KINETIC EQUATIONS WITH MAXWELL BOUNDARY CONDITION

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Abstract - We prove global stability results of *DiPerna-Lions* renormalized solutions to the initial boundary value problem for kinetic equations. The (possibly nonlinear) boundary conditions are completely or partially diffuse, which include the so-called Maxwell boundary condition, and we prove that it is realized (it is not relaxed!). The techniques are illustrated with the Fokker-Planck-Boltzmann equation and with the Vlasov-Poisson-Fokker-Planck system, but can be readily extended to the Boltzmann equation and to the Vlasov-Poisson system when linear and diffuse boundary condition are imposed. The proof uses some trace theorems of the kind previously introduced by the author for the Vlasov equations, new results concerning weak-weak convergence (the renormalized convergence and the biting L^1 weak convergence), as well as the Darroès-Guiraud information in a crucial way.

Keywords - Vlasov-Poisson, Boltzmann and Fokker-Planck equations, Maxwell or diffuse reflection, nonlinear gas-surface reflection laws, Darrozès-Guiraud information, trace Theorems, renomalized convergence, Biting Lemma, Dunford-Pettis Lemma.

1. Introduction and main results.

This paper deals with the initial boundary value problem for kinetic equations with general diffuse boundary conditions, which include the so-called Maxwell boundary condition. We treat in detail the Fokker-Planck equation type, in particular the Fokker-Planck-Boltzmann equation (FPB in short) and the Vlasov-Poisson-Fokker-Planck system (VPFP in short) for which nonlinear boundary condition can be considered. Our results extend easily to the Vlasov equation type such as the Boltzmann equation and the Vlasov-Poisson system (VP in short) with linear boundary conditions. In fact, our result can be extend to very general kinetic equation.

Our main result is a stability result from which one can deduce, in a very classical way, an existence result. Precisely, considering a sequence f^n of *DiPerna-Lions* renormalized solutions to our equation which satisfies a physical *a priori* estimate and the boundary conditions, we prove that, extracting a subsequence if necessary, f^n converges to a function f which is also a renormalized solution of the equation and satisfies the exact boundary condition.

The difficulty here is to pass to the limit at the boundary and to prove that the *exact* boundary condition holds (until now only *relaxed* boundary condition had been obtained). This difficulty is due to the very poor *a priori* estimate on the trace sequence (γf_n). The *a priori* estimate that one can derive on (γf_n) does not guarantee the L^1-weak convergence and worse, in the VP and VPFP cases, no L^1 *a priori* bound can be get. In this work, we are not able to prove any L^1-weak convergence for the sequence (γf_n), but, and this is our main result, we are able to prove a weak L^1-weak convergence in the velocity variable for the sequence $(\gamma_+ f_n)$. This gives a strong enough information in order to pass to the limit at the boundary.

The aim of this work is thus to introduce some efficient tools to deal with this boundary value problem and to establish the above convergence. On one hand we develop a trace theory adapted to the weak regularity of the force field and to the renormalized formulation of the equation in the continuity of the previous works of the author [53], [54]. This allows us to give sense to the trace function and thus to the boundary condition. As a back product, we obtain the weak-weak convergence of the sequence (γf_n). Our second tool is precisely the weak-weak type convergence (namely the biting L^1-weak convergence and the renormalized convergence) that we introduce in a L^0 setting. We say weak-weak convergence in order to express the fact that they are extremely weak sense of convergence: weaker, for instance, to the L^1-weak convergence and to the a.e. convergence, which moreover are not associated to any topogical structure. We establish new Functional Analysis results. In particular, our main convergence result mentioned above follows from this analysis, using in a crucial way the *Darrozès-Guiraud information.*

Let Ω be a smooth, open and bounded subset of $\mathbb{R}^N$ and set $\mathcal{O} = \Omega \times \mathbb{R}^N$. We consider a gas confined in $\Omega \subset \mathbb{R}^N$. The state of the gas is given by the distribution function $f(t, x, \xi) \geq 0$ of particles, which at time $t \geq 0$ and at the position $x \in \Omega$, move with the velocity $\xi \in \mathbb{R}^N$. The evolution of f is governed by a kinetic equation that we complement with boundary conditions which take into account how the particles are reflected by the wall. We assume that the boundary $\partial\Omega$ is sufficiently smooth; the exact regularity that we need is that there exists a vector field $n \in \left(W^{1,\infty}(\Omega)\right)^N$ such that $n(x)$ coincides with the outward unit normal vector at $x \in \partial\Omega$. We denote by $d\sigma_x$ the Lebesgue surface measure on $\partial\Omega$ and by $d\lambda_i$ the measure on $(0, T) \times \Sigma_\pm$ defined by $d\lambda_i = |n(x) \cdot \xi|^i \, d\xi d\sigma_x dt$, where the incoming/outgoing sets $\Sigma_\pm$ are defined by

$$\Sigma_\pm = \{(x, \xi) \in \Sigma; \pm n(x) \cdot \xi > 0\} \quad \text{with} \quad \Sigma = \partial\Omega \times \mathbb{R}^N. \tag{1.1}$$

The boundary conditions take then the form of a balance between the values of the traces $\gamma_\pm f := \mathbf{1}_{\Sigma_\pm} \gamma f$ of f on these sets. In order to describe the interaction between particles and wall, J.-C. Maxwell [52] proposed in 1879 the following phenomenological law which splits into a local reflection and a diffuse (or Maxwell) reflection

$$\gamma_- f = R(\gamma_+ f) = (1 - \alpha) \, L \, \gamma_+ f + \alpha \, D \, \gamma_+ f \quad \text{on } (0, \infty) \times \Sigma_-. \tag{1.2}$$

Here $\alpha \in [0, 1]$ is a constant, called the *accommodation coefficient*, the local reflection operator L is defined by

$$(L \, \phi) \, (t, x, \xi) = \phi(t, x, R_x \, \xi),$$

with $R_x \, \xi = -\xi$ (inverse reflection) or $R_x \, \xi = \xi - 2 \, (\xi \cdot n(x)) \, n(x)$ (specular reflection), and the diffuse reflection operator D is

$$(D \, \phi)(t, x, \xi) = M(\xi) \; \tilde{\phi}(t, x), \qquad \tilde{\phi}(t, x) = \int_{\xi \cdot n(x) > 0} \phi(t, x, \xi) \, \xi \cdot n(x) \, d\xi,$$

with M the normalized Maxwellian with temperature (of the wall) $\Theta > 0$

$$(1.3) \qquad M(\xi) = \frac{1}{(2\,\pi)^{\frac{N-1}{2}}\,\Theta^{\frac{N+1}{2}}}\,e^{-\frac{|\xi|^2}{2\,\Theta}} \quad \text{so that} \quad \int_{\xi\cdot n>0} M(\xi)\,n\cdot\xi\,d\xi = 1 \quad \forall n \in S^{N-1}.$$

This was the only model for the gas/surface interaction that appeared in the literature before the late 1960s. In order to describe with more accuracy the interaction between molecules and wall, other models have been proposed [26], [27], [49]. The boundary condition is then written $\gamma_- f = R(\gamma_+ f)$ where R is a general integral operator satisfying the so-called non-negativity, normalization and reciprocity conditions, see [30] and Remark 4.2. We do not know if our analysis can be adapted to this general kernel; however, the boundary condition can be generalized in an other direction [31], [12], and we will assume that the following non linear boundary condition holds

$$(1.4) \qquad R\,\phi = (1-\tilde\alpha)\,L\,\phi + \tilde\alpha\,D\,\phi, \qquad \tilde\alpha = \alpha(\tilde\phi),$$

where $\alpha : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies $0 < \bar\alpha \le \alpha(s) \le 1$ for all $s \in \mathbb{R}$.

We focus our analysis on the two following situations. In the FPB model, the evolution of f is governed by the equation

$$(1.5) \qquad \frac{\partial f}{\partial t} + \xi\cdot\nabla_x f - \nu\,\Delta_\xi f = Q(f,f) \;\text{ in } (0,\infty)\times\mathcal{O},$$

where $Q(f,f)$ is the quadratic Boltzmann collision operator describing the collision interactions of the particles by binary elastic shock and $\nu > 0$. We refer to [26] and [35], and the reference therein, for a physical description of the Boltzmann collision operator and of the FPB model.

In the VPFP model, the evolution of f is governed by the system of equations

$$(1.6) \qquad \frac{\partial f}{\partial t} + \xi\cdot\nabla_x f - \mathrm{div}_\xi\left((\nabla_x V_f + \lambda\,\xi)\,f\right) - \nu\,\Delta_\xi f = 0 \;\text{ in } (0,\infty)\times\mathcal{O},$$

where $\lambda \in \mathbb{R}$, $\nu > 0$ and $-\nabla_x V_f$ is a self-induced force (or mean field) which describes the fact that particles interact by the way of the two-body long range Coulomb force, so that V_f is the solution of the Poisson equation with the Dirichlet condition

$$(1.7) \qquad -\Delta V_f = \rho_f = \int_{\mathbb{R}^3} f(t,x,\xi)\,d\xi \quad \text{on} \quad (0,\infty)\times\Omega, \qquad V_f = 0 \quad \text{on} \quad (0,\infty)\times\partial\Omega.$$

We refer to S. Chandreasekahar [20] for a physical presentation. Finally, we complement these equations with a given initial condition

$$(1.8) \qquad f(0,.) = f_0 \ge 0 \;\text{ on } \mathcal{O}.$$

We have in mind to adapt the DiPerna-Lions stability theory to the boundary value problem (1.2) or (1.4). In order to do so we have to collect *a priori* bounds that one can obtain for this kind of problem. We study them in the simplest case: we assume that f is governed by the free transport equation

$$(1.9) \qquad \frac{\partial f}{\partial t} + \xi\cdot\nabla_x f = 0 \;\text{ in } (0,\infty)\times\mathcal{O}.$$

In this case, one can verify, using the *Darrozès-Guiraud* inequality [39], that

$$\frac{d}{dt}\iint_{\mathcal{O}} j(f/M)\, M\, d\xi dx = -\iint_{\Sigma} j(\gamma f/M)\, M\, \xi\cdot n(x)\ d\xi d\sigma_x \le 0, \tag{1.10}$$

for any given $j:\mathbb{R}\to\mathbb{R}$ convex. This gives the only Lyapunov functionals which are compatible with the diffuse boundary condition (1.2) and (1.4). Since we want to deal with the Vlasov-Poisson term or the Boltzmann collision term, the chose of j reduces to $j(s)=s$ and $j(s)=h(s):=s\log s$. Taking $j(s)=s$ we get the conservation of mass, and taking $j(s)=h(s)$ we obtain, coming back to (1.10),

$$\begin{aligned}
&\int_{\mathbb{R}^N} h(\gamma f/M)\, M\, \xi\cdot n(x)\, d\xi = \int_{\{\xi\cdot n(x)>0\}} \big[h(\gamma_+ f/M) - h(R\,\gamma_+ f/M)\big]\, M\, \xi\cdot n(x)\, d\xi \\
&\ge \int_{\{\xi\cdot n(x)>0\}} \big[h(\gamma_+ f/M) - (1-\tilde\alpha)\, h(L\,\gamma_+ f/M) - \tilde\alpha\, h(D\,\gamma_+ f/M)\big]\, M\, \xi\cdot n(x)\, d\xi \\
&\ge \bar\alpha\, \mathcal{E}(\frac{\gamma_+ f}{M})
\end{aligned} \tag{1.11}$$

with

$$\begin{aligned}
\mathcal{E}(\frac{\phi}{M}) &= \int_{\xi\cdot n(x)>0} \big[h(\phi/M) - h(D\,\phi/M)\big]\, d\mu_x(\xi) \\
&= \int_{\xi\cdot n(x)>0} h(\frac{\phi}{M})\, d\mu_x - h(\int_{\xi\cdot n(x)>0} \frac{\phi}{M}\, d\mu_x),
\end{aligned}$$

where $d\mu_x(\xi) := M\,\xi\cdot n(x)\, d\xi$ is a measure of probability for any $x\in\partial\Omega$, thanks to (1.3). We obviously deduce, by the Jensen inequality, that $\mathcal{E}$ is a nonnegative functional. When we assume that f_0 satisfies the following natural bounds

$$\iint_{\mathcal{O}} f_0\, \big(1+|\xi|^2+|\log f_0|\big)\, d\xi dx < \infty, \tag{1.12}$$

i.e. f_0 has finite mass, energy and entropy, we obtain from (1.10) and (1.11) that, at least formally, a solution of (1.9)-(1.4)-(1.8) satisfies the "physical" *a priori* bound

$$\sup_{[0,T]} \iint_{\mathcal{O}} f\, \big(1+|\xi|^2+|\log f|\big)\, d\xi dx + \int_0^T\int_{\partial\Omega} \mathcal{E}(\frac{\gamma_+ f}{M}) d\sigma_x dt \le C_T. \tag{1.13}$$

Of course when we consider the FPB model or the VPFP model the *a priori* bound is slightly different (additional terms in (1.13) appear due to the additional terms in (1.5) and (1.6)), but it is fundamentally of the same kind. In particular, we do not have any global *a priori* estimate on the L^p norm with $p>1$. This is an important difference with what happen for the VP and VPFP models written in the all space or provided with purely locally boundary conditions ($\alpha\equiv 0$). This implies that we must deal with the weaker sense of solution, namely the *renormalized DiPerna-Lions* solutions.

Our main result is the following stability or compactness result.

Theorem 1.1. *Let (f_n) be a sequence of renormalized solutions to the FPB equation (or to the VPFP system) which satisfies the physical bound (uniformly in n) and such that the trace $\gamma_+ f_n$ satisfies the boundary condition (1.4) and the uniform bound*

$$\int_0^T \int_{\partial\Omega} \mathcal{E}\big(\frac{\gamma_+ f_n}{M}\big) d\sigma_x dt \le C_T \qquad \forall n \ge 0. \tag{1.14}$$

Then, up to the extraction of a subsequence, $f_n \to f$ strongly in $L^p(0,T;L^1(\mathcal{O}))$ for all $T > 0$ and $p \in [1,\infty)$, and f is a renormalized solution to the FPB equation (or to the VPFP system) which satisfies the physical a priori estimate. Furthermore, for all $\varepsilon > 0$ there exists a measurable set $A \subset (0,T) \times \partial\Omega$ such that meas$((0,T)\times\partial\Omega \setminus A) < \varepsilon$ and

$$\gamma_+ f_n \longrightarrow \gamma_+ f \quad \textit{stongly in} \quad L^1(A \times \mathbb{R}^N, d\lambda_1). \tag{1.15}$$

As a consequence we can pass to the limit in the boundary condition (1.4), so that the trace condition holds.

This result can be adapted to the VP system and to the Boltzmann equation with the linear boundary conditions (1.2).

Theorem 1.2. *Let (f_n) be a sequence of renormalized solutions to the Boltzmann equation (or to the VP system) which satisfies the physical bound (uniformly in n) and such the trace $\gamma_+ f_n$ satisfies the boundary condition (1.2) and the uniform bound (1.14). Then, up to the extraction of a subsequence, $f_n \rightharpoonup f$ weakly in $L^p(0,T;L^1(\mathcal{O}))$ for all $T > 0$ and $p \in [1,\infty)$, and f is a renormalized solution to the Boltzmann equation (or to the VP system) which satisfies the physical a priori estimate. Furthermore, for all $\varepsilon > 0$ there exists a measurable set $A \subset (0,T) \times \partial\Omega$ such that meas$((0,T)\times\partial\Omega \setminus A) < \varepsilon$ and*

$$\gamma_+ f_n \rightharpoonup \gamma_+ f \quad \textit{weakly in} \quad L^1(A \times \mathbb{R}^N, d\lambda_1). \tag{1.16}$$

Then, we can pass to the limit in the linear boundary condition (1.2), so that the trace condition holds.

We do not present the proof of this second stability result since the arguments are similar to the ones we use in the proof of Theorem 1.1. We just have to combine the trace Theorem and arguments introduced in [54] with the weak convergence result presented in the section 2 (Corollary 2.4). As a standard consequence of Theorem 1.1 and 1.2, we obtain the existence of a global renormalized solution to the boundary value problem for initial data satisfying the natural "physical" bound.

The Boltzmann equation and the FPB equation for initial data satisfying the natural bound (1.12) was first studied by R. DiPerna and P.-L. Lions [35,37,38] who proved stability and existence results for weak global solutions in the case of the entire space ($\Omega = \mathbb{R}^N$). Afterwards, the corresponding boundary value problem with (partially) diffuse boundary conditions has been extensively studied in the case of the Boltzmann model [45], [4], [5], [6], [7], [28], [41], [46], [29], [54]. It has been proved, in the partial absorption case $\gamma_- f = \theta R \gamma_+ f$ with $\theta \in [0,1)$ and in the completely local reflection case (i.e. (1.2) holds with $\alpha \equiv 0$), that there exists a global renormalized solution. But in the most interesting physical case (when $\theta \equiv 1$ and $\alpha \in (0,1]$), it has only been proved that the boundary condition (1.2) hold in the relaxed form

$$\gamma_- f \ge R(\gamma_+ f) \quad \text{on } (0,\infty) \times \Sigma_-. \tag{1.17}$$

With regard to existence results for the initial value problem for the VPFP system set in the whole space, we refer to [14], [15], [16], [19], [22], [23], [34], [57], [64] and [24], [59]. The initial boundary value problem has been addressed by [13], [21]. We also refer to [6] [3], [11], [44], [56], [66] for the initial boundary value problem for the VP system and to [56] for the corresponding stationary problem. We emphasize that in all these works only local reflection or prescribed incoming data are treated, and to our knowledge, there is no result concerning the diffuse boundary condition for the VP system or for the VPFP system.

We also mention that there is a great deal of information for the boundary value problem in an abstract setting in [65], [43] with possibly non linear conditions [10], [55]. Finally, the Boltzmann equation with non linear boundary conditions has been treated in the setting of a strong but non global solution framework in [42].

Before explaining the main ideas behind our stability result, Theorem 1.1, we want to emphasize that a first fundamental question is the sense we give to the trace. The so-called trace problem has been studied by [9], [32], [2], [62], [43], [18] for the Vlasov equation with a Lipschitz force field and extended to the Vlasov-Fokker-Planck equation in [21]. In the case of the VP and the VPFP systems, the *a priori* estimate on the force field does not guarantee Lipschitz regularity but only Sobolev regularity. We follow the trace theory developed in [53], [54] for the solutions of the Vlasov equation with a force field in Sobolev space that we extend to the solutions of the Vlasov-Fokker-Planck equation. The trace is then defined by a Green formula written on the renormalized equation.

The main difficulty when we deal with this problem is the lack of a good *a priori* bound on the trace. Additionally to the *a priori* bound of the *Darozès-Guiraud information* (1.13), we can prove an L^1 *a priori* bound in the case of the Boltzmann equation and an $L^{1/2}$ *a priori* bound in the case of the VP system: in both cases, we do not have an *a priori* information on the local equi-integrability of the trace. In other words, considering a sequence of solution (f_n) satisfying the uniform natural bound, we can not say that γf_n is weakly compact in $L^1((0,T)\times\Sigma, d\lambda_1)$. In order to prove (1.16), which clearly implies that $\widetilde{\gamma_+ f_n} \rightharpoonup \widetilde{\gamma_+ f}$, and then allows to pass to the limit in the boundary condition (at least in the linear case), we proceed in several steps. First, we deduce from our trace theory that γf_n converges, up to the extraction of a subsequence, to γf in the renormalized sense. Let emphasize that this convergence yet ensures that the *relaxed* boundary condition (1.17) holds. Using the boundary condition and the *a priori* boundary estimate we deduce that $(\widetilde{\gamma_+ f_n})$ converges in the renormalized sense and that its limit belongs to L^0. Next, we extend the so-called Biting Lemma to this context, namely we prove that $\widetilde{\gamma_+ f_n}$ also converges in the biting L^1-weak sense. Last, using the uniform boundedness of the *Darozès-Guiraud information* and some convexity argument we prove a kind of Dunford-Pettis Lemma in the ξ variable; namely, we obtain the weak L^1 convergence in the ξ variable of $(\gamma_+ f_n)$, which precisely states (1.16). Finally, for Fokker-Planck type equations we propagate the a.e. convergence in the interior due to the hypoelipticity of the equation up to the outgoing boundary set. We obtain that $\gamma_+ f_n \to \gamma_+ f$ a.e. and then deduce (1.15).

The paper is organized as follows. In section 2, we introduce the weak-weak convergence and prove the main compactness results concerning renomalized convergence and the biting L^1-weak convergence. In section 3, we state some trace theorems and prove a general stability result in the interior and "up to the boundary" for sequence of solutions to the Vlasov-Fokker-Planck equation. In section 4, we state the *a priori estimates*, make precise the notion of solution we deal with, and then prove Theorem 1.1. Section 5 is devoted to the proof of the trace theorems. In the Appendix we state and prove some elementary results concerning renormalized convergence which are used through away this paper.

2. From weak-weak convergence to weak convergence.

In this section we present some Functional Analysis results which make possible to gain L^1-weak convergence in the ξ variable from weak-weak convergence and boundedness of the *Darrozès-Guiraud information*. The first notion of weak-weak convergence we deal with is the biting L^1-weak convergence. It seems to have been introduced by Kadec and Pelzyński [48] and rediscovered and developed in a L^1 and bounded measure framework by Chacon and Rosenthal in the end of the 1970's, see [40], [17]. Let us first recall the definition of biting L^1-weak convergence that we extend to an L^0 framework.

In the following Y stands for a closed and σ-compact topological space, i.e. $Y = \cup_k Y_k$ where (Y_k) is an increasing sequence of compact sets, that we provide with its σ-ring of Borel sets and with ν a Borel measure. We denote by $L(Y)$ the space of all measurable functions $\phi : Y \to \bar{\mathbb{R}}$ and by $L^0(Y)$ the subset of all measurable and almost everywhere finite functions. In order to simplify the exposition, we are only concerned with nonnegative functions of L and L^0. Thus, in this section, we also denote by L and L^0 the cone of nonnegative functions in these spaces, and we do not anymore specify it.

Definition 2.1. *We say that a sequence (ψ_n) of $L(Y)$ converges in the biting L^1-weak sense to $\psi \in L(Y)$, and we write $\psi_n \stackrel{b}{\rightharpoonup} \psi$, if for every $k \in \mathbb{N}$ we can find $A = A_k \subset Y_k$ in such a way that (A_k) is increasing, $\nu(Y_k \backslash A_k) < 1/k$, $\psi_n \in L^1(A)$ for all n large enough and $\psi_n \rightharpoonup \psi$ weakly in $L^1(A)$. In particular, this implies $\psi \in L^0(Y)$.*

The fundamental result concerning the biting L^1-weak convergence is the so-called Biting Lemma that we recall know. We refer to [25], [8], [17], [40] and [48] for a proof of this Lemma. We also refer to [1] and [33] for other developments related to the biting L^1-weak convergence. Extension of this theory to multivalued function has been done by Balder, Castaing, Valadier and others; we refer to [58] for precise references.

Theorem 2.1 (Biting Lemma). *Let (ψ_n) be a bounded sequence of $L^1(Y)$. Then, there exists $\psi \in L^1(Y)$ and a subsequence $(\psi_{n'})$ such that $\psi_{n'} \stackrel{b}{\rightharpoonup} \psi$ in the biting L^1-weak sense.*

Our first result is a kind of intermediate result between the Biting Lemma and the Dunford-Pettis Lemma. More precisely, we prove a Dunford-Pettis Lemma in the ξ variable for bounded sequences (ϕ_n) of L^1 which has a *Darrozès-Guiraud information* uniformly (in n) bounded. It is based on the Biting Lemma and a convexity argument.

Theorem 2.2. *Let consider $j : \mathbb{R}_+ \to \mathbb{R}$ a convex function of class $C^2(0,\infty)$ such that $j(s)/s \to +\infty$ when $s \nearrow +\infty$ and such that the application J from $(\mathbb{R}_+)^2$ to $\mathbb{R}$ defined by $J(s,t) = (j(t) - j(s))\,(t-s)$ is convex, ω a non negative function of $\mathbb{R}^N$ such that $\omega(\xi) \to \infty$ when $|\xi| \to \infty$ and, for any $y \in Y$, a probability measure μ_y on $\mathbb{R}^N$. Assume that (ϕ_n) is a sequence of non negative measurable functions on $Y \times \mathbb{R}^N$ such that*

$$\int_Y\int_{\mathbb{R}^N} \left[\phi_n(y,\xi)\,(1+\omega(\xi)) + \mathcal{E}(\phi_n(y,.))\right] d\mu_y(\xi)\, d\nu(y) \le C_1 < \infty, \tag{2.1}$$

where $\mathcal{E} = \mathcal{E}_{j,y}$ *is the non negative* Jensen *information functional defined by*

$$\mathcal{E}(\phi) = \int_{\mathbb{R}^N} j(\phi)\, d\mu_y - j\Big(\int_{\mathbb{R}^N} \phi\, d\mu_y\Big) \qquad \text{if} \quad 0 \le \phi \in L^1(\mathbb{R}^N, d\mu_y). \tag{2.2}$$

Then, there exists $\phi \in L^1(Y \times \mathbb{R}^N)$ *and a subsequence* $(\phi_{n'})$ *such that for every* $k \in \mathbb{N}$ *we can find* $A = A_k \subset Y_k$ *in such a way that* (A_k) *is increasing,* $\nu(Y_k \backslash A_k) < 1/k$ *and*

$$\phi_{n'} \rightharpoonup \phi \quad \text{weakly in} \quad L^1(A \times \mathbb{R}^N). \tag{2.3}$$

Furthermore, $\mathcal{E}$ *is a convex and weakly* L^1 *l.s.c. functional, and thus*

$$\int_Y \int_{\mathbb{R}^N} \big[\phi(y,\xi)\,(1+\omega(\xi)) + \mathcal{E}(\phi(y,.))\big]\, d\mu_y(\xi)\, d\nu(y) \le C_1. \tag{2.4}$$

As we say in the introduction, in general, the sequence of traces (γf_n) does not satisfy a L^1 *a priori* bound, but only an L^0 *a priori* bound (for instance $L^{1/2}$), so that Theorem 2.2 can not be applied (with $\phi_n = \gamma f_n$). Of course, the L^0 *a priori* bound is a very weak information which do not imply compactness of any kind for the sequence (γf_n). But, as we shall see in the next section, we have one more key information about our sequence of traces (γf_n) due to the fact that precisely γf_n is the trace of a solution f_n of a VFP equation and that (f_n) converges. This additional information is that (γf_n) converges in the renormalized sense, another weak weak convergence, that we define now.

Definition 2.2. *We define* $T_M(s) := s \wedge M = \min(s, M)$*, for any* $M \in \mathbb{N}$*. We say that a sequence* (ϕ_n) *of* $L(Y)$ *converges in the renormalized sense to* $\phi \in L(Y)$*, and we write* $\phi_n \overset{r}{\rightharpoonup} \phi$ *or* $\phi = r\text{-}lim\, \phi_n$*, if there exists a sequence* $(\bar{T}_M)$ *of* $L^\infty(Y)$ *such that*

$$T_M(\phi_n) \rightharpoonup \bar{T}_M \quad \sigma(L^\infty(Y), L^1(Y))\,\star \qquad \text{and} \qquad \bar{T}_M \nearrow \phi \qquad \text{a.e. in } Y. \tag{2.5}$$

We refer to the Appendix for the definitions of the lim inf and lim sup in the renormalized sense as well as basic properties concerning the renormalized convergence.

Combining renormalized convergence with the L^0 *a priori* bound, we can prove that $\gamma f_n \overset{r}{\rightharpoonup} \gamma f$ with $\gamma f \in L^0$, and we can then deduce $\widetilde{\gamma_+ f_n} \overset{r}{\rightharpoonup} \psi$ with $\psi \in L^0$. Our second Functional Analysis result gives a extension of the Biting Lemma in the L^0 framework.

Theorem 2.3. *Let* (ψ_n) *be a sequence of* $L^0(Y)$ *and assume that* $\psi_n \overset{r}{\rightharpoonup} \psi$ *in the renormalized sense with* $\psi \in L^0(Y)$*. Then, there exists a subsequence* $(\psi_{n'})$ *which converges in the biting* L^1*-weak sense to* ψ *and moreover* (ψ_n) *is asymptotically bounded in* $L^0(Y)$*: for any* $k \in \mathbb{N}$ *there exists* $\delta_k : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $\delta_k(M) \searrow 0$ *when* $M \nearrow +\infty$ *and for any* M *there is* $n_{k,M}$ *such that*

$$meas\{y \in Y_k,\ \psi_n(y) \ge M\} \le \delta_k(M) \qquad \forall k \in \mathbb{N},\ \forall n \ge n_{k,M}. \tag{2.6}$$

Remark 2.1. In the L^1 framework, J. Ball & F. Murat [8] have already proved that the biting L^1-weak convergence implies, up to the extraction of a subsequence, the convergence in the renormalized sense. Their proof readily extends to the L^0 framework. As a consequence, combining Ball & Murat's result with Theorem 2.3 we get the equivalence between the biting L^1-weak convergence and the renormalized convergence. More precisely, considering a sequence (ψ_n) of $L(Y)$, it is equivalent to say that, up to the extraction of a subsequence,

$$\psi_n \stackrel{b}{\rightharpoonup} \psi \quad \text{in the biting } L^1\text{-weak sense (so that } \psi \in L^0(Y)), \tag{2.7}$$

$$\psi_n \stackrel{r}{\rightharpoonup} \psi \quad \text{in the renormalised sense and } \psi \in L^0(Y). \tag{2.8}$$

Furthermore, in both cases, the full sequence (ψ_n) is asymptotically bounded in L^0.

Remark 2.2. Let emphasize that, if (ψ_n) satisfies (2.8) then there is a subsequence $(\psi_{n'})$ which biting L^1-weak converges to ψ, but in general, the full sequence do not biting L^1-weak converges. A similarly situation holds for the implication (2.7) to (2.8) and we refer to the Appendix for details.

Let also emphasize that the hypothesis $\psi \in L^0(Y)$ in (2.8) is fundamental, since for example, the sequence (ψ_n) defined by $\psi_n \equiv +\infty \ \forall n$ does converge in the renormalized sense to $\psi \equiv +\infty$, but (ψ_n) does not converge (and none of its subsequence!) in the biting L^1-weak sense.

Let emphasize once more, that the (asymptotically) boundedness of (ψ_n) in L^0 does not guarantee that (ψ_n) satisfies, up to the extraction of a subsequence, (2.7) or (2.8). An instructive example is the following: we define $u(y) = 1/y$ on $Y = [0, 1]$ that we extend by 1-periodicity to $\mathbb{R}$, and we set $\psi_n(y) = u(n\,y)$ for $y \in Y$. Therefore, (ψ_n) is obviously bounded in $L^a(Y)$ for all $a \in [0, 1)$ and converges to $\psi \equiv +\infty$ in the renormalized sense.

A simple consequence of the two preceding results is the following.

Corollary 2.4. *Consider a function $m : \mathbb{R}^N \to \mathbb{R}$ and a family of measures $d\varpi_y$ on $\mathbb{R}^N$ such that*

$$\int_{\mathbb{R}^N} m(\xi)\, d\varpi_y(\xi) = 1, \quad \int_{\mathbb{R}^N} m(\xi)^{1/4}\, d\varpi_y(\xi) \le C_4 \ \forall y \quad \textit{and} \quad m(0) \ge m(\xi) \underset{|\xi|\to\infty}{\longrightarrow} 0. \tag{2.9}$$

Let (ϕ_n) be a sequence of $L^0(Y \times \mathbb{R}^N)$ which satisfies

$$\int_Y \mathcal{E}\Big(\frac{\phi_n(y, .)}{m(.)}\Big)\, d\nu(y) \le C_1 < \infty, \tag{2.10}$$

with $\mathcal{E}$ just like in Theorem 2.2 with $d\mu_y(\xi) = m(\xi)\, d\varpi_y(\xi)$, and assume that

$$\psi_n(y) := \int_{\mathbb{R}^N} \phi_n(y, \xi)\, d\varpi_y(\xi) \ \stackrel{r}{\rightharpoonup} \ \psi \quad \textit{with} \quad \psi \in L^0(Y). \tag{2.11}$$

Then, there exists $\phi \in L^1(Y \times \mathbb{R}^N, d\nu d\varpi)$ and a subsequence $(\phi_{n'})$ such that for every $k \in \mathbb{N}$ we can find $A = A_k \subset Y_k$ in such a way that (A_k) is increasing, $\nu(Y_k \backslash A_k) < 1/k$ and

$$\phi_{n'} \rightharpoonup \phi \quad \textit{weakly in} \quad L^1(A \times \mathbb{R}^N, d\nu d\varpi). \tag{2.12}$$

As a consequence $\psi = \int_{\mathbb{R}^N} \phi\, d\varpi$ and $\mathcal{E}(\phi/m) \in L^1(Y)$.

Proof of Theorem 2.2. From bound (2.1) and the Biting Lemma we know that there exists a subsequence n' such that for every $k \in \mathbb{N}$ we can find a Borel set $A = A_k \subset Y_k$ with $\nu(Y_k \backslash A) < 1/k$ such that

$$\int_{\mathbb{R}^N} \phi_{n'} \, d\mu_y(\xi) \quad \text{weakly converges in } \; L^1(A). \tag{2.13}$$

Thanks to the Dunford Pettis Lemma and (2.13) there is a convex function $\Phi = \Phi_k$ such that $\Phi(s)/s \to \infty$ when $s \to \infty$ and

$$\int_A \Phi\Big(\int_{\mathbb{R}^N} \phi_{n'} \, d\mu_y(\xi)\Big) \, d\nu(y) \le C_2 = C_2(k) < \infty. \tag{2.14}$$

Furthermore, we can assume that $\Phi(0) = 0$, $\Phi' = a_m$ in $[m, m+1]$ with $j'(s_0) \le a_m \nearrow +\infty$, where $s_0 \in \mathbb{N}^\star$ is such that $j(s_0) \ge 0$ and $j'(s_0) \ge 0$.

Then we define $\Psi = \Psi_k$ by $\Psi(s) = j(s)$ for $s \in [0, s_0]$ and by induction on $m \in \mathbb{N}$, we consider t_m such that $j'(t_m) = a_m - \Psi'(s_m) + j'(s_m)$ and we set $s_{m+1} = [t_m] + 1$, $\Psi'' := j''$ on $[s_m, t_m]$ and $\Psi'' := 0$ on $[t_m, s_{m+1}]$ so that $t_m \ge s_m \ge m$ and $\Psi'(s_{m+1}) \ge a_m \ge \Psi'(s_m)$. Therefore, we have built a convex function Ψ such that the function $s \mapsto j(s) - \Psi(s)$ is convex, $\Psi(s)/s \nearrow \infty$ since $\Psi'(s) \nearrow \infty$, and $\Psi \le \Phi$ since $\Psi' \le \Phi'$, so that

$$\int_A \Psi\Big(\int_{\mathbb{R}^N} \phi_{n'} \, d\mu\Big) \, d\nu \le C_2. \tag{2.15}$$

The Jensen inequality, written for the function $s \mapsto j(s) - \Psi(s)$, gives

$$\int_{\mathbb{R}^N} \Psi(\phi_{n'}) \, d\mu - \Psi\big(\int_{\mathbb{R}^N} \phi_{n'} \, d\mu\big) \le \mathcal{E}(\phi_{n'}),$$

and combining it with (2.1) and (2.15) we get

$$\iint_{A \times \mathbb{R}^N} \Psi(\phi_{n'}) \, d\mu \, d\nu \le C_1 + C_2,$$

and thus

$$\begin{aligned} \iint_{A \times \mathbb{R}^N} \Psi^+(\phi_{n'}) \, d\mu_y \, d\nu &\le C_1 + C_2 + \iint_{A \times \mathbb{R}^N} \Psi^-(\phi_{n'}) \, d\mu_y \, d\nu \\ &\le C_3(k) := C_1 + C_2 + \nu(A) \sup j^- < \infty. \end{aligned} \tag{2.16}$$

Therefore, thanks to estimates (2.1) and (2.16) and thanks to the Dunford-Pettis Lemma we get that $(\phi_{n'})$ falls in a relatively weakly compact set of $L^1(A_k \times \mathbb{R}^N)$ for any $k \in \mathbb{N}$. We conclude, by a diagonal process, that there is a function $\phi \in L^1(Y \times \mathbb{R}^N)$ and a subsequence $(\phi_{n''})$ which converges to ϕ in the sense stated in Theorem 2.2.

In order to prove that $\mathcal{E}$ is a convex functional, we begin by assuming that $j \in C^1(\mathbb{R}_+, \mathbb{R})$, so that $\mathcal{E}$ is Gâteaux differentiable. By definition of the G-differential

$$\begin{aligned} \nabla\mathcal{E}(\phi) \cdot \psi &:= \lim_{t \to 0} \frac{\mathcal{E}(\phi + t\,\psi) - \mathcal{E}(\phi)}{t} \\ &= \int_{\mathbb{R}^N} j'(\phi) \, \psi \, d\mu - j'\big(\int_{\mathbb{R}^N} \phi \, d\mu\big) \int_{\mathbb{R}^N} \psi \, d\mu, \end{aligned}$$

for any $0 \le \phi, \psi \in L^\infty(\mathbb{R}^N)$. Therefore, by the Jensen inequality, we have

$$< \nabla\mathcal{E}(\psi) - \nabla\mathcal{E}(\phi), \psi - \phi > = \int_{\mathbb{R}^N} J(\phi,\psi)\, d\mu - J\Big(\int_{\mathbb{R}^N} \phi\, d\mu, \int_{\mathbb{R}^N} \psi\, d\mu\Big) \ge 0,$$

so that $\nabla\mathcal{E}$ is monotone and thus $\mathcal{E}$ is convex on $L^\infty(\mathbb{R}^N)$: for any $0 \le \phi, \psi \in L^\infty(\mathbb{R}^N)$ and any $t \in (0,1)$

$$\mathcal{E}(\phi + (1-t)\,\psi) \le t\,\mathcal{E}(\phi) + (1-t)\,\mathcal{E}(\psi). \tag{2.17}$$

When $j \notin C^1(\mathbb{R}_+, \mathbb{R})$ we define, for any $\varepsilon > 0$, the function $j_\varepsilon(s) = j(s+\varepsilon) - j(\varepsilon)$ which belongs to $C^1(\mathbb{R}_+, \mathbb{R})$, and the above computations for the associated functional $\mathcal{E}_\varepsilon$ are correct, so that inequality (2.17) holds for $\mathcal{E}$ replaced by $\mathcal{E}_\varepsilon$. Then, writing inequality (2.17) for $\mathcal{E}_\varepsilon$ and fixed $0 \le \phi, \psi \in L^\infty(\mathbb{R}^N)$, $t \in (0,1)$ and passing to the limit $\varepsilon \to 0$ we obtain that $\mathcal{E}$ is convex on $L^\infty(\mathbb{R}^N)$. Now let $0 \le \phi, \psi \in L^1(\mathbb{R}^N)$, $t \in (0,1)$. If $j(\phi)$ or $j(\psi) \notin L^1(\mathbb{R}^N)$ then $t\,\mathcal{E}(\phi) + (1-t)\,\mathcal{E}(\psi) = +\infty$ and the convex inequality (2.17) obviously holds. In the other case, we have $j(\phi)$, $j(\psi) \in L^1(\mathbb{R}^N)$, we can choose two sequences $0 \le (\phi_n)$, (ψ_n) of $L^\infty(\mathbb{R}^N)$ such that $\phi_n \nearrow \phi$ and $\psi_n \nearrow \psi$ a.e., and passing to the limit $\varepsilon \to 0$ in the convex inequality (2.17) written for ϕ_ε and ψ_ε we get, by the Lebesgue convergence dominated Theorem and the Fatou Lemma,

$$\begin{aligned}\int_{\mathbb{R}^N} j(t\,\phi + (1-t)\,\psi) &\le \liminf_{\varepsilon\to 0} \int_{\mathbb{R}^N} j(t\,\phi_\varepsilon + (1-t)\,\psi_\varepsilon) \\ &\le t\,\mathcal{E}(\phi) + (1-t)\,\mathcal{E}(\psi) + j\Big(\int_{\mathbb{R}^N} t\,\phi + (1-t)\,\psi\Big),\end{aligned}$$

which exactly means that $\mathcal{E}$ is a convex functional in $L^1(\mathbb{R}^N)$. Finally, if $0 \le \phi,\, \psi \in L^1(Y \times \mathbb{R}^N)$ and $t \in (0,1)$, then $\phi(y,.)$, $\psi(y,.) \in L^1(\mathbb{R}^N)$ for almost every $y \in Y$ and, integrating the convex inequality (2.17), we obtain that the functional

$$0 \le \phi \in L^1(Y \times \mathbb{R}^N) \mapsto \mathcal{F}(\phi) = \int_Y \mathcal{E}(\phi)\, d\nu$$

is convex. Furthermore, by Fatou Lemma, $\mathcal{F}$ is l.s.c. and then $\mathcal{F}$ is l.s.c. for the biting L^1-weak convergence, so that (2.4) holds. □

Proof of Theorem 2.3. We first prove the asymptotic L^0 bound. To do so, we argue by contradiction. For an arbitrary $\varepsilon > 0$ we know that there exists $B \subset Y_k$ such that $\nu(Y_k \backslash B) < \varepsilon/2$ and $\psi \in L^1(B)$. If there is no $m \in \mathbb{N}$ such that meas $\{\psi_n \ge m\} < \varepsilon/2$ for all n large enough, this means that there exists an increasing sequence (n_m) such that

$$\text{meas}\,\{\psi_{n_m} \ge m\} \ge \varepsilon/2 \qquad \forall m \ge 0.$$

Therefore, for any $\ell \in \mathbb{N}$ and any $m \ge \ell$ we have

$$\int_B T_\ell(\psi_{n_m}) \ge \text{meas}\,\{\psi_{n_m} \ge \ell\}\,\ell \ge \frac{\varepsilon}{2}\,\ell,$$

and passing to the limit $m \to \infty$ we get

$$\int_B \psi \geq \int_B \bar{T}_\ell \geq \frac{\varepsilon}{2}\,\ell \qquad \forall \ell \geq 0.$$

Letting $\ell \nearrow \infty$ we get a contradiction with the fact that $\psi \in L^1(B)$.

Next, we pass to the convergence in the biting L^1-weak sense. For any $k \in \mathbb{N}$ we can choose A'' such that $\nu(Y_k \backslash A'') < 1/3k$ and $\psi \in L^1(A'')$. Setting $\displaystyle\int_{A''} \psi \, dy = C_0$ we construct a first subsequence (n_ℓ) such that

$$\int_{A''} T_\ell(\psi_{n_\ell}) \, dy \leq C_0 + \frac{1}{\ell}.$$

Then, for any $M \in \mathbb{N}$ we have $T_M(\psi_{n_\ell}) \leq T_\ell(\psi_{n_\ell})$ for $\ell \geq M$ so that $\bar{T}_M \leq \liminf T_\ell(\psi_{n_\ell})$ and thus

$$\psi \leq \liminf T_\ell(\psi_{n_\ell}).$$

Here, the lim inf of $T_\ell(\psi_{n_\ell})$ is taken in the biting L^1-weak sense, what we can do since $(T_\ell(\psi_{n_\ell}))_\ell$ is a bounded sequence of L^1. But since

$$\int_{A''} \limsup T_\ell(\psi_{n_\ell}) \, dy \leq \limsup \int_{A''} T_\ell(\psi_{n_\ell}) \leq \int_{A''} \psi \, dy,$$

we see that $T_\ell(\psi_{n_\ell}) \overset{b}{\rightharpoonup} \psi$ in $L^1(A'')$. Using Theorem 2.1 there is A' such that $|A'' \backslash A'| < 1/3k$ and

$$T_\ell(\psi_{n_\ell}) \rightharpoonup \psi \quad \text{ weakly in } L^1(A').$$

Furthermore, since (ψ_n) is asymptotically bounded in $L^0(Y)$ we have, up to the extraction of a subsequence again,

$$\text{meas}\{\psi_{n_\ell} \neq T_\ell(\psi_{n_\ell})\} = \text{meas}\{\psi_{n_\ell} > \ell\} \leq \delta_k(\ell) \underset{\ell \to \infty}{\longrightarrow} 0.$$

Therefore, we can choose an other subsequence, still noted (ψ_{n_ℓ}), such that $Z_L := \{\forall \ell \geq L \;/\; \psi_{n_\ell} \neq T_\ell(\psi_{n_\ell})\}$ and satisfies

$$\text{meas}(Z_L) \leq \sum_{\ell \geq L} \text{meas}\{\psi_{n_\ell} > \ell\} \underset{L \to \infty}{\longrightarrow} 0.$$

Finally, choosing L large enough such that meas $(Z_L) < 1/3k$ and setting $A_k := A' \cap Z_L^c$, we have $|Y_k \backslash A| < 1/k$, $\psi_{n_\ell} \in L^1(A)$ for all $\ell \geq L$ and

$$\psi_{n_\ell} = T_\ell(\psi_{n_\ell}) \rightharpoonup \psi \quad \text{ weakly in } L^1(A).$$

We conclude thanks to a diagonal process. □

Proof of Corollary 2.4. Thanks to Theorem 2.3 we know that there exists a subsequence $(\psi_{n'})$ such that for every $k \in \mathbb{N}$ we can find $A = A_k \subset Y_k$ satisfying (A_k) is increasing, $\nu(Y_k \backslash A_k) < 1/k$ and

$$\psi_{n'} \quad \text{is weakly compact in} \quad L^1(A). \tag{2.18}$$

Next, we come back to the proof of Theorem 2.2, and estimates (2.16). Written with the new notation, we have

$$\iint_{A\times\mathbb{R}^N} \phi_{n'}\,\Xi\big(\frac{\phi_{n'}}{m(\xi)}\big)\,d\varpi_y d\nu \le C_3, \tag{2.19}$$

where we have set $\Psi^+(s) = s\,\Xi(s)$. Of course we can assume, without loss of generality, that Ξ is not decreasing, $\Xi(s) \nearrow \infty$ when $s \nearrow \infty$ and $\Xi(s) \le s^{1/2}$. Then, we deduce from (2.19)

$$\iint_{A\times\mathbb{R}^N} \phi_{n'}\,\Xi\big(\frac{\phi_{n'}}{m(0)}\big)\,d\varpi_y\, d\nu \le C_3, \tag{2.20}$$

and

$$\begin{aligned}
&\iint_{A\times\mathbb{R}^N} \phi_{n'}\,\Xi(m(\xi)^{-1/2})\,d\varpi_y\,d\nu \le \\
&\le \iint_{A\times\mathbb{R}^N} \phi_{n'}\,\Xi(m(\xi)^{-1/2})\,\big(\mathbf{1}_{\{\phi_{n'}\le m(\xi)^{1/2}\}} + \mathbf{1}_{\{\phi_{n'}\ge m(\xi)^{1/2}\}}\big)\,d\varpi_y\,d\nu \\
&\le \iint_{A\times\mathbb{R}^N} m(\xi)^{1/4}\,d\varpi_y\,d\nu + \iint_{A\times\mathbb{R}^N} \phi_{n'}\,\Xi\big(\frac{\phi_{n'}}{m(\xi)}\big)\,d\varpi_y\,d\nu \le C_4\,|Y_k| + C_3.
\end{aligned} \tag{2.21}$$

Combining (2.18), (2.20) and (2.21), we deduce by the Dunford-Pettis Lemma that $(\phi_{n'})$ belongs to a weak compact set of $L^1(A\times\mathbb{R}^N, d\nu d\varpi)$, and we conclude as in the end of the proof of Theorem 2.2. □

3. Trace theorems for solutions of the Vlasov-Fokker-Planck equation.

In this section we extend to the VFP equation the trace results established in [53], [54] for the Vlasov equation. Given a vector field $E = E(t,x,\xi)$, a source term $G = G(t,x,\xi)$, a constant $\nu \in \mathbb{R}$ and a solution $g = g(t,x,\xi)$ of the Vlasov-Fokker-Planck equation

$$\Lambda_E\, g = \frac{\partial g}{\partial t} + \xi\cdot\nabla_x g + E\cdot\nabla_\xi g - \nu\,\Delta_\xi g = G \ \text{ in } D, \tag{3.1}$$

we show that g has a trace γg on the boundary $(0,T)\times\Sigma$ and a trace $\gamma_t g$ on the section $\{t\}\times\mathcal{O}$ for all $t\in[0,T]$. These trace functions are defined thanks to a Green formula. We note indifferently $\gamma_t g = g(t,.)$.

The meaning of equation (3.1) is of two kinds. In the first case, we assume that $g \in L^\infty(0,T;L^p_{loc}(\bar{\mathcal{O}}))$ with $p\in[1,\infty]$ is a solution of (3.1) in the sense of distributions, i.e.,

$$\iiint_D (g\,\Lambda^\star_E\phi + G\,\phi)\,d\xi dx dt = 0, \tag{3.2}$$

for all test functions $\phi \in \mathcal{D}(D)$, where we have set

$$\Lambda^\star_E\,\phi = \frac{\partial\phi}{\partial t} + \xi\cdot\nabla_x\phi + E\cdot\nabla_\xi\phi + \nu\,\Delta_\xi\phi + (\mathrm{div}_\xi E)\,\phi. \tag{3.3}$$

In this case we assume

$$
\begin{aligned}
(3.4)\qquad & E \in L^1\big(0,T; W^{1,p'}_{loc}(\mathcal{O}) \cap W^{1,1}_{loc}(\bar{\mathcal{O}}) \cap L^{p'}_{loc}(\bar{\mathcal{O}})\big),\\
& \mathrm{div}_\xi E \in L^1\big(0,T; L^{p'}_{loc}(\bar{\mathcal{O}})\big), \quad G \in L^1_{loc}([0,T]\times\bar{\mathcal{O}}),
\end{aligned}
$$

where $p' \in [1,\infty]$ stands for the conjugate exponent of p, given by $1/p + 1/p' = 1$, and we make one of the two additional hypothesis

$$
(3.5)\qquad \int_0^T \int_{\mathcal{O}_R} |\nabla_\xi g|^2 \, d\xi dx dt \le C_{T,R}
$$

or

$$
(3.6)\qquad \int_0^T \int_{\mathcal{O}_R} |\nabla_\xi g|^2 \, \mathbf{1}_{\{M \le |g| \le M+1\}} \, d\xi dx dt \le C_{T,R} \quad \forall M \ge 0.
$$

Remark 3.1. The bound (3.6) is the natural bound that appears when we consider, for example, the initial value problem with initial datum $g_0 \in L^p(\mathcal{O})$ when $\Omega = \mathbb{R}^N$ or when Ω is an open subset of $\mathbb{R}^N$ and specular reflections are imposed at the boundary.

In the second case, we assume that g is a renormalized solution of (3.1). In order to make precise the meaning of such a solution, we must define some notation. We denote by $\mathcal{B}_1$ the class of all functions $\beta \in W^{2,\infty}(\mathbb{R})$ such that β' has a compact support and by $\mathcal{B}_2$ the class of all functions $\beta \in W^{2,\infty}_{loc}(\mathbb{R})$ such that β'' has a compact support. Remark that for every $u \in L(Y)$ and $\beta \in \mathcal{B}_1$ one has $\beta(u) \in L^\infty(Y)$. We shall write $g \in C([0,T]; L(\mathcal{O}))$ if $\beta(g) \in C([0,T]; L^1_{loc}(\bar{\mathcal{O}}))$ for every $\beta \in \mathcal{B}_1$.

We say that $g \in L((0,T)\times\mathcal{O})$ is a renormalized solution of (3.1) if for all $\beta \in \mathcal{B}_1$ we have

$$
(3.7)\quad E \in L^1\big(0,T; W^{1,1}_{loc}(\bar{\mathcal{O}})\big), \; \beta'(g)\, G \in L^1_{loc}([0,T]\times\bar{\mathcal{O}}), \; \beta''(g)\, |\nabla_\xi\, g|^2 \in L^1_{loc}([0,T]\times\bar{\mathcal{O}}),
$$

and $\beta(g)$ is solution of

$$
(3.8)\qquad \Lambda_E\, \beta(g) = \beta'(g)\, G - \nu\, \beta''(g)\, |\nabla_\xi\, g|^2 \;\text{ in } \mathcal{D}'(D).
$$

We can now state the trace Theorems that we use in this paper and that we prove in section 5.

Theorem 3.1. (The case $p = \infty$). *Let $g \in L^\infty([0,T]\times\mathcal{O})$ be a solution of equation (3.2)-(3.4)-(3.5). Then for every $t \in [0,T]$ there exists $\gamma_t g \in L^\infty(\mathcal{O})$ and γg defined on $(0,T)\times\Sigma)$ such that*

$$
(3.9)\qquad \gamma_t g \in C([0,T]; L^a_{loc}(\bar{\mathcal{O}})) \quad \forall a \in [1,\infty) \quad \text{and} \quad \gamma\, g \in L^\infty([0,T]\times\Sigma),
$$

and satisfying the Green formula

$$
\begin{aligned}
(3.10)\qquad & \int_{t_0}^{t_1} \iint_{\mathcal{O}} \big(\beta(g)\, \Lambda^\star_E \phi + (\beta'(g)\, G - \nu\, \beta''(g)\, |\nabla_\xi\, g|^2)\, \phi\big)\, d\xi dx dt =\\
&\qquad = \Big[\iint_{\mathcal{O}} \beta(g(\tau,.))\, \phi\, dx d\xi \Big]_{t_0}^{t_1} + \int_{t_0}^{t_1} \iint_{\Sigma} \beta(\gamma\, g)\, \phi\; n(x)\cdot\xi \; d\xi d\sigma_x d\tau,
\end{aligned}
$$

for all t_0, $t_1 \in [0,T]$, all $\beta \in W^{2,\infty}_{loc}(\mathbb{R})$ and all test function $\phi \in \mathcal{D}([0,T] \times \bar{\mathcal{O}})$.

Remark 3.2. A fundamental point, which is a consequence of Green formula (3.10), is the possibility of renormalizing the trace function, i.e.

$$\gamma\,\beta(g) = \beta(\gamma\, g) \tag{3.11}$$

for all $\beta \in W^{2,\infty}(\mathbb{R})$. More generally, (3.11) holds as soon as $\gamma\,\beta(g)$ is defined. This is the property that will allow us to define the trace of a renormalized solution.

Theorem 3.2. (The case $p \in [1,\infty)$). *Let $g \in L^\infty(0,T;L^p_{loc}(\bar{\mathcal{O}}))$ be a solution of equation (3.2)-(3.4)-(3.6). Then for every $t \in [0,T]$ there exists $\gamma_t g \in L^p(\mathcal{O})$ and γg defined on $(0,T)\times \Sigma$ such that*

$$\gamma_t g \in C([0,T];L^1_{loc}(\mathcal{O})) \quad \text{and} \quad \gamma\, g \in L^1_{loc}\big([0,T]\times\Sigma, d\lambda_2\big), \tag{3.12}$$

and which satisfies the Green formula (3.10) for every t_0, $t_1 \in [0,T]$, every $\beta \in \mathcal{B}_1$ and every test function $\phi \in \mathcal{D}([0,T] \times \bar{\mathcal{O}})$, and also for every t_0, $t_1 \in [0,T]$, every $\beta \in \mathcal{B}_2$ and every test functions $\phi \in \mathcal{D}_0([0,T] \times \bar{\mathcal{O}})$, the space of functions $\phi \in \mathcal{D}([0,T] \times \bar{\mathcal{O}})$ such that $\phi = 0$ on $(0,T)\times\Sigma_0$.

Theorem 3.3. (The renormalized case). *Let $g \in L((0,T)\times\mathcal{O})$ satisfy (3.7) and the equation (3.8). Then for every $t \in [0,T]$ there exists $\gamma_t g \in C([0,T];L(\mathcal{O}))$ and $\gamma g \in L([0,T]\times \Sigma)$, satisfying the Green formula (3.10) for all t_0, $t_1 \in [0,T]$, all $\beta \in \mathcal{B}_1$ and all test functions $\phi \in \mathcal{D}([0,T] \times \bar{\mathcal{O}})$. Furthermore, if (3.8) make sense for at least one function β such that $\beta(s) \nearrow \infty$ when $s \nearrow \infty$, then of course $\gamma_t g \in L^0(\mathcal{O})$ for any $t \in [0,T]$ and $\gamma g \in L^0([0,T]\times\Sigma)$.*

We present now a quite general stability result in both the interior and at the boundary for a sequence of renormalized solutions to the Vlasov-Fokker-Planck equation on a bounded domain. This will be a key argument in the proof of Theorem 1.1. In some sense, this result says that renormalized convergence, as well as a.e. convergence, can be propagated from the interior to the boundary. Notice that this propagation property does not obviously hold for the L^1-weak convergence.

Theorem 3.4. *Consider three sequences (g_n), (E_n) and (G_n) which satisfy, for all $\beta \in \mathcal{B}_3$ the class of functions of $W^{2,\infty}_{loc}(\mathbb{R})$ such that $|\beta'(s)|\,(1+s)^{-1} \in L^\infty(\mathbb{R})$ and $|\beta''(s)|\,(1+s)^{-2} \in L^\infty(\mathbb{R})$,*

$$g_n \to g \text{ strongly in } L^1(0,T)\times\mathcal{O}) \text{ and is uniformly bounded in } L^\infty(0,T;L^1(\mathcal{O})), \tag{3.13}$$

$$E_n \rightharpoonup E \quad \text{weakly in } L^1(0,T;W^{1,1}_{loc}(\bar{\mathcal{O}})), \tag{3.14}$$

$$\beta'(g_n)\,G_n \rightharpoonup \beta'(g)\,G \quad \text{weakly in } L^1((0,T)\times\mathcal{O}_R) \quad \forall \beta \in \mathcal{B}_3,\ \forall R \geq 0, \tag{3.15}$$

$$\int_0^T\!\!\int_{\mathcal{O}} \frac{|\nabla_\xi g_n|^2}{1+g_n}\,d\xi dx dt \leq C_T, \tag{3.16}$$

and the renormalized Vlasov-Fokker-Planck equation

$$\Lambda_{E_n}\,\beta(g_n) = \beta'(g_n)\,G_n - \beta''(g_n)\,|\nabla_\xi g_n|^2 \quad \text{in } \mathcal{D}'((0,T)\times\mathcal{O}), \tag{3.17}$$

for which clearly each term make sense thanks to (3.13)–(3.16). Then $g \in L^\infty(0,T;L^1(\mathcal{O}))$ is a solution of

$$\Lambda_E\,\beta(g) = \beta'(g)\,G - \beta''(g)\,|\nabla_\xi g|^2 \quad \text{in } \mathcal{D}'((0,T)\times\mathcal{O}) \tag{3.18}$$

for all $\beta \in \mathcal{B}_3$. Furthermore, the traces γg_n and γg defined thanks to the Theorem 3.3 satisfy

$$\gamma g_n \overset{r}{\rightharpoonup} \gamma g \quad \text{in the renormalized sens, and} \quad \gamma_+ g_n \to \gamma_+ g \quad \text{a.e.} \tag{3.19}$$

We shall need the following auxiliary results in the proof of Theorem 3.4.

Lemma 3.1. *Let (u_n) be a bounded sequence of $L^2(Y)$ such that $u_n \rightharpoonup u$ weakly in $L^2(Y)$. Then, there exists $\mu \in (C_c(Y))'$, a nonnegative measure, such that, up to the extraction of a subsequence,*

$$|u_n|^2 \rightharpoonup |u|^2 + \mu \qquad \text{weakly in} \quad (C_c(Y))'.$$

We note $C_c(Y)$ the space of continuous functions on Y with compact support and $C_b(Y)$ the space of continuous and bounded functions on Y.

Lemma 3.2. *For any $\theta \in (0,1)$ and $M \in (0,\infty)$ we set*

$$\Phi(s) = \Phi_{M,\theta}(s) := \begin{cases} 1/\theta\,(e^{\theta\, s} - 1) & \text{if } s \le M \\ (s-M)\, e^{\theta\, M} + 1/\theta\,(e^{\theta\, M} - 1) & \text{if } s \ge M. \end{cases}$$

Then

$$\begin{cases} \Phi'(s) \ge 1, \qquad \Phi \circ \beta(s) \nearrow s \text{ when } M \nearrow \infty,\ \theta \nearrow 1, \\ \text{and} \qquad 0 \le -(\Phi\circ\beta)''(s) \le \dfrac{1-\theta + e^{(\theta-1)\,M}}{1+s} \end{cases} \qquad \forall s \ge 0.$$

Lemma 3.3. *Let $g \in L^\infty(0,T;L^p_{loc}(\mathcal{O}))$ be a solution to the Vlasov-Fokker-Planck equation*

$$\Lambda_E\, g = G + \mu \quad \text{in } \mathcal{D}'((0,T)\times\mathcal{O}),$$

with $E \in L^1(0,T;W^{1,p'}_{loc}(\mathcal{O}))$, $G \in L^1_{loc}((0,T)\times\mathcal{O}))$ and $\mu \in \mathcal{D}'((0,T)\times\mathcal{O})$, $\mu \ge 0$. For a given mollifer ρ_k in $\mathbb{R}^N$, we set

$$g_k := g *_t \rho_k *_x \rho_k *_\xi \rho_k \qquad \text{and} \qquad \mu_k := \mu *_t \rho_k *_x \rho_k *_\xi \rho_k.$$

Then g_k satisfies the Vlasov-Fokker-Planck equation

$$\Lambda_E\, g_k = G_k + \mu_k \quad \text{in all compact set of } (0,T)\times\mathcal{O},$$

with $G_k \to G$ strongly in $L^1_{loc}([0,T]\times\mathcal{O}))$.

The proof of Lemma 3.1 is classical, the one of Lemma 3.2 is elementary, and we refer to [33,34] for the proof of Lemma 3.3.

Proof of the Theorem 3.4. Step 1: Proof of (3.18). This step is inspired from [35] and it is clear from the theory of renormalized solution [36] that it is enough to prove (3.18) only for $\beta(s) := \log(1+s)$. With the notation $h_n := \beta(g_n)$ and $h = \beta(g)$ we have $\nabla_\xi h_n = \sqrt{-\beta''(g_n)}\,\nabla_\xi g_n \rightharpoonup \sqrt{-\beta''(g)}\,\nabla_\xi g = \nabla_\xi h$ weakly in $L^2((0,T)\times\mathcal{O})$ so that, thanks to Lemma 3.1, there is a bounded measure $\mu \ge 0$ such that, up to the extraction of a subsequence, $|\nabla_\xi h_n|^2 \rightharpoonup |\nabla_\xi h|^2 + \mu$ weakly in $\mathcal{D}'([0,T]\times\bar{\mathcal{O}})$. Passing to the limit $n\to\infty$ in (3.17) we get

$$\Lambda_E\, \beta(g) = \beta'(g)\, G - \beta''(g)\,|\nabla_\xi g|^2 + \mu \quad \text{in } \mathcal{D}'((0,T)\times\mathcal{O}).$$

We just point out that

$$E_n\,\beta(g_n) \rightharpoonup E\,\beta(g) \quad \text{weakly in } L^1((0,T)\times\mathcal{O}),$$

since $\beta(g_n) \to \beta(g)$ strongly in $L^\infty(0,T;L^p(\mathcal{O}))$ for all $p<\infty$ and $E_n \rightharpoonup E$ weakly in $L^1(0,T;L^q(\mathcal{O}))$ for every $q\in[1,N/(N-1))$. We prove now that $\mu=0$ in $(0,T)\times\mathcal{O}$.
With the notation introduced in Lemma 3.3 we have

$$\Lambda_E\,\Phi(h_k) = \Phi'(h_k)\left(\beta'(g)\,G - \beta''(g)\,|\nabla_\xi g|^2\right) *_{t,x,\xi} \rho_k - \Phi''(h_k)\,|\nabla_\xi h_k|^2 + \Phi'(h_k)\,\mu_k.$$

Using that $\Phi'\geq 1$ and passing to the limit $k\to\infty$ we get

$$\Lambda_E\,(\Phi\circ\beta)(g) \geq \Phi'(\beta(g))\,\beta'(g)\,G - (\Phi'(\beta(g))\,\beta''(g) + \Phi''(\beta(g))\,(\beta'(g))^2)\,|\nabla_\xi g|^2 + \mu$$

and then

$$\Lambda_E\,(\Phi\circ\beta)(g) - (\Phi\circ\beta)'(g)\,G \geq (\Phi\circ\beta)''(g)\,|\nabla_\xi g|^2 + \mu \quad \text{in } \mathcal{D}'((0,T)\times\mathcal{O}). \tag{3.20}$$

In order to have an estimate of the left hand side we come back to equation (3.17), and we write

$$\Lambda_{E_n}\,\Phi\circ\beta(g_n) = (\Phi\circ\beta)'(g_n)\,G_n - (\Phi\circ\beta)''(g_n)\,|\nabla_\xi g_n|^2 \quad \text{in } \mathcal{D}'((0,T)\times\mathcal{O})$$

since $\Phi\circ\beta\in\mathcal{B}_3$. Then, for all $\chi\in\mathcal{D}((0,T)\times\mathcal{O}$ such that $0\leq\chi\leq 1$ we have

$$\begin{aligned}\Big|\int_0^T\!\!\int_{\mathcal{O}} \big(\Phi\circ\beta(g_n)\,\Lambda_{E_n}\chi + (\Phi\circ\beta)'(g_n)\,G_n\,\chi\big)\,d\xi dx dt\Big| &=\\ &= -\int_0^T\!\!\int_{\mathcal{O}} (\Phi\circ\beta)''(g_n)\,|\nabla_\xi g_n|^2\,\chi\,d\xi dx dt\\ &\leq [1-\theta+e^{(\theta-1)\,M}]\int_0^T\!\!\int_{\mathcal{O}} \frac{|\nabla_\xi g_n|^2}{1+g_n}\,d\xi dx dt.\end{aligned}$$

Passing to the limit $n\to\infty$ we get, thanks to (3.16),

$$\Big|\int_0^T\!\!\int_{\mathcal{O}} \big(\Phi\circ\beta(g)\,\Lambda_E\chi + (\Phi\circ\beta)'(g)\,G\,\chi\big)\,d\xi dx dt\Big| \leq [1-\theta+e^{(\theta-1)\,M}]\,C_T.$$

Then, coming back to (3.20), we have

$$\begin{aligned}\int_0^T\!\!\int_{\mathcal{O}} \chi\,d\mu &\leq -\int_0^T\!\!\int_{\mathcal{O}} \big(\Phi\circ\beta(g)\,\Lambda_E\chi + (\Phi\circ\beta)'(g)\,G\,\chi + (\Phi\circ\beta)''(g)\,|\nabla_\xi g|^2\big)\,d\xi dx dt\\ &\leq 2\,[1-\theta+e^{(\theta-1)\,M}]\,C_T \qquad \forall\theta\in[0,1],\ M>0,\end{aligned}$$

and letting $M\to\infty$ and then $\theta\to 1$ we obtain $\mu=0$ on $\operatorname{supp}\chi$, which is precisely to say that $\mu=0$ in $(0,T)\times\mathcal{O}$.

Step 2: Proof of (3.19). We fix $\phi \in \mathcal{D}((0,T)\times\bar{\mathcal{O}})$ such that $0 \le \phi \le 1$. By definition of γg_n we have

$$\Big| \int_0^T \iint_\Sigma \Phi\circ\beta(\gamma\, g_n)\,\phi\; n(x)\cdot\xi \; d\xi d\sigma_x dt \\ - \int_0^T\int_{\mathcal{O}} \big(\Phi\circ\beta(g_n)\,\Lambda_{E_n}\chi + (\Phi\circ\beta)'(g_n)\,G_n\,\chi\big)\, d\xi dx dt \Big| = \\ = \int_0^T\int_{\mathcal{O}} (\Phi\circ\beta)''(g_n)\,|\nabla_\xi g_n|^2\,\chi\, d\xi dx dt \le [1-\theta+e^{(\theta-1)\,M}]\int_0^T\int_{\mathcal{O}} \frac{|\nabla_\xi g_n|^2}{1+g_n}\, d\xi dx dt.$$

We note $\overline{\Phi\circ\beta}$ the L^1-weak limit of $\Phi\circ\beta(\gamma\, g_n)$. Passing to the limit $n\to\infty$ we get

$$\Big| \int_0^T \iint_\Sigma \overline{\Phi\circ\beta}\,\phi\; n(x)\cdot\xi \; d\xi d\sigma_x dt - \int_0^T\int_{\mathcal{O}} \big(\Phi\circ\beta(g)\,\Lambda_E\,\chi + (\Phi\circ\beta)'(g)\,G\,\chi\big)\, d\xi dx dt \Big| \le \\ \le [1-\theta+e^{(\theta-1)\,M}]\, C_T,$$

and thus

$$\Big| \int_0^T \iint_\Sigma \overline{\Phi\circ\beta}\,\phi\; n(x)\cdot\xi \; d\xi d\sigma_x dt - \int_0^T\int_{\mathcal{O}} \big[\Phi\circ\beta(g)\,\Lambda_E\,\chi \\ + \big((\Phi\circ\beta)'(g)\,G - (\Phi\circ\beta)''(g)\,|\nabla_\xi g|^2\big)\,\chi\big]\, d\xi dx dt \Big| \le 2\,[1-\theta+e^{(\theta-1)\,M}]\, C_T.$$

Once again, by definition of γg, we obtain

$$\Big| \int_0^T \iint_\Sigma \big(\overline{\Phi\circ\beta} - \Phi\circ\beta(\gamma\, g)\big)\,\phi\; n(x)\cdot\xi \; d\xi d\sigma_x dt \Big| \le 2\,[1-\theta+e^{(\theta-1)\,M}]\, C_T \underset{M\to\infty,\theta\to 1}{\longrightarrow} 0,$$

and $\overline{\Phi\circ\beta} \nearrow r\text{-}lim\,\gamma g_n$ since $\Phi\circ\beta(s)\nearrow s$ when $M\nearrow\infty$, $\theta\searrow 1$, so that $\gamma g = r\text{-}lim\,\gamma g_n$.

In order to prove the a.e. convergence we only have to show, thanks to Proposition 6.3.3, that, up to the extraction of a subsequence,

$$r\text{-}liminf\,\beta(\gamma_+ g_n) \ge \beta(\gamma_+ g). \tag{3.21}$$

Using Lemma 3.1 and the first step, we can pass to the limit in (3.17), up to the extraction of a subsequence, and we get

$$\int_0^T \iint_\Sigma \overline{\beta}\,\phi\; n(x)\cdot\xi \; d\xi d\sigma_x dt = \int_0^T\int_{\mathcal{O}} \big(\beta(g)\,\Lambda_E\,\chi + (\beta'(g)\,G + \beta''(g)\,|\nabla_\xi g|^2 + \mu)\,\chi\big)\, d\xi dx dt \\ = \int_0^T \iint_\Sigma (\beta(\gamma g)\; n(x)\cdot\xi + \mu)\,\phi\; d\xi d\sigma_x dt,$$

where $\overline{\beta} = \text{w-lim}\,\beta(\gamma g_n)$ is the weak limit in $L^1((0,T)\times\Sigma)$ of $\beta(\gamma g_n)$. We deduce that $\overline{\beta}\; n(x)\cdot\xi = \beta(\gamma g)\; n(x)\cdot\xi + \mu$ on $(0,T)\times\Sigma$, and in particular

$$\overline{\beta} \ge \beta(\gamma_+ g) \quad \text{on} \quad (0,T)\times\Sigma_+.$$

Since $r\text{-}liminf\, \beta(\gamma_+ g_n) = \bar\beta$ this prove (3.21). $\square$

4. A priori estimates and proof of the main result.

In this section we derive the *a priori* physical bound, then make precise the exact meaning of renormalized solution we deal with and finally present a proof of Theorem 1.1. In order to do not repeat twice the exposition, we consider the full Vlasov-Poisson-Fokker-Planck-Boltzmann system (VPFPB in short)

$$\frac{\partial f}{\partial t} + \xi\cdot\nabla_x f - \operatorname{div}_\xi\big((\nabla_x V_f + \lambda\,\xi)\,f\big) - \nu\,\Delta_\xi f = Q(f,f) \ \text{ in } \ (0,\infty)\times\mathcal{O}, \tag{4.1}$$

where V_f is given by the Poisson equation (1.7). We assume that f satisfies the boundary condition (1.4) and the initial condition (1.8), where f_0 is assumed to verify (1.12) and

$$\int_\Omega |\nabla V_{f_0}|^2\,dx < \infty \qquad \text{with} \qquad -\Delta V_{f_0} = \int_{\mathbb{R}^3} f_0(x,\xi)\,d\xi \text{ on } \Omega, \quad V_0 = 0 \text{ on } \partial\Omega. \tag{4.2}$$

We claim that a solution f of (4.1)-(1.4)-(1.8), which is sufficiently regular and decreasing at the infinity in such a way that all integration (by parts) that we shall perform are allowed, satisfies

$$\begin{aligned}\sup_{[0,T]}\Big\{\iint_{\mathcal{O}} f\,\big(1+|\xi|^2+|\log f|\big)\,d\xi dx + \int_\Omega |\nabla_\xi V_f|^2\,dx\Big\}&\\ + \int_0^T\!\!\iint_{\mathcal{O}} \big(e(f) + \frac{|\nabla_\xi f|^2}{f}\big)\,d\xi dx dt + \int_0^T\!\!\int_{\partial\Omega} \mathcal{E}\big(\frac{\gamma_+ f}{M}\big)\,d\sigma_x dt &\le C_T < +\infty,\end{aligned} \tag{4.3}$$

where $e(f) \geq 0$ denotes the usual entropy dissipation term. We do not give the explicit expression for the Boltzmann collision operator that we find in [26] or [37] for example. The precise assumptions we make on the cross section are those introduced in [37]. We only recall that the collision operator has the following remarkable properties

$$\int_{\mathbb{R}^3} Q(f,f)\begin{pmatrix}1\\ \xi\\ |\xi|^2\end{pmatrix} d\xi = 0, \tag{4.4}$$

and the entropy production term $e(f)$ satisfies

$$\int_{\mathbb{R}^3} e(f)\,d\xi = -\int_{\mathbb{R}^3} Q(f,f)\,\log f\,d\xi. \tag{4.5}$$

First, simply integrate equation (4.1) in all the variables, and we get the conservation of the mass

$$\iint_{\mathcal{O}} f(t,.)\,d\xi dx = \iint_{\mathcal{O}} f_0\,d\xi dx \qquad \forall t\geq 0. \tag{4.6}$$

Next, setting $h_M(s) = s\,\log(s/M)$, we compute

$$\begin{aligned}\frac{\partial}{\partial t} h_M(f) + \xi\cdot\nabla_x h_M(f) + \mathrm{div}_\xi\big((E+\lambda\,\xi) h_M(f)\big) - \nu\,\Delta_\xi h_M(f) = \\ = h_M'(f)\,Q(f,f) - \nu\, h_M''(f)\,|\nabla_\xi f|^2 - f\,(E+\lambda\,\xi)\cdot\nabla_\xi(\log M) \\ + \lambda\,(h_M(f) - f\,h_M'(f)) + 2\,\nu\,\nabla_\xi f\cdot\nabla_\xi(\log M) + \nu\, f\,\Delta_\xi(\log M),\end{aligned}$$

where we denote $h_M'(s) = 1 + \log(s/M)$. We integrate this equation using collision invariants (4.4) and entropy production (4.5), to obtain

$$\begin{aligned}\frac{d}{dt}\iint_{\mathcal{O}} h_M(f)\,d\xi dx + \iint_{\mathcal{O}} \big(e(f) + \nu\,\frac{|\nabla_\xi f|^2}{f}\big)\,d\xi dx + \iint_{\Sigma} h_M(\gamma f)\,\xi\cdot n(x)\,d\xi d\sigma_x = \\ = \int_\Omega E\cdot\frac{j}{\Theta}\,dx + \iint_{\mathcal{O}} \big\{\lambda\,\big(\frac{|\xi|^2}{\Theta} - 1\big) + \frac{\nu}{\Theta}\big\}\,f\,d\xi dx,\end{aligned} \tag{4.7}$$

where

$$j(t,x) = \int_{\mathbb{R}^3} \xi\,f(t,x,\xi)\,d\xi.$$

We first remark that integrating equation (4.1) in the velocity variable we have

$$\frac{\partial}{\partial t}\rho + div_x\,j = 0 \quad \text{ on } \ (0,\infty)\times\Omega,$$

and therefore

$$-\int_\Omega E\cdot\frac{j}{\Theta}\,dx = \int_\Omega \nabla V_f\cdot\frac{j}{\Theta}\,dx = \int_\Omega \frac{V_f}{\Theta}\,\frac{\partial\rho}{\partial t}\,dx = \frac{d}{dt}\int_\Omega \frac{|\nabla_x V_f|^2}{2\,\Theta}\,dx. \tag{4.8}$$

Next, combining (4.7), (4.8) and the boundary estimate (1.11) we obtain

$$\begin{aligned}\frac{d}{dt}\Big\{\iint_{\mathcal{O}} h_M(f)\,d\xi dx + \int_\Omega \frac{|E_f|^2}{2\,\Theta}\,dx\Big\} + \iint_{\mathcal{O}} \big[e(f) + \frac{|\nabla_\xi f|^2}{f}\big]\,d\xi dx \\ + \bar{\alpha}\int_{\partial\Omega} \mathcal{E}(\gamma_+ f)\,d\sigma_x \le C_{\lambda,\nu}\iint_{\mathcal{O}} (1+|\xi|^2)\,f\,d\xi dx.\end{aligned} \tag{4.9}$$

Finally, using the elementary estimate

$$\int_{\mathbb{R}^N} f\,\big(\frac{|\xi|^2}{4\,\Theta} + |\log f|\big)\,d\xi \le C_M + \int_{\mathbb{R}^N} h_M(f)\,d\xi,$$

(that one can find in [51] for example), we obtain (4.3) thanks to the Gronwall Lemma.

We can now specify the sense of the solution we deal with. With DiPerna and Lions [35], [37,38], [50] we say that $0 \le f \in C([0,\infty); L^1(\mathcal{O}))$ is a renormalized solution of (4.1)-(1.4)-(1.8) if first f satisfies the *a priori* physical bound (4.3) and is a solution of

$$\begin{aligned}\frac{\partial}{\partial t}\beta(f) + \xi\cdot\nabla_x\beta(f) + (\nabla_x V_f + \lambda\,\xi)\cdot\nabla_\xi\beta(f) - \nu\,\Delta_\xi\beta(f) = \\ = \beta'(f)\,(Q(f,f) + \lambda\,N\,f) - \nu\,\beta''(f)\,|\nabla_\xi f|^2 \quad \text{in} \quad \mathcal{D}'((0,T)\times\mathcal{O}),\end{aligned} \tag{4.10}$$

for all time $T > 0$, and all $\beta \in \mathcal{B}_4$, the class of all functions $\beta \in C^2(\mathbb{R})$ such that $|\beta''(s)| \le C/(1+s)$, $|\beta'(s)| \le C/\sqrt{1+s}$, $\forall s \ge 0$. Since for any $R > 0$ there is a constant C_R such that

$$\int_{B_R} \frac{|Q(f,f)|}{\sqrt{1+f}}\, d\xi \le C_R \int_{\mathbb{R}^N} \big[(1+|\xi|^2)\, f + e(f)\big]\, d\xi,$$

(we refer to [63] for the proof of this claim and to [50] for a related result), we see that each term in equation (4.10) makes sense. Next, the trace functions $f(0,.)$ and γf defined by Theorem 3.3 and the Green formula (3.10) must satisfy (1.8) and (1.4), say almost everywhere.

Before passing to the proof of Theorem 1.1, we would like to emphasize that $\gamma_+ f$ satisfies an additional bound which is a consequence of the *a priori* bound (4.3), the boundary conditions (1.4) and the Green formula (3.10).

Proposition 4.1. *If f is a renormalized solution of (4.1)-(1.4) then there exists a constant C'_T which only depends on the physical bound C_T in (4.3) such that*

$$\int_0^T \int_{\partial\Omega} \sqrt{\widetilde{\gamma_+ f}}\; d\sigma_x dt \le C'_T. \tag{4.11}$$

Proof of Proposition 4.1. We fix $\chi \in \mathcal{D}(\mathbb{R}^N)$ such that $0 \le \chi \le 1$, $\chi = 1$ on B_1 and $\operatorname{supp} \chi \subset B_2$. Then, the Green formula (3.10) written with $\phi = n(x)\cdot\xi\,\chi(\xi)$ and $\beta(s) = \sqrt{1+s}$ gives

$$\int_0^T \iint_\Sigma \sqrt{1+\gamma f}\,\chi\, d\lambda_2(t,x,\xi) \le C''_T$$

with C''_T which depends on C_T. But, from (1.4) we have $\gamma_- f \ge \bar\alpha\, M(\xi)\, \widetilde{\gamma_+ f}$ on $(0,T)\times\Sigma_-$, and therefore there is a constant $\kappa > 0$ such that

$$\begin{aligned}\kappa \int_0^T \int_{\partial\Omega} \sqrt{\widetilde{\gamma_+ f}}\; d\sigma_x dt &\le \int_0^T \iint_{\Sigma_-} \sqrt{\widetilde{\gamma_+ f}}\; \bar\alpha^{1/2}\, M^{1/2}(\xi)\,\chi\, d\lambda_2(t,x,\xi)\\ &\le \int_0^T \iint_{\Sigma_-} \sqrt{\gamma_- f}\,\chi\, d\lambda_2(t,x,\xi) \le C''_T.\end{aligned}$$

□

Proof of the Theorem 1.1. Let (f_n) be a sequence of renormalized solutions to the VPFPB system (4.1)-(1.4)-(1.8) such that for any $T > 0$ there is a constant C_T

$$\begin{aligned}&\sup_{[0,T]}\Big\{\iint_{\mathcal{O}} f_n\,\big(1+|\xi|^2+|\log\, f_n|\big)\, d\xi dx + \int_\Omega |\nabla_x V_{f_n}|^2\, dx\Big\}\\ &\qquad + \int_0^T \iint_{\mathcal{O}} \big(\nu\, \frac{|\nabla_\xi f_n|^2}{f_n} + e(f_n)\big)\, d\xi dx dt + \int_0^T \int_{\partial\Omega} \mathcal{E}(\gamma_+ f_n) d\sigma_x dt \le C_T;\end{aligned} \tag{4.12}$$

thus, we may assume without loss of generality, and extracting a subsequence if necessary, that f_n converges weakly in $L^p(0,T;L^1(\mathcal{O}))$ ($\forall p \in [1,\infty)$) to a function f. Furthermore, since the renormalized term on the right hand side of equation (4.10) is bounded in L^1, thanks to the bound (4.12), and since Λ_E is an hypoelliptic operator (see [35], [15], [47]), we obtain that, say,

$\log(1+f^n)$ and next f^n converge a.e.. We conclude that $f^n \to f$ strongly in $L^p(0,T;L^1(\mathcal{O}))$, $\forall p \in [1,\infty)$. Moreover, we can show from (4.12) that

$$\sup_{[0,T]} \int_\Omega \rho_n(1+|\log \rho_n|)\,dx \le C_T,$$

(see [50]) and then that $\nabla_x V_{f_n} \to \nabla_x V_f$ strongly in $L^1(0,T;W^{1,1}(\Omega))$ (see for instance [50] and [60]). It is also shown in [35] that

$$\frac{Q(f_n,f_n)}{1+f_n} \to \frac{Q(f,f)}{1+f} \quad \text{strongly in} \quad L^1((0,T)\times\mathcal{O}).$$

Therefore, we are able to apply Theorem 3.4, and we obtain that f satisfies the renormalized equation (4.10) and that

$$(4.13) \qquad \gamma f_n \overset{r}{\rightharpoonup} \gamma f \quad \text{in the renormalized sense, and} \quad \gamma_+ f_n \to \gamma_+ f \quad \text{a.e. on } (0,T)\times\Sigma_+.$$

Next, from (1.4) we have

$$(4.14) \qquad \widetilde{\gamma_+ f_n} \le \bar{\alpha}^{-1}\, M^{-1}(\xi)\, \gamma_- f_n \quad \text{on} \quad (0,T)\times\Sigma_-,$$

so that

$$\widetilde{\gamma_+ f_n} \overset{r}{\rightharpoonup} \psi \quad \text{in} \quad (0,T)\times\partial\Omega, \quad \text{with} \quad \psi \le \bar{\alpha}^{-1}\, M^{-1}(\xi)\, \gamma_- f.$$

Furthermore, repeating the proof of Proposition 4.1 we get $\psi \in L^{1/2}((0,T)\times\partial\Omega)$. Now, we can apply Corollary 2.4, which says that for every $\varepsilon > 0$ there is $A = A_\varepsilon \subset (0,T)\times\partial\Omega$ such that meas$\,((0,T)\times\partial\Omega\backslash A) < \varepsilon$ and

$$\gamma_+ f_n \rightharpoonup \gamma_+ f \quad \text{weakly in} \quad L^1(A\times\mathbb{R}^N).$$

Since we already know the a.e. convergence, this convergence is in fact strong in $L^1(A\times\mathbb{R}^N)$. There is no difficulty in passing to the limit in the boundary condition so that f satisfies (1.4) and f satisfies the same physical estimate (4.3) thanks the convexity argument of Corollary 2.4, [35], [38]. □

Remark 4.1. For the Boltzmann equation and the FPB equation, as well as for the VP system and the VPFP system when the Poisson equation (1.7) is provided with Neumann condition, we can prove the following additional *a priori* estimate on the trace

$$(4.15) \qquad \int_0^T \iint_\Sigma \gamma f\,(1+|\xi|^2)\,|\xi\cdot n(x)|\,d\xi d\sigma_x dt \le C_T.$$

As a consequence, we also establish the *a priori* physical bound (4.3) for time and position dependent wall temperature $\Theta = \Theta(t,x)$ which satisfies

$$0 < \Theta_0 \le \Theta(t,x) \le \Theta_1 < \infty.$$

Therefore, the stability result and the corresponding existence result can be generalized to these kind of boundary conditions. We refer to [6] and [54] for more details.

Also notice that thanks to estimate (4.15) we can propose, in this case, a simpler proof of Theorem 1.1, using directly Theorem 2.2 instead of the Corollary 2.4.

Remark 4.2. Just a word about the general reflection operator

$$R\,\phi = \int_{\xi'\cdot n(x)>0} k(\xi,\xi')\,\phi(\xi')\,\xi'\cdot n(x)\,d\xi' \tag{4.16}$$

when k satisfies the usual properties

$$(i) \qquad k \ge 0,$$

$$(ii) \qquad \int_{\xi\cdot n(x)<0} k(\xi,\xi')\,d\xi = 1,$$

$$(iii) \qquad R\,M = M,$$

where M is the normalized Maxwellian (1.3). In this case, we can prove that a solution f of (4.1)-(1.2) provided with the reflection condition (1.1)-(4.15) formally satisfies the *a priori* physical estimate (4.3) with $\mathcal{E}$ replaced by

$$\mathcal{E}_k(\phi/M) := \int_{\xi\cdot n(x)>0} \big[\, h(\frac{\phi}{M}) - h(\frac{R\,\phi}{M})\,\big]\, M\,\xi\cdot n(x)\,d\xi.$$

By Jensen inequality one can prove that $\mathcal{E}_k$ is non negative, see [39], [30], [41]. It is not so clear how to adapt the analysis of section 2 in order to get weak L^1 convergence in the ξ variable for sequence (ϕ_n) such that $\mathcal{E}_k(\phi_n)$ is bounded in $L^1((0,T)\times\partial\Omega)$. Nevertheless, considering a sequence (f_n) of solutions of one of the kinetic equations (Boltzmann, VP, FPB or VPFP) just like in Theorem 1.1 and without assuming that (1.14) holds, we obtain, using Proposition 3.4, that $\gamma f_n \stackrel{r}{\rightharpoonup} \gamma f$. Passing to the limit $n \to +\infty$ thanks to Proposition 6.3.4 we have that γf satisfies the relaxed boundary condition (1.17). As a conclusion, with the analysis presented here, we are able to prove existence result with relaxed boundary condition (1.17) for all theses equations and for general collision operator k satisfying (i), (ii), (iii). This extends and generalizes previous results known for the Boltzmann equation, see for instance [6], [29], [54].

5. Proof of the trace theorems.

We begin with some notation. For a given real $R > 0$, we define $B_R = \{y \in R^N \;/\; |y| < R\}$, $\Omega_R = \Omega \cap B_R$, $\mathcal{O}_R = \Omega_R \times B_R$, $D_R = (0,T)\times\mathcal{O}_R$, $\Sigma_R = (\partial\Omega\cap B_R)\times B_R$ and $\Gamma_R = (0,T)\times\Sigma_R$. We also denote by $L^{a,b}_R$ the spaces $L^a(0,T;L^b(\mathcal{O}_R))$ or $L^a(0,T;L^b(\Omega_R))$, and $L^{a,b}_{loc}$ the spaces $L^a(0,T;L^b_{loc}(\bar{\mathcal{O}}))$ or $L^a(0,T;L^b_{loc}(\bar\Omega))$.

Proof of Theorem 3.1. First step: a priori bounds. In this step we assume that g is a solution of (3.1) and is "smooth". Precisely, $g \in W^{1,1}\big(0,T;W^{1,\infty}(\Omega;W^{2,\infty}(\mathbb{R}^N))\big)$, in such a way that the Green formula (3.10) holds. The trace γg in (3.10) is defined thanks to the usual trace theorem in the Sobolev spaces. We shall prove two *a priori* bounds on g. Let define $\beta \in W^{2,\infty}_{loc}(\mathbb{R})$ by

$$\beta(s) = \begin{cases} |s| - 1/2 & \text{if } |s|\ge 1 \\ s^2/2 & \text{if } |s| \le 1 \end{cases} \quad\text{so that } \beta'(s) = \begin{cases} 1 & \text{if } s \ge 1 \\ s & \text{if } |s| \le 1 \\ -1 & \text{if } s \le -1 \end{cases} \quad\text{and } \beta''(s) = \begin{cases} 0 & \text{if } |s|\ge 1 \\ 1 & \text{if } |s| \le 1 \end{cases},$$

and thus $\beta \in \mathcal{B}_1$. Fix $R > 0$ and consider $\chi \in \mathcal{D}(\bar{\mathcal{O}})$ such that $0 \le \chi \le 1$, $\chi = 1$ on $\mathcal{O}_R$ and $\operatorname{supp} \chi \subset \bar{\mathcal{O}}_{R+1}$. We set $\phi = \chi\, n(x) \cdot \xi$. The Green formula (3.10) is then written

$$\int_0^T \iint_\Sigma \beta(\gamma\, g)\, \chi\, (n(x)\cdot\xi)^2\, d\xi d\sigma_x d\tau = -\Big[\iint_{\mathcal{O}} \beta(g(\tau,.))\, \phi\, dx d\xi \Big]_0^T + \int_0^T \iint_{\mathcal{O}} \big(\beta(g)\, \Lambda_E^\star \phi + (\beta'(g)\, G - \nu\, \beta''(g)\, |\nabla_\xi\, g|^2)\, \phi\big)\, d\xi dx dt.$$

We deduce from it a first *a priori* bound: there are some constants γ_R and C_R such that

$$\begin{aligned}\gamma_R \int_0^T \iint_{\Sigma_R} |\gamma\, g|\, (n(x)\cdot\xi)^2\, d\xi d\sigma_x d\tau &\le \int_0^T \iint_{\Sigma_R} \beta(\gamma\, g)\, (n(x)\cdot\xi)^2\, d\xi d\sigma_x d\tau \\ &\le C_R \int_0^T \iint_{\mathcal{O}_{R+1}} \big(g^2\,(1+|E|) + |G| + \nu\, |\nabla_\xi\, g|^2\big)\, d\xi dx dt \\ &\quad + C_R \iint_{\mathcal{O}_{R+1}} \big(g^2(0,.) + g^2(T,.)\big)\, dx d\xi,\end{aligned} \tag{5.1}$$

where we have used the fact that for $u \in L^\infty(Y_R)$ with $Y_R = \mathcal{O}_R$ or Σ_R one has

$$\gamma_R \int_{Y_R} |u| \;\le\; \int_{Y_R} \beta(u) \;\le\; \gamma_R^{-1} \int_{Y_R} u^2.$$

Let $K \subset \mathcal{O}$ be a compact set and consider $\phi \in \mathcal{D}(\mathcal{O})$ such that $0 \le \phi \le 1$, $\phi = 1$ on K and $R > 0$ such that $\operatorname{supp} \phi \subset \mathcal{O}_R$. We fix $t_0 \in [0,T]$. The Green formula (3.10) becomes

$$\begin{aligned}\iint_{\mathcal{O}} \beta(g(t_1,.))\, \phi\, dx d\xi &= \iint_{\mathcal{O}} \beta(g(t_0,.))\, \phi\, dx d\xi \\ &\quad + \int_{t_0}^{t_1} \iint_{\mathcal{O}} \big(\beta(g)\, \Lambda_E^\star \phi + (\beta'(g)\, G - \nu\, \beta''(g)\, |\nabla_\xi\, g|^2)\, \phi\big)\, d\xi dx dt,\end{aligned} \tag{5.2}$$

and we get a second *a priori* bound

$$\begin{aligned}\gamma_R \iint_K |g|(t_1,.)\, dx d\xi &\le C_R \iint_{\mathcal{O}_R} g^2(t_0,.)\, dx d\xi \\ &\quad + C_R \int_0^T \iint_{\mathcal{O}_R} \big(g^2\,(1+|E|) + |G| + \nu\, |\nabla_\xi\, g|^2\big)\, d\xi dx dt.\end{aligned} \tag{5.3}$$

Second step: regularization and passing to the limit. Let us now consider a function g which satisfies the assumption of Theorem 3.1. We define the mollifer ρ_k by

$$\rho_k(z) = k^N\, \rho(k\, z) \ge 0, \quad k \in \mathbb{N}^\star, \quad \rho \in \mathcal{D}(\mathbb{R}^N), \quad \operatorname{supp} \rho \subset B_1, \quad \int_{\mathbb{R}^N} \rho(z)\, dz = 1,$$

and we introduce the regularized functions $g_k = g \star_{x,k} \rho_k *_\xi \rho_k$, where $*$ stands for the usual convolution and $\star_{x,k}$ for the convolution-translation defined by

$$(u \star_{x,k} h_k)(x) = [\tau_{2\,n(x)/k}(u * h_k)](x) = \int_{\mathbb{R}^N} u(y)\, h_k(x - \frac{2}{k}\, n(x) - y)\, dy,$$

for all $u \in L^1_{loc}(\bar{\Omega})$ and $h_k \in L^1(\mathbb{R}^N)$ with $\operatorname{supp} h_k \subset B_{1/k}$.

Lemma 5.1. *With this notation one has* $g_k \in W^{1,1}\big(0,T; W^{1,\infty}(\Omega; W^{2,\infty}(\mathbb{R}^N))\big)$ *and*

$$\Lambda_E g_k = G_k \quad \textit{dans} \quad \mathcal{D}'(D), \tag{5.4}$$

with $G_k \in L^1_{loc}((0,T) \times \bar{\mathcal{O}})$ *for all* $k \in \mathbb{N}$. *Moreover, the sequences* (g_k) *and* (G_k) *satisfy*

$$\begin{cases} (g_k) \text{ is bounded in } L^\infty((0,T)\times\mathcal{O}), & g_k \longrightarrow g \text{ a.e. in } (0,T)\times\mathcal{O}, \\ \nabla_\xi g_k \longrightarrow \nabla_\xi g \text{ in } L^2_{loc}([0,T]\times\bar{\mathcal{O}}) & \text{and} \quad G_k \longrightarrow G \text{ in } L^1_{loc}([0,T]\times\bar{\mathcal{O}}). \end{cases} \tag{5.5}$$

The proof of Lemma 5.1 is similar to the proof of Lemma 1 in [53] and of Lemma II.1 in [36]. We refer to [53] and [36] for details.

From Lemma 5.1 we have that for all $k, \ell \in \mathbb{N}^\star$ the difference $g_k - g_\ell$ belongs to $W^{1,1}\big(0,T; W^{1,\infty}(\Omega; W^{2,\infty}(\mathbb{R}^N))\big)$ and is a solution of

$$\Lambda_E(g_k - g_\ell) = G_k - G_\ell \quad \text{in} \quad \mathcal{D}'(D). \tag{5.6}$$

We know, thanks to (5.5), that $g_k(t,.)$ converges to $g(t,.)$ in $L^2_{loc}(\bar{\mathcal{O}})$ for a.e. $t \in [0,T]$; we fix t_0 such that $g_k(t_0,.) \to g(t_0,.)$. Moreover, up to a choice for the continuous representation of g_k, we can assume that $g_k \in C([0,T], L^1_{loc}(\bar{\mathcal{O}}))$. Therefore, the estimate (5.2) applied to $g_k - g_\ell$ in t_0 and the convergence (5.5) imply that for all compact sets $K \subset \mathcal{O}$ we have

$$\sup_{t\in[0,T]} \|(g_k - g_\ell)(t,.)\|_{L^1(K)} \underset{k,\ell\to+\infty}{\longrightarrow} 0. \tag{5.7}$$

We deduce from this, that there exists, for any time $t \in [0,T]$, a function $\gamma_t g$ such that $g_k(t,.)$ converges to $\gamma_t g$ in $C([0,T]; L^1_{loc}(\mathcal{O}))$; in particular,

$$g(t,x,\xi) = \gamma_t g(x,\xi) \text{ for a.e. } (t,x,\xi) \in D.$$

Thus, we also have $g_k(t,.) = (\gamma_t\, g) \star_{x,k} \rho_k *_\xi \rho_k$ a.e. in D, and since these two functions are continuous, the equality holds for all $(t,x,\xi) \in [0,T] \times \bar{\mathcal{O}}$ and $k \in \mathbb{N}^\star$, so that $g_k(t,.) \to \gamma_t g$ in $L^p_{loc}(\bar{\mathcal{O}})$ for all $t \in [0,T]$.

Using now the estimate (5.1), applied to $g_k - g_\ell$, and the convergence (5.5) and (5.7) we get that

$$\int_0^T \iint_{\Sigma_R} |\gamma g_k - \gamma g_\ell|\, (n(x)\cdot\xi)^2\, d\xi d\sigma_x dt \underset{k,\ell\to+\infty}{\longrightarrow} 0,$$

for all $R > 0$. We deduce that there exists a function $\gamma g \in L^1_{loc}([0,T]\times\Sigma, (n(x)\cdot\xi)^2\, d\xi d\sigma_x dt)$, which is the limit of γg_k in this space. Moreover, since $\|\gamma g_k\|_{L^\infty} \le \|g_k\|_{L^\infty}$ is bounded, we have $\gamma g \in L^\infty((0,T)\times\mathcal{O})$.

Finally, we obtain the Green formula (3.10) writing it first for g_k and then passing to the limit $k \to \infty$ thanks to the convergence previously obtained. Uniqueness of the trace function follows from the Green formula. □

Proof of Theorem 3.3. The proof is based on Theorem 1 and on an monotony argument. This is exactly the same as the one presented in [54] in the case of Vlasov equation. Let $(\beta_M)_{M\geq 1}$ be a sequence of odd functions of $\mathcal{B}_1$ such that

$$\beta_M(s) = \begin{cases} s & \text{if } s \in [0, M] \\ M + 1/2 & \text{if } s \geq M+1, \end{cases}$$

and $|\beta_M(s)| \leq |s|$ for all $s \in \mathbb{R}$. The function $\alpha_M(s) := \beta_M(\beta_{M+1}^{-1}(s))$ is well defined, with the convention $\alpha_M(s) = M + 1/2$ if $s \geq M + 3/2$ and $\alpha_M(s) = -M - 1/2$ if $s \leq -M - 3/2$, and also belongs to $\mathcal{B}_1$. Furthermore, one has $\alpha_M(s) \leq s$ for all $s \geq 0$ and $\alpha_M(s) \geq s$ for all $s \leq 0$. We will construct the trace function γg as the limit of $(\gamma\beta_M(g))$ when $M \to \infty$, that one being defined thanks to Theorem 3.1. Indeed, the condition (3.6) implies that

$$\nabla_\xi g \, \mathbf{1}_{|g|\leq M+1} \in L^2_{loc}([0,T] \times \bar{\mathcal{O}}),$$

and then $\nabla_\xi \beta_M(g) = \beta_M'(g) \nabla_\xi g \in L^2_{loc}([0,T] \times \bar{\mathcal{O}})$ in such a way that $\beta_M(g)$ satisfies the assumption on Theorem 3.1. We define $\Gamma_M^{(\pm)} = \{(t,x,\xi) \in (0,T) \times \Sigma, \pm\gamma\beta_M(g)(t,x,\xi) > 0\}$ and $\Gamma_M^{(0)} = \{(t,x,\xi) \in (0,T) \times \Sigma, \gamma\beta_M(g)(t,x,\xi) = 0\}$. Thanks to the definition of α_M and the renormalization property (3.11) of the trace, one has $\gamma\,\beta_M(g) = \gamma\,\alpha_M(\beta_{M+1}(g)) = \alpha_M(\gamma\,\beta_{M+1}(g))$. We deduce that, up to a set of measure zero,

$$\Gamma_M^{(+)} = \Gamma_1^{(+)}, \quad \Gamma_M^{(-)} = \Gamma_1^{(-)} \quad \text{and} \quad \Gamma_M^{(0)} = \Gamma_1^{(0)} \quad \text{for all } M \geq 1.$$

Therefore the sequence $(\gamma\,\beta_M(g))_{M\geq 1}$ is not decreasing on $\Gamma_1^{(+)}$ and not increasing on $\Gamma_1^{(-)}$. This implies that $\gamma\,\beta_M(g)$ converges a.e. to a limit denoted by γg and which belongs to $L([0,T]\times\Sigma)$. Obviously, if (3.8) holds for one function β such that $\beta(s) \nearrow +\infty$ when $s \nearrow \pm\infty$, then $\beta(\gamma g) \in L^1((0,T) \times \Sigma, d\lambda_2)$ and $\gamma g \in L^0((0,T) \times \Sigma)$. In order to establish the Green formula (3.10) we fix $\beta \in \mathcal{B}_1$ and $\phi \in \mathcal{D}((0,T] \times \bar{\mathcal{O}})$. We write the Green formula for the function $\beta(\beta_M(g))$, and using the fact that $\gamma\big[\beta \circ \beta_M(g)\big] = \beta(\gamma\beta_M(g))$, we find

$$\int_0^T \iint_{\mathcal{O}} (\beta \circ \beta_M(g) \, (\frac{\partial \phi}{\partial t} + \xi \cdot \nabla_x \phi + E \cdot \nabla_\xi \phi) + (\beta \circ \beta_M)'(g) \, G \, \phi) \, d\xi dx dt =$$
$$= \int_0^T \iint_{\Sigma} \beta(\gamma\,\beta_M(g)) \, \phi \, n(x) \cdot \xi \, d\xi d\sigma_x ds.$$

We get (3.10) by letting $M \to \infty$ and noting that $\beta \circ \beta_M(s) \to \beta(s)$ for all $s \in \mathbb{R}$. □

Remark 5.1. Theorem 3.2 is now a quite simple consequence of Theorem 3.3 using the *a priori* bounds stated in the proof of Theorem 3.1. We emphasize that with the additional assumption (3.5), it is possible to give a "direct" proof of Theorem 3.2 (following the proof of Theorem 3.1) instead of passing by the renormalization step. See [53]for details.

Proof of Theorem 3.2. For all $\beta \in \mathcal{B}_1$ it is clear that $\beta(g) \in L^\infty$, $\nabla_\xi \beta(g) \in L^2$ and that $\beta(g)$ is solution of (1.6) using Lemma 3.3 (we just have to multiply equation (5.9), in the case $\mu \equiv 0$, by $\beta'(g_k)$ and pass to the limit $k \to \infty$). Thanks to Theorem 3.3, we already know that g has a trace $\gamma_t g \in L(\mathcal{O})$ and $\gamma g \in L((0,T) \times \mathcal{O})$ which satisfies the Green formula (3.10) for all

$\beta \in \mathcal{B}_1$ and $\phi \in \mathcal{D}([0,T]\times\bar{\mathcal{O}})$. We just have to prove that γg and $\gamma_t g$ belong to the appropriate space. On one hand, for all $\beta \in \mathcal{B}_1$ such that $|\beta(s)| \le |s|$ one has

$$\|\beta(\gamma_t g)\|_{L^p_R} \le \sup_k \sup_{[0,T]} \|\beta(g_k(t,.))\|_{L^p_R} \le \sup_{[0,T]} \|g_k(t,.)\|_{L^p_R} \le \|g\|_{L^{\infty,p}_R},$$

and thus, choosing $\beta = \beta_M$, defined in the proof of Theorem 3.3, one gets, passing to the limit $M \to \infty$,

$$\sup_{[0,T]} \|\gamma_t g\|_{L^p_R} \le \|g\|_{L^{\infty,p}_R} < \infty.$$

In the same way and using (5.1), we show that

$$\|\gamma g\|_{L^1([0,T]\times\Sigma_R, d\lambda_2)} < \infty.$$

We still have to prove that $\gamma_t g \in C([0,T], L^1_{loc}(\bar{\mathcal{O}}))$, which is an immediate consequence of the following Lemma. □

Lemma 5.2. *Let (u_n) be a bounded sequence of $L^1_{loc}(\mathcal{O})$ such that $\beta(u_n) \rightharpoonup \beta(u)$ in $\big(C_c(\mathcal{O})\big)'$ for all $\beta \in \mathcal{B}_2$. Then $u_n \to u$ in $L^1_{loc}(\mathcal{O})$.*

Proof of Lemma 5.2. We fix $j : \mathbb{R} \to \mathbb{R}$ a nonnegative function of class C^2, strictly convex on the interval $[-M,M]$ and such that $j''(t) = 0$ for all $t \notin [-M,M]$; in particular $j \in \mathcal{B}_2$. We also consider $\chi \in C_c(\mathcal{O})$ such that $0 \le \chi \le 1$. By assumption

$$\int_{\mathcal{O}} j(u_n)\,\chi \to \int_{\mathcal{O}} j(u)\,\chi \tag{5.10}$$

and by convexity of j one also has

$$\liminf_{n\to\infty} \int_{\mathcal{O}} j\big(\frac{u_n+u}{2}\big)\,\chi \ge \int_{\mathcal{O}} j(u)\,\chi \quad \text{since} \quad \frac{u_n+u}{2} \rightharpoonup u \ \text{ in } \ \big(C_c(\mathcal{O})\big)'. \tag{5.11}$$

Remarking that

$$\frac{1}{2}\,j(t) + \frac{1}{2}\,j(s) - j\big(\frac{t+s}{2}\big) \ge 0 \qquad \forall t,\ s \in \mathbb{R}, \tag{5.12}$$

we deduce from (5.10) and (5.11) that

$$\int_{\mathcal{O}} \big[\frac{1}{2}\,j(u_n) + \frac{1}{2}\,j(u) - j\big(\frac{u_n+u}{2}\big)\big]\,\chi \to 0. \tag{5.13}$$

From the fact that in (5.12) the inequality is strict whenever $t,\ s \in [-M,M]$ and $t \ne s$, we obtain from (5.13) that there exists a subsequence (u_{n_k}) such that $u_{n_k} \to u$ a.e. on $\operatorname{supp}\chi \cap [\,|u| < M]$. The preceding argument being valuable for arbitrary M and χ, we obtain, by a diagonal process, a subsequence of (u_n), still denoted by (u_{n_k}), such that $u_{n_k} \to u$ a.e. in $\mathcal{O}$.

We now set $j_\pm(s) = s_\pm$. We first remark that we can write $j_\pm = j_{\pm,1} + j_{\pm,2}$ with $j_{\pm,1} \in \mathcal{B}_2$ and $j_{\pm,2} \in W^{2,\infty}(\mathbb{R})$ in such a way that

$$\int_{\mathcal{O}} j_\pm(u_{n_k})\,\chi \to \int_{\mathcal{O}} j_\pm(u)\,\chi.$$

On the other hand, the elementary inequality $\big|\, b - |a-b| \,\big| \le a \;\; \forall a,\, b \ge 0$ and the dominated convergence Theorem imply $j_\pm(u_{n_k}) - |j_\pm(u_{n_k}) - j_\pm(u)| \to j_\pm(u)$ in $L^1_{loc}(\mathcal{O})$. It follows that

$$\limsup_{k\to\infty} \int_{\mathcal{O}} \big|\, j_\pm(u_{n_k}) - j_\pm(u) \,\big|\, \chi = \int_{\mathcal{O}} j_\pm(u)\, \chi - \lim_{k\to\infty} \int_{\mathcal{O}} j_\pm(u_{n_k})\, \chi = 0.$$

We conclude that $u_{n_k} = j_+(u_{n_k}) - j_-(u_{n_k}) \to j_+(u) - j_-(u) = u$ strongly in $L^1_{loc}(\mathcal{O})$ and that, in fact, it is the entire sequence (u_n) which converges. □

6. Appendix: on the convergence in the renormalized sense.

The main basic properties concerning the notion of convergence in the renormalized sense are presented in this Appendix. Once again, we only deal with nonnegative functions of $L := L(Y)$ and $L^0 := L^0(Y)$, but we do not specify it anymore in what follows.

Definition 6.1. We say that α is a renormalizing function if $\alpha \in C_b(\mathbb{R})$ is not decreasing and $0 \le \alpha(s) \le s$ for any $s \ge 0$. We say that (α_M) is a renormalizing sequence if α_M is a renormalizing function for any $M \in \mathbb{N}$ and $\alpha_M(s) \nearrow s$ for all $s \ge 0$ when $M \nearrow \infty$. Given any renormalizing sequence (α_M), we say that (ϕ_n) (α_M)-renormalized converges if there exists a sequence $(\bar\alpha_M)$ of $L^\infty(Y)$ such that

$$\alpha_M(\phi_n) \rightharpoonup \bar\alpha_M \quad \sigma(L^\infty(Y), L^1(Y)) \star \qquad \text{and} \qquad \bar\alpha_M \nearrow \phi \qquad \text{a.e. in } Y. \tag{6.1}$$

Notice that the renormalized convergence is nothing but the (T_M)-renormalized convergence.

Proposition 6.1. *1. The (α_M)-renormalized limit in the definition 6.1 does not depend on the renormalizing sequence (α_M), but only on the sequence (ϕ_n).*

2. For any sequence (ϕ_n) of L and any renormalizing sequence (α_M) there exists a subsequence $(\phi_{n'})$ of (ϕ_n) and $\phi \in L$ such that $(\phi_{n'})$ (α_M)-renormalized converges to ϕ. Of course in general, we can not exclude that $\phi \equiv +\infty$, see Remark 2.2.

3. As a consequence, given two renormalized sequences (α_M) and (β_M), if (ϕ_n) (α_M)-renormalized converges, then (ϕ_n) also (β_M)-renormalized converges to the same limit, up to the extraction of a subsequence.

4. In fact, if for a given renormalized sequence (α_M), the sequence (ϕ_n) (α_M)-renormalized converges to ϕ then

(6.2)

for any sub-sequence $(\phi_{n'})$ there exists a sub-sequence $(\phi_{n''})$ of $(\phi_{n'})$ such that $\phi_{n''} \stackrel{r}{\rightharpoonup} \phi$.

The inverse implication if false, even with $\alpha_M = T_M$. In particular, the renormalized convergence is not associated to any topological structure.

5. Finally, assume that $\phi_n \stackrel{r}{\rightharpoonup} \phi$. We can construct a subsequence $(\phi_{n'})$ in such a way that for any renormalizing function α there exists $\bar\alpha \in L^\infty$ such that $\alpha(\phi_{n'}) \rightharpoonup \bar\alpha$. As a consequence, $(\phi_{n'})$ (α_M)-renormalized converges to ϕ for any renormalizing sequence (α_M).

Remark 6.1. Instead of Definition 2.2 there is a lot of possible definition for the notion of renormalized convergence of (ϕ_n) to ϕ. We should take, as a definition, the assertion (6.2) or also the fact that (ϕ_n) (α_M)-renormalized converges to ϕ for an other fixed renormalizing sequence (α_M), for at least one renormalizing sequence (α_M) or for every renormalizing sequence (α_M).

But since all these definitions are equivalent, up to the extraction of a subsequence, there is no importance to specify which one we choose.

Let emphasize that the definition of (α_M)-renormalized convergence with $\alpha_M \neq T_M$ is important in order to obtain the renormalized convergence of the trace sequence in Theorem 3.4. Indeed, T_M is not smooth enough in order to be taken as a renormalizing function for the VFP equation and we have to introduce the "smooth" renormalizing function $\alpha := \Phi_{M,\theta}$.

We come back to the link between renormalized convergence and biting L^1-weak converge.

Proposition 6.2. *1. Let (ϕ_n) be a sequence which converge to ϕ a.e., strongly or weakly in L^p, $p \in [1,\infty]$, or in the biting L^1-weak sense. Then, there is a subsequence $(\phi_{n'})$ of (ϕ_n) such that $\phi_n \overset{r}{\rightharpoonup} \phi$. In general, the all sequence (ϕ_n) does not renormalized converge.*

2. Coming back to Theorem 2.3. There exists (ϕ_n) which renormalized converges but does not biting L^1-weak converge.

3. The biting L^1-weak convergence is not associated to any topological structure.

Let now define the limit superior and the limit inferior in the renormalized sense.

Definition 6.2. *Let (ϕ_n) be a sequence of L. Consider I the set of all the increasing applications $\imath : \mathbb{N} \to \mathbb{N}$ such that the subsequence $(\phi_{\imath(k)})_{k\geq 0}$ of $(\phi_n)_{n\geq 0}$ converges in the renormalized sense and note $\phi_\imath = r\text{-}lim\, \phi_{\imath(k)}$. Thanks to the Proposition 6.1.2, we know that I is not empty. We defined the limit sup and the limit inf of (ϕ_n) in the renormalized sense by*

$$r\text{-}limsup\, \phi_n := \sup_{\imath \in I} \phi_\imath \quad \text{and} \quad r\text{-}liminf\, \phi_n := \inf_{\imath \in I} \phi_\imath. \tag{6.3}$$

Proposition 6.3. *1. If $\phi_n \overset{r}{\rightharpoonup} \phi$, $\psi_n \overset{r}{\rightharpoonup} \psi$ and $\lambda_n \to \lambda$ in $\mathbb{R}_+^\star$ then $\phi_n + \lambda\, \psi_n \overset{r}{\rightharpoonup} \phi + \lambda\, \psi$. If $\lambda_n \to 0$ and (ϕ_n) is bounded in L^0 then $\lambda_n\, \phi_n \overset{r}{\rightharpoonup} 0$. As a consequence, if $\phi_n \overset{r}{\rightharpoonup} \phi \in L^0$ and $\psi_n \to \psi \in L^0$ a.e., then $\phi_n\, \psi_n \overset{r}{\rightharpoonup} \phi\, \psi$.*

2. Let $\phi_n \overset{r}{\rightharpoonup} \phi$ and β be a nonnegative and concave function. Then $\beta(\phi) \geq r\text{-}limsup\, \beta(\phi_n)$.

3. Let β be a strictly concave function, and (ϕ_n) be a sequence such that $\phi_n \overset{r}{\rightharpoonup} \phi$ and $\beta(\phi) \leq r\text{-}liminf\, \beta(\phi_n)$. Then, up to extraction a subsequence, $\phi_n \to \phi$ a.e. in Y.

4. Let $\phi_n \overset{r}{\rightharpoonup} \phi$ and let S be a bounded and nonnegative operator of L^1. Then we have $S\, \phi \leq r\text{-}liminf\, S\, \phi_n$.

Proof of the Proposition 6.1. 2. Considering a renormalizing sequence (α_M) we can find a subsequence $(n') = (n_k^M)_k$ and $\bar\alpha_M$ such that $\alpha_M(\phi_{n'}) \rightharpoonup \bar\alpha_M$ weakly in L^∞. By a diagonal process we can obtain a unique subsequence (n'') such that the weak convergence holds for any $M \in \mathbb{N}$. Furthermore, since (α_M) is not decreasing, we get that $(\bar\alpha_M)$ is a not decreasing sequence of non negative measurable functions, so that it converges. The limit ϕ belongs to L.

1. Let assume that for a renormalizing sequence (α_K) we have $\alpha_K(\phi_n) \rightharpoonup \bar\alpha_K \nearrow \psi$. Thanks to part 2., there exits a sub-sequence $(\phi_{n'})$, a sequence $\bar T_M \in L^\infty$ and a function $\phi \in L$ such that $T_M(\phi_{n'}) \rightharpoonup \bar T_M \nearrow \phi$. It is clear that $\forall K, M \in \mathbb{N}$ $\forall \varepsilon > 0$ there is $k_{M,\varepsilon}, m_K \in \mathbb{N}$ such that $\alpha_K \leq T_{m_K}$ and $T_M \leq \alpha_{k_{M,\varepsilon}} + \varepsilon$. Therefore, writing $\alpha_K(\phi_n) \leq T_{m_K}(\phi_n)$ and $T_M(\phi_n) \leq \alpha_{k_{M,\varepsilon}}(\phi_n) + \varepsilon$, and passing to the limit $n \to +\infty$, we get

$$\bar\alpha_K \leq \bar T_{m_K} \leq \phi \quad \text{and} \quad \bar T_M \leq \bar\alpha_{k_{M,\varepsilon}} + \varepsilon \leq \psi + \varepsilon.$$

Then passing to the limit $M, K \nearrow \infty$ and $\varepsilon \to 0$ we obtain that $\psi = \phi$.

Point *3* and the first part of Point *4* are immediate consequences of points *1* and 2. Let consider $\mu_z = \mu$ and $\nu_z = \nu$ two Young measures on $Y = [0,1]$ such that

$$\int_{\mathbb{R}} \lambda\, \mu(\lambda) = \int_{\mathbb{R}} \lambda\, \nu(\lambda) =: \phi$$
$$\int_{\mathbb{R}} T_{2\,M}(\lambda)\, \mu(\lambda) = \int_{\mathbb{R}} T_{2\,M}(\lambda)\, \nu(\lambda) =: \bar{T}_{2\,M}$$
$$\int_{\mathbb{R}} T_{2\,M+1}(\lambda)\, \mu(\lambda) \neq \int_{\mathbb{R}} T_{2\,M+1}(\lambda)\, \nu(\lambda).$$

For instance, take

$$\mu(\lambda) = \frac{1}{2} \sum_{\ell=0}^{\infty} \frac{1}{2^{\ell}}\, \delta_{\lambda=\ell} \quad \text{ and } \quad \nu(\lambda) = \frac{1}{2} \sum_{\ell=0}^{\infty} \Big(\frac{1}{2^{2\,\ell}} + \frac{1}{2^{2\,\ell+1}} \Big)\, \delta_{\lambda=\theta_\ell},$$

with $\theta_\ell \in [2\,\ell, 2\,(\ell+1)[$ well chosen. Now define (u_n) (resp. (v_n)) a sequence of L^1 associated to μ (resp. ν), see [61], [67], and define $\phi_{2\,n} = u_n$, $\phi_{2\,n+1} = v_n$. In such a way, we have constructed a sequence (ϕ_n) which does not converge in the renormalized sense but $(T_{2\,M})$-renormalized converge, since $T_{2\,M}(\phi_n) \rightharpoonup \bar{T}_{2\,M} \nearrow \phi$. Moreover, thanks to part 3, (ϕ_n) satisfies (6.2). As a conclusion, (6.2) does not imply the renormalized convergence and therefore the renormalized convergence is not associated to any topology.

5. Let remark that the class of renormalizing functions is separable for the uniform norm of $C(\mathbb{R}_+)$. For instance, the family $\mathcal{A} = \{\alpha^k\}$ of functions α such that

$$0 \leq \alpha(s) \leq s \quad \text{ and } \quad \alpha'(s) = \sum_{j=1}^{J} \theta_j\, \mathbf{1}_{[a_i, a_{i+1}[}(s), \quad a_i,\, \theta_i \in \mathbb{Q}_+$$

is countable and dense. By a diagonal process, we can find a subsequence $(\phi_{n'})$ in such a way that for any $\alpha \in \mathcal{A}$ there exists $\bar{\alpha} \in L^\infty$ such that $\alpha(\phi_{n'}) \rightharpoonup \bar{\alpha}$. On one hand, for any renormalizing sequence α there exists a sequence (α_k) of $\mathcal{A}$ such that $\alpha_k \leq \alpha \leq \alpha_k + 1/k$ for any $k \in \mathbb{N}$ and $\alpha_k \nearrow \alpha$. We yet know that $\alpha_k(\phi_{n'}) \rightharpoonup \bar{\alpha}_k$. Since $(\bar{\alpha}_k)$ is not decreasing, it converges a.e. and we set $\alpha^* = \lim \bar{\alpha}_k$. On the other hand, thanks to part 3, there exists a subsequence $(\phi_{n''})$ and a function $\bar{\alpha}$ such that $\alpha(\phi_{n''}) \rightharpoonup \bar{\alpha}$. This implies $\bar{\alpha}_k \leq \bar{\alpha} \leq \bar{\alpha}_k + 1/k$. Passing to the limit $k \to +\infty$ we get $\bar{\alpha} = \alpha^*$. Therefore, by uniqueness of the limit, it is the all sequence $\alpha(\phi_{n'})$ which converges to $\bar{\alpha}$. □

Proof of the Proposition 6.2. 1. If $\phi_n \to \phi$ a.e. then clearly $\alpha_M(\phi_n) \rightharpoonup \alpha_M(\phi)$ L^∞-weak and $\alpha_M(\phi) \nearrow \phi$, so that $\phi_n \overset{r}{\rightharpoonup} \phi$.

If (ϕ_n) converges strongly or weakly in L^p $p \in [1, \infty]$ then it obviously converges in the biting L^1-weak sense. We now follow the proof of [8]. Assuming that $\phi_n \overset{b}{\rightharpoonup} \phi$, there exists for any $k \in \mathbb{N}$ a Borel set A_k such that $\text{meas}\,(Y_k \backslash A_k) < 1/k$ and $\phi_n \rightharpoonup \phi$ weakly in $L^1(A_k)$. Thanks to Dunford-Pettis Lemma there is a function $\delta_k : \mathbb{R} \to \mathbb{R}$ such that $\delta_k(M) \to 0$ when $M \to +\infty$ and

$$\int_{A_k} \phi_n\, \mathbf{1}_{\{\phi_n \geq M\}}\, dy \leq \delta_k(M).$$

Therefore, up to the extraction of a subsequence, we have $T_M(\phi_n) \rightharpoonup \bar T_M$ weakly in $L^\infty(A_k)$, so that

$$\int_{A_k} |\phi - \bar T_M| \, dy \le \liminf \int_{A_k} |\phi_n - T_M(\phi_n)| \, dy \le \delta_k(M).$$

This implies that $\bar T_M \nearrow \phi$ a.e. in A_k, and thus a.e. in Y, when $M \to +\infty$.

The sequence (ϕ_n) constructed in Proposition 6.1.4 clearly converges L^1-weak and therefore biting L^1-weak, but does not converge in the renormalized sense.

2. & 3. Let ϕ_n be the sequence defined by $\phi_n = \phi_{p,k} = p\, \mathbf{1}_{[k/p,(k+1)/p]}$ where $n = 1{+}2{+}...{+}p{+}k$. Then (ϕ_n) is bounded in L^1 and clearly converges to 0 in the renormalized sense, but does not converge in the biting L^1-weak sense. Moreover, for any subsequence $(\phi_{n'})$ we can find a second subsequence $(\phi_{n''})$ of $(\phi_{n'})$ such that $\phi_{n''} \overset{b}{\rightharpoonup} 0$: the biting L^1-weak convergence is not associated to any topology. □

Proof of the Proposition 6.3. 1. For any $\lambda, M, s, t \ge 0$ the elementary inequalities

$$M \wedge (s+t) \le M \wedge s + M \wedge t \le (2\,M) \wedge (s+t) \quad \text{and} \quad \lambda\,(M \wedge s) = (\lambda\,M) \wedge (\lambda\,s)$$

holds. We first deduce

$$\text{w-lim}\,[M \wedge (\phi_n + \psi_n)] \le \text{w-lim}\,[M \wedge \phi_n] + \text{w-lim}\,[M \wedge \psi_n] \le \text{w-lim}\,[(2\,M) \wedge (\phi_n + \psi_n)]$$

so that *r-lim* $(\phi_n + \psi_n) = \phi + \psi$. Next, since for n large enough $0 < a \le \lambda_n \le A < \infty$, we can write

$$(a\,M) \wedge (\lambda_n\,\phi_n) \le (\lambda_n\,M) \wedge (\lambda_n\,\phi_n) = M\,(\lambda_n \wedge \phi_n) \le (A\,M) \wedge (\lambda_n\,\phi_n).$$

Then passing to the limit $n \to +\infty$, $M \to +\infty$ we obtain

$$\textit{r-limsup}\,(\lambda_n\,\phi_n) \le \lambda\,\phi \le \textit{r-liminf}\,(\lambda_n\,\phi_n).$$

We assume now that $\lambda_n \to 0$. We define $Y_{k,m} := \{y \in Y_k,\ \phi(y) \le m\}$, so that $Y_{k,m} \nearrow Y_k$ when $m \nearrow +\infty$. Since (ϕ_n) is bounded in L^0, for all fixed k and m and all sequences (M_n) tending to $+\infty$ we have meas$\,\{Y_{k,m},\ \phi_n \ge M_n\} \to 0$ when $n \to \infty$. Let $\varepsilon > 0$ and set $\varepsilon_n = \varepsilon_n(k,m) :=$ mes$\,\{Y_{k,m},\ \phi_n \ge \varepsilon/\lambda_n\}$. Up to the extraction of a subsequence

$$Z_L := \{Y_k,\ \phi \le m_\ell,\ \lambda_{m_\ell}\,\phi_{m_\ell} \ge \frac{1}{L},\ \forall \ell \ge L\}, \quad \text{satisfies} \quad \text{meas}\,(Z_L) \underset{L\to\infty}{\longrightarrow} 0.$$

Therefore, we have

$$\begin{aligned}\limsup_{\ell\to\infty} \int_{Y_k} T_M(\lambda_{m_\ell}\,\phi_{m_\ell}) &\le \limsup_{\ell\to\infty} \int_{(Y_k\backslash Y_{k,L})\cup Z_{k,L}} T_M(\lambda_{m_\ell}\,\phi_{m_\ell}) + \limsup_{\ell\to\infty} \int_{\{\lambda_{m_\ell}\,\phi_{m_\ell} \le 1/L\}} T_M(\lambda_{m_\ell}\,\phi_{m_\ell}) \\ &\le M\,\big(\nu(Y\backslash Y_k) + \nu(Z_{k,L})\big) + \frac{1}{L}\,\nu(Y_k),\end{aligned}$$

for all $L \in \mathbb{N}^\star$. Thus $T_M(\lambda_{m_\ell}\,\phi_{m_\ell}) \to 0$ in $L^1(Y_k)$.

2. We know that

$$\beta(s) = \inf_{\ell \ge \beta} \ell(s), \tag{6.3}$$

where the inf is taken over all affine applications $\ell(t) = a\,t + b$ which satisfy $a, b \geq 0$ and $\beta(t) \leq \ell(t)$ for any $t \geq 0$. Furthermore, for any ℓ and m, there clearly exists K_0 such that

$$T_M(\ell(s)) \leq \ell(T_K(s)) \qquad \text{and} \qquad \ell(T_M(s)) \leq T_K(\ell(s)) \qquad \text{for all} \qquad K \geq K_0,\ s \geq 0.$$

We deduce that for any $\ell \geq \beta$ we have

$$T_M(\beta(\phi_n)) \leq \ell(T_K(\phi_n)).$$

Thus

$$\limsup_n T_M(\beta(\phi_n)) \leq \ell(\lim_n T_K(\phi_n)) \leq \ell(\phi).$$

Finally, thanks to (6.2),

$$\limsup_n T_M(\beta(\phi_n)) \leq \beta(\phi) \qquad \text{for any } M,$$

which exactly means that $r\text{-}limsup\, \beta(\phi_n) \leq \beta(\phi)$.

3. For any subsequence (n') such that $\beta(\phi_{n'})$, $\beta(\phi_{n'}/2+\phi/2))$ and $\beta(\phi_{n'}/2+\phi/2))-\beta(\phi_{n'})/2-\beta(\phi)/2 \geq 0$ converge in the renormalized sense we have, thanks to c),

$$r\text{-}lim\, \big[\beta\big(\frac{\phi_{n'}+\phi}{2}\big) - \frac{\beta(\phi_{n'})}{2} - \frac{\beta(\phi)}{2}\big] + \frac{\beta(\phi)}{2} + r\text{-}lim\, \frac{\beta(\phi_{n'})}{2} = r\text{-}lim\, \beta\big(\frac{\phi_{n'}+\phi}{2}\big),$$

so that

$$\begin{aligned} 0 \leq r\text{-}lim\, \big[\beta\big(\frac{\phi_{n'}+\phi}{2}\big) - \frac{\beta(\phi_{n'})}{2} - \frac{\beta(\phi)}{2}\big] &= r\text{-}lim\, \beta\big(\frac{\phi_{n'}+\phi}{2}\big) - \frac{\beta(\phi)}{2} - r\text{-}lim\, \frac{\beta(\phi_{n'})}{2} \\ &\leq \beta(\phi) - \frac{\beta(\phi)}{2} - \frac{\beta(\phi)}{2} = 0, \end{aligned}$$

thanks to the point 2. and since $\phi_n/2+\phi/2 \overset{r}{\rightharpoonup} \phi$. Therefore, for any k, we have

$$\begin{aligned} 0 \leq \lim_{n\to\infty} \int_{Y_k} T_1\Big(\beta\big(\frac{\phi_n+\phi}{2}\big) - \frac{\beta(\phi_n)}{2} - \frac{\beta(\phi)}{2}\Big)\, d\mu \leq \\ \leq \int_{Y_k} r\text{-}limsup\, \big[\beta\big(\frac{\phi_n+\phi}{2}\big) - \frac{\beta(\phi_n)}{2} - \frac{\beta(\phi)}{2}\big]\, d\mu = 0, \end{aligned}$$

so that, up to extraction a subsequence,

$$\beta\big(\frac{\phi_n+\phi}{2}\big) - \frac{\beta(\phi_n)}{2} - \frac{\beta(\phi)}{2} \to 0 \quad \text{a.e. on } Y \qquad \text{and} \qquad \phi_n \to \phi \quad \text{a.e. on } Y.$$

4. Fix $\chi \in C_c(Y)$ such that $0 \leq \chi \leq 1$. Since $T_M(\phi_n)\,\chi \rightharpoonup \bar T_M\,\chi$ weakly in L^1, we have

$$S(T_M(\phi_n)\,\chi) \rightharpoonup S(\bar T_M\,\chi) \quad \text{weakly in } L^1. \tag{6.4}$$

Furthermore, we have $T_K(S(T_M(\phi_n)\,\chi)) \leq T_K(S(\phi_n))$, and using (6.4) and part 2 it follows

$$S(\bar{T}_M\,\chi) = r\text{-}\mathop{liminf}_{n\to\infty}\; S(T_M(\phi_n)\,\chi) \leq r\text{-}\mathop{liminf}_{n\to\infty}\; S(\phi_n).$$

We conclude letting $\chi \nearrow 1$ and $M \to /\infty$. $\square$

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