt \le H(\mathcal{B}_m) - H(F_{in}).$$

We can pass to the limit $n \to \infty$ using the fact that $F \mapsto D(F)$ is s.c.i. and that the averages in k of F_n strongly converge. We obtain a solution F which satisfies

$$0 \le F \le g_0 + \alpha_{in}\,\delta_0 \qquad \text{and} \qquad \int_0^T D(F)\,dt \le H(\mathcal{B}_m) - H(F_{in}),$$

but we do not know whether the entropy is not decreasing or F is unique. □

6. Asymptotic behavior when $t \to \infty$.

We consider in this section the solution $F(t)$ of (2.2) given by Theorem 3, 4 or 5 and associated to the initial datum F_{in} with $M(F_{in}) = m > 0$. For a given sequence (t_n) such that $t_n \nearrow +\infty$ and $T > 0$ we set

$$F_n(t,k) := F(t + t_n, k).$$

We first prove, following the arguments introduced by Arkeryd [3], [4], that F_n weakly converges to the appropriate Bose-Einstein state (the one corresponding to the mass of the initial datum). It is clear that F_n is still a solution of (2.2) and satisfies

$$\begin{cases} M(F_n(t,.)) = m \qquad \forall t \in [0,T], \ \forall n \geq 0, \\ -H(F_n(t,.)) \leq -H(F_{in}) \qquad \forall t \in [0,T], \ \forall n \geq 0, \\ \displaystyle\int_0^T D(F_n(s,.))\, ds \underset{n\to\infty}{\longrightarrow} 0. \end{cases} \tag{6.1}$$

Moreover if $\eta \in (0,1)$, there exists $0 < \theta < \min(\eta, 1-\eta)$ and C_θ such that

$$\sup_{[0,T]} Y_\theta(F_n) \leq C_\theta, \qquad \int_0^T Y_{\eta+\theta}(F_n)\, dt \leq C_\theta.$$

Lemma 6.1 *The sequence (F_n) with $F_n = g_n + G_n$ satisfies*

$$G_n \ \to \ 0 \quad \textit{in} \quad \big(C_c([0,T]\times\mathbb{R}_+)\big)' \qquad \textit{and} \qquad F_n \ \rightharpoonup \ \mathcal{B}_m \quad \textit{weakly in} \quad \big(C_c([0,T]\times\mathbb{R}_+)\big)'.$$

Proof of Lemma 6.1. By (6.1) we know that, for a subsequence $(t_{n'})$ there exists $F_{1,\infty}, F_{2,\infty} \in L^\infty(0,T;M^1)$ such that

$$g_{n'} \ \rightharpoonup \ F_{1,\infty} \quad \text{and} \quad G_{n'} \ \rightharpoonup \ F_{2,\infty} \quad \text{weakly in} \quad \big(C_c([0,T]\times\mathbb{R}_+)\big)', \tag{6.2}$$

with $M(F_{1,\infty}) + M(F_{2,\infty}) = m$ for almost every $t \in [0,T]$. Moreover, by lower semi continuity

$$\int_0^T D_3(F_{2,\infty})\, dt \leq \liminf_{n'\to\infty} \int_0^T D_3(G_{n'})\, dt = 0,$$

which implies that $F_{2,\infty}$ is supported in a single point. By Theorem 3,

$$\int_0^T D(F_{1,\infty} + F_{2,\infty})\, dt \leq \liminf_{n'\to\infty} \int_0^T D(F_{n'})\, dt = 0.$$

Therefore, $F_{1,\infty} + F_{2,\infty} = \mathcal{B}_m$ for almost every $t \in [0,T]$. This implies $\operatorname{supp} F_{2,\infty} \subset \{0\}$ and $M(G_{n'}\,\mathbf{1}_{[k_0,\infty)}) \to 0$ for all $k_0 > 0$. Moreover, since the limit is uniquely identified, the limits are taken by the whole sequences (F_n) and (G_n). □

Proof of Theorem 6. The first part of (2.25) has been proved in Lemma 6.1. In order to prove that $g_n(t,\cdot) \to g_\mu$ in $L^1([k_0,\infty))$ for every $k_0 > 0$, we follow the approach of P.-L. Lions [26]. We claim first that

$$\frac{Q^+(g_n, g_n)}{k^2 + g_n} = e^{-k}\, L(g_n) \underset{n\to\infty}{\longrightarrow} e^{-k}\, L(g_\mu) \quad \text{strongly in } L^1([0,T]\times\mathbb{R}_+). \tag{6.3}$$

To show this we consider the system (2.3) and multiply the first equation by any function $\chi \in L^\infty(\mathbb{R}_+)$ to obtain

$$\frac{d}{dt}\int_{\mathbb{R}_+} g_n\,\chi\,dk =< Q_1(g_n,G_n),\chi >,$$

which is bounded in $L^\infty(0,T)$. This implies that $< g_n,\chi >\to< g_\mu,\chi >$ strongly in $L^1(0,T)$. Moreover, if the solution is given by Theorem 3, $M(kg_n)$ is bounded in $L^1(0,T)$. If it is given by Theorem 4 or 5, $Y_{\theta+2\tau}(g_n)$ is bounded in $L^1(0,T)$. Therefore,

$$\begin{aligned}
\|e^{-k}(L(g_n)-L(g_\mu))\|_{L^1((0,T)\times\mathbb{R}_+)} &= \int_0^T\int_{\mathbb{R}_+} e^{-k}\left|\int_{\mathbb{R}_+} b(k,k')(g_n(k',t)-g_\mu(k'))\,dk'\right| dtdk\\
&\le \int_0^T\int_{\mathbb{R}_+} e^{-k}\left|\int_0^R b(k,k')(g_n(k',t)-g_\mu(k'))\,dk'\right| dtdk\\
&+\sigma^\star\int_0^T\int_{\mathbb{R}_+} e^{-k+\eta k}\int_R^\infty e^{\eta k'}\left|g_n(k',t)-g_\mu(k')\right| dk'dtdk \equiv E_1+E_2.
\end{aligned}$$

Observe that

$$\begin{aligned}
E_2 &\le \sigma^\star\, e^{-(\theta+2\tau)R}\int_0^T\int_{\mathbb{R}_+} e^{-k+\eta k}\int_R^\infty e^{\eta k'+(\theta+2\tau)k'}\left|g_n(k',t))-g_\mu(k')\right| dk'dtdk\\
&\le \sigma^\star\, e^{-(\theta+2\tau)R}\int_{\mathbb{R}_+} e^{-k+\eta k}\left(Y_{\theta+2\tau+\eta}(g_n)+Y_{\theta+2\tau+\eta}(g_\mu)\right) dk.
\end{aligned} \tag{6.4}$$

On the other hand, since (g_n) is bounded in $L^1((0,T)\times\mathbb{R}_+)$ we deduce from the above remark that for every $R>0$ and almost every $k>0$,

$$\lim_{n\to\infty}\int_0^T|\int_0^R b(k,k')\,(g_n(k',t)-g_\mu(k'))\,dk'|\,dtdk=0. \tag{6.5}$$

We deduce (6.3) from (6.4) and (6.5).

The same arguments prove that

$$L((k^2+g_n)\,e^{-k})\underset{n\to\infty}{\longrightarrow} L((k^2+g_\mu)\,e^{-k})\quad\text{strongly in } L^1([0,T]\times\mathbb{R}_+). \tag{6.6}$$

On the other hand, we use the elementary inequality

$$|b-a|\le\varepsilon\,b+\frac{1}{\ln(1+\varepsilon)}(a-b)\,(\ln\,a-\ln\,b)\qquad\forall a,\,b,\,\varepsilon>0,$$

with $a=(k^2+g)\,e^{-k}\,g'$ and $b=(k'^2+g')\,e^{-k'}=g$. We obtain, for all $\varepsilon>0$,

$$\begin{aligned}
\left\|e^{-k}\,L(g_n)-\frac{g_n}{k^2+g_n}\,L((k^2+g_n)\,e^{-k})\right\|_{L^1(\mathbb{R}_+)} &\le\varepsilon\,\left\|e^{-k}\,L(g_n)\right\|_{L^1(\mathbb{R}_+)}\\
&+\frac{1}{\ln(1+\varepsilon)}\,D(g_n)\qquad a.e.\ t\in[0,T].
\end{aligned} \tag{6.7}$$

Since $e^{-k}\,L(g_n)$ converges in $L^1(\mathbb{R}_+)$ its L^1-norm is bounded. Since $D(g_n)\to0$ as $n\to\infty$, we deduce from (6.7) that

$$\lim_{n\to\infty}||\frac{g_n}{k^2+g_n}\,L((k^2+g_n)\,e^{-k})-e^{-k}\,L(g_\mu)||_{L^1((0,T)\times\mathbb{R}_+)}=0. \tag{6.8}$$

From (6.6) and (6.8) we deduce that there exists a subsequence $(g_{n'})$ such that

$$\begin{cases} \dfrac{g_{n'}}{k^2+g_{n'}}\, L((k^2+g_{n'})\, e^{-k}) \to e^{-k}\, L(g_\mu) & \text{for almost every } k>0,\ t\in(0,T), \\ L((k^2+g_{n'})\, e^{-k}) \to L((k^2+g_\mu)\, e^{-k}) & \text{for almost every } k>0,\ t\in(0,T). \end{cases}$$

Therefore

$$\frac{g_{n'}}{k^2+g_{n'}} \to \frac{e^{-k}\, L(g_\mu)}{L((k^2+g_\mu)\, e^{-k})}$$

and

$$g_{n'} \to g_\mu \quad \text{for almost every } k>0,\ t\in(0,T).$$

In order to conclude we use the following classical lemma, which we state below and prove at the end of the section.

Lemma 6.2. *Let (X,d) be a metric space with its Borel sets. Assume that (u_n) is a sequence of L^1 functions such that for a given $u\in L^1$ one has*

$$u_n \geq 0, \qquad u_n \to u \text{ a.e.}, \qquad u_n \rightharpoonup u\ \sigma(M^1, C_c).$$

Therefore, $u_n \to u$ strongly in L^1_{loc}.

Using Lemma 6.2, we have thus proved that, for every $k_0>0$

$$g_n \to g_\mu \quad \text{in} \quad L^1((0,T)\times[k_0,\infty).$$

We deduce that there exists $t_0\in[0,T]$ such that

$$g_n(t_0,.) \to g_\mu \quad \text{in} \quad L^1([k_0,\infty)).$$

Moreover,

$$Q_1(g_n,G_n) \to 0 \quad \text{in} \quad L^1((0,T)\times[k_0,\infty)).$$

Therefore,

$$\begin{aligned} \sup_{[0,T]} \big\|g_n(t,.)-g_\mu\big\|_{L^1([k_0,\infty))} &\leq \big\|g_n(t_0,.)-g_\mu\big\|_{L^1([k_0,\infty))} + \sup_{[0,T]}\big\|g_n(t,.)-g_n(t_0,.)\big\|_{L^1([k_0,\infty))} \\ &\leq \big\|g_n(t_0,.)-g_\mu\big\|_{L^1([k_0,\infty))} + \int_0^T \big\|Q_1(g_n,G_n)(s,.)\big\|_{L^1([k_0,\infty))}\, ds \to 0, \end{aligned}$$

which ends the proof of Theorem 6. $\square$

Proof of the Lemma 6.2. Remark that $-a\leq b-|b-a|\leq a$ for all $a,b\geq 0$ so that we can apply the dominated convergence theorem with $v_n := u_n - |u_n-u|$. We deduce that $v_n\to u$ strongly in $L^1(X)$. Then for any given $\chi\in\mathcal{D}(X)$, say $0\leq\chi\leq 1$, one has

$$\int_X \big(u_n-|u_n-u|\big)\,\chi \to \int_X u\,\chi \quad \text{and} \quad \int_X u_n\,\chi \to \int_X u\,\chi,$$

so that

$$\int_X |u_n-u|\,\chi \to 0.$$

$\square$

7. The Kompaneets Limit: proof of Theorem 7.

This section is devoted to the proof of Theorem 7. Let (g_ε) be the sequence of solutions of equation (2.2) defined in the hypothesis of Theorem 7. It satisfies then the following properties

$$0 \le g_\varepsilon \le g_0 \qquad \forall t \ge 0,\ \forall \varepsilon > 0, \tag{7.1}$$

and

$$\int_0^T D_\varepsilon(g_\varepsilon)\,dt \ \le\ C_0 \ :=\ H(\mathcal{B}_m) - H(F_{in}) \qquad \forall \varepsilon > 0,\ \forall T \ge 0, \tag{7.2}$$

where

$$D_\varepsilon(g) = \iint_{\mathbb{R}_+^2} b(k,k')\,\frac{1}{\varepsilon^2}\,\sigma_\varepsilon(k'-k)\,j\big(g'\,(k^2+g)\,e^{-k}, g\,(k'^2+g')\,e^{-k'}\big)\,dkdk'. \tag{7.3}$$

Finally we define

$$Q_\varepsilon(h,h) = \iint_{\mathbb{R}_+^2} b_\varepsilon(k,k')(h'\,(k^2+h)\,e^{-k} - h\,(k'^2+h')\,e^{-k'})dk'.$$

Theorem 7 is now a direct consequence of the two following results.

Proposition 7.1. *Let (h_ε) be a sequence such that $0 \le h_\varepsilon \le g_0$, $h_\varepsilon \to h$ in $L^2([0,T]\times\mathbb{R}_+)$, and define $f = k^{-2}h$. Then*

$$\int_{\mathbb{R}_+} Q_\varepsilon(h_\varepsilon,h_\varepsilon)\,\psi\,dk \quad \underset{\varepsilon\to 0}{\longrightarrow} \quad < Q_0(f,f),\psi >, \tag{7.4}$$

for all $\psi \in \mathcal{D}((0,T)\times\mathbb{R}_+)$.

Remark 7.1. With the notations and assumption of Proposition 7.1, and if α_ε is a bounded sequence of $\mathbb{R}_+$, we can prove, following the same demonstration that we will present, that

$$\int_{\mathbb{R}_+} Q_\varepsilon(h_\varepsilon + \alpha_\varepsilon\,\delta_0, h_\varepsilon + \alpha_\varepsilon\,\delta_0)\,\psi\,dk \quad \underset{\varepsilon\to 0}{\longrightarrow} \quad < Q_0(f,f),\psi > \qquad \text{for all} \qquad \psi \in \mathcal{D}((0,T)\times\mathbb{R}_+).$$

Therefore, combining this result with Proposition 7.2 we should be able to prove that when $F_{in} = g_{in} + \alpha_{in}\,\delta_0$ with $0 \le g_{in} \le g_0$, the corresponding solution $F_\varepsilon = g_\varepsilon + \alpha_\varepsilon\,\delta_0$ to the Boltzmann equation (2.5) satisfies (with notations of Theorem 7) $g_\varepsilon \to g = k^2\,f$ strongly in $C([0,T];L^2(\mathbb{R}_+))$ and $\alpha_\varepsilon \to \alpha_{in}$ strongly in $C([0,T])$, where f is the solution to the Kompaneets equation (1.14) corresponding to the initial datum $f_{in} = g_{in}/k^2$.

Proposition 7.2. *The sequence of solutions (g_ε) defined in the statement of Theorem 7 which satisfy (7.1) and (7.2) is relatively strongly compact in $L^p([0,T]\times\mathbb{R}_+)$, for every $1 \le p < \infty$.*

Remark 7.2. The fact that it is possible to get compactness or regularity using the dissipation of entropy term is reminiscent in the literature on (classical) Boltzmann equation. It have first be obtained by P.-L. Lions [27] and C. Villani [40] in the case of Boltzmann equation without cut-off, see also [1] for more precise result. More recently, and independently to our work, R. Alexandre and C. Villani [2] have obtained a similar result to Proposition 7.2, but in a much more complicated situation: they prove strong compactness for a sequence of solutions to the Boltzmann equation with cut-off in the grazing collision asymptotic.

Proof of Proposition 7.1. We define

$$f_\varepsilon = k^{-2}\,h_\varepsilon,$$

and write

$$\begin{aligned}
\int_0^T\!\!\int_{\mathbb{R}_+} Q_\varepsilon(h_\varepsilon,h_\varepsilon)\,\psi\,dkdt &= \frac{1}{2}\int_0^T\!\!\iint_{\mathbb{R}_+^2} b\,k^2\,k'^2\,\sigma_\varepsilon\,f_\varepsilon{}'\,f_\varepsilon\,\frac{e^{-k}-e^{-k'}}{\varepsilon}\,\frac{\psi-\psi'}{\varepsilon}\,dk'dkdt \quad (= I_1^\varepsilon)\\
&\quad - \frac{1}{2}\int_0^T\!\!\iint_{\mathbb{R}_+^2} b\,k^2\,k'^2\,\frac{\sigma_\varepsilon}{\varepsilon}\,f_\varepsilon\,(e^{-k}+e^{-k'})\,\frac{\psi-\psi'}{\varepsilon}\,dk'dkdt \quad (= I_2^\varepsilon)\\
&\quad + \frac{1}{2}\int_0^T\!\!\iint_{\mathbb{R}_+^2} b\,k^2\,k'^2\,\sigma_\varepsilon\,f_\varepsilon\,\frac{e^{-k}-e^{-k'}}{\varepsilon}\,\frac{\psi-\psi'}{\varepsilon}\,dk'dkdt \quad (= I_3^\varepsilon).
\end{aligned}$$

In order to pass to the limit $\varepsilon \to 0$ in I_1^ε and I_3^ε, we remark that for all $k>0$, $k'>0$, since $\psi \in \mathcal{D}((0,T)\times\mathbb{R}_+)$,

$$\big(e^{-k}-e^{-k'}\big)\,(\psi-\psi') = -e^{-k}\,(k'-k)^2\,\frac{\partial\psi}{\partial k}(k) + \mathbb{R}_+^2\big((k'-k)^3\big),$$

from where

$$\begin{aligned} I_3^\varepsilon &= \frac12\int_0^T\!\!\int_{\mathbb{R}_+} k^2\, f^\varepsilon\, e^{-k}\Big\{\int_{\mathbb{R}_+} k'^2\, b(k,k')\,\sigma_\varepsilon\,\Big(\frac{k'-k}{\varepsilon}\Big)^2\,\big[-\frac{\partial\psi}{\partial k}(k)+\varepsilon\,\mathcal{O}\big(\frac{k'-k}{\varepsilon}\big)\big]\,dk'\Big\}\,dkdt \\ &\underset{\varepsilon\to0}{\longrightarrow}\quad -\frac{\Sigma}{2}\int_0^T\!\!\int_{\mathbb{R}_+} b(k,k)\,k^4\,f\,e^{-k}\,\frac{\partial\psi}{\partial k}\,dkdt = -\int_0^T\!\!\int_{\mathbb{R}_+} k^4\,\alpha(k)\,f\,\frac{\partial\psi}{\partial k}\,dkdt, \end{aligned} \tag{7.5}$$

since for all $k>0$ we have

$$\int_{\mathbb{R}_+} k'^2\, b(k,k')\,\Big(\frac{k'-k}{\varepsilon}\Big)^2\,\sigma_\varepsilon\,dk' \quad \underset{\varepsilon\to0}{\longrightarrow}\quad \Sigma\,k^2\,b(k,k).$$

In the same way, we write

$$I_1^\varepsilon = \frac12\int_0^T\!\!\int_{\mathbb{R}_+} k^2\, f\, e^{-k}\,\Big\{\int_{\mathbb{R}_+} k'^2\, b(k,k')\, f^{\varepsilon\prime}\sigma_\varepsilon\,\Big(\frac{k'-k}{\varepsilon}\Big)^2\,\big[-\frac{\partial\psi}{\partial k}(k)+\varepsilon\,\mathcal{O}\big(\frac{k'-k}{\varepsilon}\big)\big]\,dk'\Big\}\,dkdt,$$

and, since

$$\int_{\mathbb{R}_+} k'^2\, b(k,k')\, f^{\varepsilon\prime}\,\Big(\frac{k'-k}{\varepsilon}\Big)^2\,\sigma_\varepsilon\,dk' \quad\rightharpoonup\quad \Sigma\,k^2\,b(k,k)\,f$$

weakly in $L^2_{loc}([0,T]\times\mathbb{R}_+)$, we obtain

$$I_1^\varepsilon \quad\underset{\varepsilon\to0}{\longrightarrow}\quad -\int_0^T\!\!\int_{\mathbb{R}_+} k^4\,\alpha(k)\,f^2\,\frac{\partial\psi}{\partial k}\,dkdt. \tag{7.6}$$

The limit of I_2^ε is slightly more delicate. First of all, since the support of σ is contained in $[-2,2]$,

$$\begin{aligned} -2\,I_2^\varepsilon &= \int_0^T\!\!\int_0^{3\,\varepsilon}\!\!\int_0^{5\,\varepsilon} b\,k^2\,k'^2\,\frac{\sigma_\varepsilon}{\varepsilon}\,f^\varepsilon\,(e^{-k}+e^{-k'})\,\frac{\psi-\psi'}{\varepsilon}\,dk'dkdt \quad (=I_4^\varepsilon) \\ &\quad+\int_0^T\!\!\int_{3\,\varepsilon}^{\infty}\!\!\int_{\varepsilon}^{\infty} b\,k^2\,k'^2\,\frac{\sigma_\varepsilon}{\varepsilon}\,f^\varepsilon\,(e^{-k}+e^{-k'})\,\frac{\psi-\psi'}{\varepsilon}\,dk'dkdt \quad (=I_5^\varepsilon). \end{aligned} \tag{7.7}$$

We first remark that $I_4^\varepsilon \to 0$ since

$$|I_4| \le 5\,\varepsilon\,\|b\|_{L^\infty}\,2\,\Big\|\frac{\partial\psi}{\partial k}\Big\|_{L^\infty}\int_0^T\!\!\int_0^1 h^\varepsilon\,\Big(\int_{\mathbb{R}}\frac{|k-k'|}{\varepsilon}\,\sigma_\varepsilon(k'-k)\,dk'\Big)\,dkdt \quad\underset{\varepsilon\to0}{\longrightarrow}\quad 0. \tag{7.8}$$

Concerning I_5^ε, observe that

$$k'^2 = k^2 + 2\,k\,(k'-k) + \mathcal{O}\big((k'-k)^2\big),$$

$$e^{-k}+e^{-k'} = e^{-k}\,(2-(k'-k)+\mathcal{O}\big((k'-k)^2\big),\quad \psi'-\psi = (k'-k)\,\frac{\partial\psi}{\partial k} + \frac{(k'-k)^2}{2}\,\frac{\partial^2\psi}{\partial k^2} + \mathcal{O}\big((k'-k)^3\big).$$

Under the current assumptions on b we also have

$$b(k,k') = b(k,k) + (k'-k)\int_0^1 \frac{\partial b}{\partial k'}(k,\theta k' + (1-\theta)k)d\theta.$$

We deduce then by straightforward computation

$$
\begin{aligned}
I_5^\varepsilon = & \int_0^T\!\!\int_{2\,\varepsilon}^\infty f^\varepsilon\, k^2\, e^{-k} \Big\{ b(k,k)\, k^2\, \frac{\partial^2 \psi}{\partial k^2} + \big(-b(k,k)\, k^2 + b(k,k)\, 4\, k\big)\, \frac{\partial \psi}{\partial k} \Big\} \\
& \qquad\qquad\qquad\qquad\qquad\qquad \times \int_{\mathbb{R}} \Big(\frac{k'-k}{\varepsilon}\Big)^2 \sigma_\varepsilon\, dk'\, dkdt \\
& + \int_0^T\!\!\int_{2\,\varepsilon}^\infty\!\!\int_{\mathbb{R}} f^\varepsilon\, k^2\, e^{-k}\, k^2\, \frac{\partial \psi}{\partial k} \Big(\frac{k'-k}{\varepsilon}\Big)^2 \sigma_\varepsilon \int_0^1 \frac{\partial b}{\partial k'}(k, \theta k' + (1-\theta)k) d\theta\, dk'\, dkdt \\
& - \int_0^T\!\!\int_{2\,\varepsilon}^\infty b(k,k)\, k^4\; f^\varepsilon e^{-k}\, \frac{\partial \psi}{\partial k} \Big(\int_{\mathbb{R}} \frac{k'-k}{\varepsilon^2}\, \sigma_\varepsilon\, dk' \Big)\, dkdt \quad + \quad \mathcal{O}(\varepsilon).
\end{aligned}
$$

Since σ is even, the third term in the right hand side is zero. The second term satisfies

$$
\begin{aligned}
& \lim_{\varepsilon\to 0} \int_0^T\!\!\int_{2\,\varepsilon}^\infty\!\!\int_{\mathbb{R}} f^\varepsilon\, k^2\, e^{-k}\, k^2\, \frac{\partial \psi}{\partial k} \Big(\frac{k'-k}{\varepsilon}\Big)^2 \sigma_\varepsilon \int_0^1 \frac{\partial b}{\partial k'}(k, \theta k' + (1-\theta)k)\, d\theta\, dk'\, dkdt = \\
& = \lim_{\varepsilon\to 0} \int_0^T\!\!\int_{2\,\varepsilon}^\infty\!\!\int_0^1 f^\varepsilon\, k^2\, e^{-k}\, k^2\, \frac{\partial \psi}{\partial k} \int_{\mathbb{R}} \Big(\frac{k'-k}{\varepsilon}\Big)^2 \sigma_\varepsilon \frac{\partial b}{\partial k'}(k, \theta k' + (1-\theta)k)\, dk'\, d\theta\, dkdt \\
& = 2\,\Sigma \int_0^T\!\!\int_{\mathbb{R}_+} f\, e^{-k}\, k^4\, \frac{\partial \psi}{\partial k} b'(k,k)\, dk\, dt,
\end{aligned}
$$

where $b'(k,k) = 1/2(d/dk)(b(k,k))$. Then, since

$$
\frac{d}{dk}(k^4\, \alpha) = \Sigma\, k^4\, b'(k,k)\, e^{-k} - k^4\, \alpha + 4\, k^3\, \alpha,
$$

we obtain

$$
I_5^\varepsilon \quad \underset{\varepsilon\to 0}{\longrightarrow} \quad 2 \int_0^T\!\!\int_{\mathbb{R}_+} f \Big(\frac{d}{dk}(k^4\, \alpha)\, \frac{\partial \psi}{\partial k} + k^4\, \alpha\, \frac{\partial^2 \psi}{\partial k^2} \Big)\, dkdt. \tag{7.9}
$$

Finally, by (7.5)–(7.9) we have

$$
\begin{aligned}
\lim_{\varepsilon\to 0} \int_0^T\!\!\int_{\mathbb{R}_+} Q_\varepsilon(f^\varepsilon, f^\varepsilon)\, \psi\, dkdt = & \int_0^T\!\!\int_{\mathbb{R}_+} \Big\{ f\, \frac{\partial}{\partial k}\Big(k^4\, \alpha\, \frac{\partial \psi}{\partial k}\Big) - (f + f^2)\, k^4\, \alpha\, \frac{\partial \psi}{\partial k} \Big\}\, dkdt \\
& =: < Q_0(f,f), \psi > .
\end{aligned}
$$

□

Proof of the Proposition 7.2. We proceed in three steps.

Step 1. We remark that $(t-s)\,(\ln\, t - \ln\, s) \geq 4\,(\sqrt{t} - \sqrt{s})^2$. This follows from the fact that the function $\phi(u) = (u-1)\, \ln\, u - 4\,(\sqrt{u} - 1)^2$ is strictly convex, $\phi'(1) = 0$ and $\phi(u) \geq \phi(1) = 0$. It follows from (7.2) that

$$
\begin{aligned}
& \int_0^T\!\!\iint_{\mathbb{R}_+^2} b(k,k')\, \frac{1}{\varepsilon^2}\, \sigma_\varepsilon(k'-k) \\
& \qquad \times \Big(\sqrt{g'_\varepsilon\, (k^2 + g_\varepsilon)\, e^{-k}} - \sqrt{g_\varepsilon\, (k'^2 + g'_\varepsilon)\, e^{-k'}} \Big)^2 dk' dkdt \leq \int_0^T D_\varepsilon(g_\varepsilon) dt \leq C_0,
\end{aligned} \tag{7.10}
$$

and then, if we define

$$
v_\varepsilon := \sqrt{\frac{g_\varepsilon\, e^k}{k^2 + g_\varepsilon}},
$$

$$
(7.11) \qquad 0 \le v_\varepsilon \le \min\left(e^{k/2}, \sqrt{\frac{e^k}{e^k-1}}\right),
$$

and we have

$$
(7.12) \qquad \int_0^T\!\!\iint_{\mathbb{R}_+^2} \beta(k,k')\,\sigma_\varepsilon\left(\frac{v'_\varepsilon - v_\varepsilon}{\varepsilon}\right)^2 dk'dkdt \le C_0,
$$

where $\beta(k,k') := b(k,k')\,k^2\,e^{-k}\,k'^2\,e^{-k'}$.

We fix now $k_0 \in (0,1)$, $\chi \in \mathcal{D}(\mathbb{R}_+)$ such that $\chi = 1$ on $[k_0, k_0^{-1}]$, $0 \le \chi \le 1$, $\operatorname{supp}\chi \subset [k_0/2, 2\,k_0^{-1}]$ and define $u_\varepsilon := \chi\,v_\varepsilon$. It is clear that (u_ε) is bounded in L^∞ and that for all $\varepsilon > 0$, $\operatorname{supp} u_\varepsilon \subset [k_0/2, 2\,k_0^{-1}]$. Moreover

$$
\begin{aligned}
&\int_0^T\!\!\iint_{\mathbb{R}_+^2} \beta(k,k')\,\sigma_\varepsilon\left(\frac{u'_\varepsilon - u_\varepsilon}{\varepsilon}\right)^2 dk'dkdt \le \\
&\qquad\qquad \le \int_0^T\!\!\iint_{\mathbb{R}_+^2} \beta(k,k')\,\sigma_\varepsilon\left\{\chi'^2\left(\frac{v'_\varepsilon - v_\varepsilon}{\varepsilon}\right)^2 + v_\varepsilon^2\left(\frac{\chi'-\chi}{\varepsilon}\right)^2\right\} dk'dkdt,
\end{aligned}
$$

and since $\sigma_\varepsilon\,(\chi'-\chi) = 0$ if $(k,k') \notin \mathcal{K}_0 := [k_0/4, 1+2/k_0]^2$ for every $\varepsilon < k_0/7$, we deduce

$$
\int_0^T\!\!\iint_{\mathbb{R}_+^2} \beta(k,k')\,\sigma_\varepsilon\,v_\varepsilon^2\left(\frac{\chi'-\chi}{\varepsilon}\right)^2 dk'dkdt \le \|\beta\,e^k\|_{L^\infty(\mathcal{K}_0)}\,\Sigma\,\left\|\frac{\partial\psi}{\partial k}\right\|_{L^\infty} =: C_\chi < \infty.
$$

On the other hand, for every $\eta > 0$ consider the function

$$
(7.13) \qquad \rho_\eta(z) = \frac{1}{\eta}\,\rho\Big(\frac{z}{\eta}\Big) \quad \text{and} \quad \rho(z) = \frac{1}{2}\,\mathbf{1}_{[-1,1]}(z),
$$

define $\beta_\star := \inf_{\mathcal{K}_0}\beta > 0$ and

$$
(7.14) \qquad \Delta_\eta(u) := \int_0^T\!\!\iint_{\mathbb{R}^2} \big(u(t,x-z) - u(t,x)\big)^2\,\rho_\eta(z)\,dzdx.
$$

Finally for every $\varepsilon \in (0, k_0/8)$ one has

$$
(7.15) \qquad
\begin{aligned}
\Delta_\varepsilon(u_\varepsilon) &\le \frac{1}{\sigma_\star}\int_0^T\!\!\iint_{\mathbb{R}^2} \sigma_\varepsilon(z)\,\big(u_\varepsilon(x-z) - u_\varepsilon(x)\big)^2\,dxdzdt \\
&\le \frac{1}{\sigma_\star\,\beta_\star}\int_0^T\!\!\iint_{\mathbb{R}_+^2} \beta(k,k')\,\sigma_\varepsilon(k'-k)\,\big(u'_\varepsilon - u'_\varepsilon\big)^2\,dk'dkdt \;\le\; C_1\,\varepsilon^2,
\end{aligned}
$$

where $\beta_\star := \inf_{\mathcal{K}_0}\beta > 0$, $\sigma_\star$ is a positive constant such that $\sigma \ge \sigma_\star\,\rho$ and $C_1 := (C_0 + C_\chi)/(\sigma_\star\,\beta_\star)$.

Step 2. This step is dedicated in proving how one can deduce from (7.15) that the sequence (u_ε) is strongly relatively compact in the x variable. More precisely, we prove the following.

Proposition 7.3. *Let (u_ε) be a sequence of $L^2([0,T]\times\mathbb{R}_+)$ satisfying*

$$
\Delta_\varepsilon(u_\varepsilon) \le C_1\,\varepsilon^2.
$$

Therefore, for every $\alpha > 0$ there exists $h_\alpha > 0$ such that

$$
\int_0^T\!\!\int_{\mathbb{R}} |u_\varepsilon(t,x+h) - u_\varepsilon(t,x)|^2\,dxdt \le \alpha, \qquad \textit{for every } |h| \le h_\alpha.
$$

We need the following two lemmas, which we state below and prove at the end of the section.

Lemma 7.1. *For every $u \in L^2([0,T]\times\mathbb{R})$ and all $\eta > 0$ we have*

$$\|\rho_\eta *_x u - u\|_{L^2}^2 \;\le\; \Delta_\eta(u). \tag{7.16}$$

Lemma 7.2. *For every $u \in L^2([0,T]\times\mathbb{R})$ and all $0 < \varepsilon \le \eta/2$*

$$\Delta_\eta(u) \;\le\; 64\left(\frac{\eta}{\varepsilon}\right)^2 \Delta_\varepsilon(u). \tag{7.17}$$

Proof Proposition 7.3. Fix $\alpha > 0$ and write

$$u_\varepsilon = \rho_\eta * u_\varepsilon + (u_\varepsilon - \rho_\eta * u_\varepsilon).$$

Fix now $\eta > 0$ small enough to have, by Lemma 7.1, Lemma 7.2 and (7.15)

$$\|u_\varepsilon - \rho_\eta *_x u_\varepsilon\|_{L^2}^2 \le \Delta_\eta(u_\varepsilon) \le C_2\left(\frac{\eta}{\varepsilon}\right)^2 \Delta_\varepsilon(u_\varepsilon) \le C_1\, C_2\, \eta^2 \le \alpha/3. \tag{7.18}$$

We now observe that for every $\eta > 0$ fixed, the set $(\rho_\eta *_x u_\varepsilon)_{\varepsilon>0}$ is strongly relatively compact in the x variable, i.e. given $\alpha > 0$, there exists $h_\alpha > 0$ such that for every $|h| \le h_\alpha$

$$\int_0^T\!\!\int_{\mathbb{R}} |(\rho_\eta * u_\varepsilon)(t,x+h) - (\rho_\eta * u_\varepsilon)(t,x)|^2\, dxdt \le \alpha/3. \tag{7.19}$$

Proposition 7.3 follows from (7.18) and (7.19). □

Step 3 of the proof of Proposition 7.2. In this step we deduce from Proposition 7.3 that (g_ε) satisfies the Frechet-Kolmogorov criteria in $L^2((0,T)\times\mathbb{R}_+)$, and we start proving that it satisfies the following property :

$$\forall \alpha \in (0,1)\ \exists h_\alpha > 0 \quad \text{s.t.} \quad \int_0^T\!\!\int_{\Omega_\alpha} |g_\varepsilon(t,x+h) - g_\varepsilon(t,x)|^2\, dxdt \le \alpha \quad \text{for all } |h| \le h_\alpha, \tag{7.20}$$

where we have set $\Omega_\alpha = [\alpha, 1/\alpha]$. Let define w_ε and z_ε by

$$z_\varepsilon \;:=\; \frac{g_\varepsilon}{k^2} \qquad \text{and} \qquad w_\varepsilon \;:=\; v_\varepsilon^2\, e^{-k} = \frac{z_\varepsilon}{1+z_\varepsilon}.$$

We prove successively that (7.20) is satisfied by v_ε, w_ε, z_ε and then finally g_ε. It is quite easy to show that (7.20) holds for v_ε using Proposition 7.3, and then for w_ε. We only show that if we already know that w_ε satisfies (7.20) then so do z_ε since it is the only delicate step.

Remark that

$$z_\varepsilon = \frac{w_\varepsilon}{1-w_\varepsilon} \qquad \text{so that} \qquad \frac{1}{1-w_\varepsilon} = 1 + z_\varepsilon.$$

So that

$$\begin{aligned}\frac{\tau_h w_\varepsilon}{1-\tau_h w_\varepsilon} - \frac{w_\varepsilon}{1-w_\varepsilon} &= \left[\frac{1}{1-\tau_h w_\varepsilon} + \frac{w_\varepsilon}{(1-\tau_h w_\varepsilon)\,(1-w_\varepsilon)}\right](\tau_h w_\varepsilon - w_\varepsilon)\\ &= \Big[(1+\tau_h z_\varepsilon) + w_\varepsilon\,(1+\tau_h z_\varepsilon)\,(1+z_\varepsilon)\Big]\,(\tau_h w_\varepsilon - w_\varepsilon).\end{aligned}$$

Now, by (7.1), $z_\varepsilon \le g_0/k^2$, and therefore

$$\int_0^T\int_{\Omega_\alpha} |z_\varepsilon(t,x+h)-z_\varepsilon(t,x)|^2\,dxdt \le 2\left(1+\|\frac{g_0}{k^2}\|_{L^\infty(\Omega_\alpha)}\right)^2 \int_0^T\int_{\Omega_\alpha} |w_\varepsilon(t,x+h)-w_\varepsilon(t,x)|^2\,dxdt,$$

and this ends the proof of (7.20) for z_ε. It is now easy to deduce that (g_ε) also satisfies (7.20). Finally, in order to prove that it is relatively compact in $L^p([0,T]\times\mathbb{R}_+)$ for every $1\le p<\infty$ we argue as follows. For every $\psi\in\mathcal{D}(\mathbb{R}_+^\star)$,

$$\frac{d}{dt}\int_{\mathbb{R}_+} g_\varepsilon\,\psi\,dk = \int_{\mathbb{R}_+} Q_\varepsilon(g_\varepsilon,g_\varepsilon)\,\psi\,dkdt.$$

By (7.5), (7.6), (7.8) and (7.9) we deduce

$$\frac{d}{dt}\int_{\mathbb{R}_+} g_\varepsilon\,\psi\,dk \le C_4\left(\|\frac{\partial\psi}{\partial k}\|_{L^\infty}+\|\frac{\partial^2\psi}{\partial k^2}\|_{L^\infty}+\|\frac{\partial^3\psi}{\partial k^3}\|_{L^\infty}\right). \tag{7.21}$$

By (7.20) and (7.21), the family

$$\left\{\int_{\mathbb{R}_+} g_\varepsilon(k')\,\rho_\eta(k-k')\,\psi(k')\,dk'\right\}_{\{\varepsilon>0\}}$$

belongs to a compact subset of $L^\infty((0,T)\times\Omega_\alpha)$ $\forall T,\ \alpha>0$. Now writing

$$g_\varepsilon \;=\; g_\varepsilon *_x \rho_\eta + (g_\varepsilon - g_\varepsilon *_x \rho_\eta),$$

we see that (g_ε) is relatively compact in $L^2((0,T)\times\Omega_\alpha)$ $\forall\alpha>0$ and therefore in $L^p((0,T)\times\mathbb{R}_+)$ $\forall p\in[1,\infty)$ thanks to (7.1). □

Proof of Lemma 7.1. For every $u\in L^2([0,T]\times\mathbb{R})$ and all $\eta>0$ we have, for almost every $x\in\mathbb{R}$ and $t\in[0,T]$

$$|\,u *_x \rho_\eta(t,x)-u(t,x)\,|^2 = \left(\int_{\mathbb{R}}(u(t,x-z)-u(t,x))\,\rho_\eta(z)\,dz\right)^2 \le \int_{\mathbb{R}}(u(t,x-z)-u(t,x))^2\,\rho_\eta(z)\,dz,$$

by Cauchy-Schwartz; (7.16) follows. □

Proof of Lemma 7.2. We start proving (7.17) when $\eta=n\,\varepsilon$, $n\in\mathbb{N}^\star$ and $\varepsilon\le\eta/2$. We have in that case

$$\begin{aligned}\Delta_\eta(u) &= \frac{1}{2\,n\,\varepsilon}\int_0^T\int_{\mathbb{R}}\Big[\int_{-n\,\varepsilon}^{n\,\varepsilon}(u(t,x-z)-u(t,x))^2\,dz\Big]\,dxdt\\ &= \frac{1}{2\,n\,\varepsilon}\sum_{k=-n}^{n-1}\int_0^T\int_{\mathbb{R}}\int_0^\varepsilon(u(t,x-(z+k\,\varepsilon))-u(t,x))^2\,dzdxdt,\end{aligned} \tag{7.22}$$

and we are going to prove that for every $k\in\{-n,...,n-1\}$ one has

$$\int_{\mathbb{R}}\int_0^\varepsilon(u(x-z-k\,\varepsilon)-u(x))^2\,dzdx \le (2n)^2\int_{\mathbb{R}}\int_{-\varepsilon}^\varepsilon(u(x-z)-u(x))^2\,dxdz. \tag{7.23}$$

We only treat in detail the case $k=-n$ since the other cases can be handled in the same way. For the sake of brevity we do not write the dependence on the t variable. We start writing

$$\begin{aligned}u(x+n\,\varepsilon-z)-u(x) = {}&(u(x+n\,\varepsilon-z)-u(x+(n-1)\,\varepsilon))\\ &+(u(x+(n-1)\,\varepsilon)-u(x+(n-1)\,\varepsilon-z))\\ &+(u(x+(n-1)\,\varepsilon-z)-u(x+(n-2)\,\varepsilon))\;+\;...\\ &...\quad+(u(x+\varepsilon)-u(x+\varepsilon-z))+(u(x+\varepsilon-z)-u(x)),\end{aligned}$$

and, since the number of terms in the right hand side is $2\,(n-1)$,

$$\begin{aligned}(u(x+n\,\varepsilon-z)-u(x))^2 \leq 2\,n\,\big[&(u(x+n\,\varepsilon-z)-u(x+(n-1)\,\varepsilon))^2\\ &+(u(x+(n-1)\,\varepsilon)-u(x+(n-1)\,\varepsilon-z))^2\\ &\quad+(u(x+(n-1)\,\varepsilon-z)-u(x+(n-2)\,\varepsilon))^2\ +\ \ldots\\ &\quad\ldots\quad+(u(x+\varepsilon)-u(x+\varepsilon-z))^2+(u(x+\varepsilon-z)-u(x))^2\big].\end{aligned}$$

Now, it follows from a change of variables that for every $\ell\in\mathbb{Z}$

$$\int_{\mathbb{R}}\int_0^{\varepsilon}(u(x+\ell\,\varepsilon)-u(x+\ell\,\varepsilon-z))^2\,dzdx=\int_{\mathbb{R}}\int_0^{\varepsilon}(u(y-z)-u(y))^2\,dzdy,$$

$$\int_{\mathbb{R}}\int_0^{\varepsilon}(u(x+\ell\,\varepsilon-z)-u(x+(\ell-1)\,\varepsilon))^2\,dzdx=\int_{\mathbb{R}}\int_{-\varepsilon}^{0}(u(y-\theta)-u(y))^2\,d\theta dy.$$

We finally obtain (7.23). Then, by (7.22),

$$\Delta_{n\,\varepsilon}(u)\leq 4\,n^2\,\Delta_{\varepsilon}(u)\equiv 4(\frac{\eta}{\varepsilon})^2\Delta_{\varepsilon}(u).$$

Consider now the general case, $0<\varepsilon\leq\eta$. Let $n\in\mathbb{N}$ such that $n\,\varepsilon\leq\eta<(n+1)\,\varepsilon$. Since $n\,\varepsilon\geq\eta-\varepsilon\geq\eta/2$ and $(n+1)\,\varepsilon\leq\eta+\varepsilon\leq 2\,\eta$ we have

$$\Delta_{\eta}(u)\leq\frac{1}{2\,n\,\varepsilon}\int_{\mathbb{R}}\int_{-(n+1)\,\varepsilon}^{(n+1)\,\varepsilon}(u(x-z)-u(x))^2\,dzdx=\frac{(n+1)}{n}\triangle_{(n+1)\,\varepsilon}(u),$$

by the previous case, and moreover

$$\frac{(n+1)}{n}\triangle_{(n+1)\,\varepsilon}(u)\leq 4\,\frac{(n+1)^3}{n}\,\Delta_{\varepsilon}(u)\leq 64\,\big(\frac{\eta}{\varepsilon}\big)^2\,\Delta_{\varepsilon}(u).$$

□

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