THE CONTINUOUS COAGULATION-FRAGMENTATION EQUATIONS WITH DIFFUSION

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Abstract

Existence of global weak solutions to the continuous coagulation-fragmentation equations with diffusion is investigated when the kinetic coefficients satisfy a detailed balance condition or the coagulation coefficient enjoys a monotonicity condition. Our approach relies on weak and strong compactness methods in L^1 in the spirit of the DiPerna-Lions theory for the Boltzmann equation. Under the detailed balance condition the large time behaviour is also studied.

1 Introduction

Coagulation and fragmentation processes occur in the dynamics of cluster growth and describe the mechanisms by which clusters can coalesce to form larger clusters or fragment into smaller ones. Such processes are met in a variety of physical contexts, for instance in aerosol science for the description of the evolution of a system of solid or liquid particles suspended in a gas [16], but also in astrophysics [31], colloidal chemistry [35, 36] and polymer science. Other situations in which coagulation-fragmentation processes arise include hematology (red blood cell aggregation [30]) and population dynamics (animal grouping [29]). In these situations, the clusters are assumed to be fully identified by their size (or volume or number of particles) which might be either a positive real number (continuous model) or a positive integer (discrete model). Coagulation-fragmentation models then aim at providing a description of the cluster size distribution as a function of space and time as the system of clusters undergoes various physical influences. In the model to be studied in this paper, the only reactions taken into account are the binary coagulation and fragmentation of clusters and the approach of two clusters leading to aggregation takes place only by brownian movement or diffusion (thermal coagulation). Other effects such as multiple coagulation or multiple fragmentation, condensation, together with the influence of other external force fields (such as electric fields for charged particles) are neglected. More precisely, denoting by C_y the clusters of size $y \in \mathbb{R}_+ := (0,+\infty)$, the basic reactions taken into account herein are

$$C_y + C_{y'} \stackrel{a(y,y')}{\longrightarrow} C_{y+y'} \quad \text{(binary coagulation)},$$

and

$$C_y \stackrel{b(y-y',y')}{\longrightarrow} C_{y-y'} + C_{y'} \quad \text{(binary fragmentation)}.$$

Here the rates a and b of these reactions are assumed to depend only on the sizes of the clusters involved in the reactions and satisfy

$$a(y,y') = a(y',y) \geq 0 \quad \text{and} \quad b(y,y') = b(y',y) \geq 0. \tag{1.1}$$

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Similarly, the diffusion of the clusters is assumed to be only size-dependent with a diffusion coefficient $d = d(y)$. Denoting by $f(t,x,y)$ the size distribution function at time t and position x, the continuous coagulation-fragmentation equations with diffusion read

$$\begin{aligned}
&(1.2) & \partial_t f - d(y)\,\Delta_x f &= Q(f), \quad (t,x,y) \in (0,+\infty)\times\Omega\times\mathbb{R}_+,\\
&(1.3) & \partial_n f &= 0, \quad (t,x,y) \in (0,+\infty)\times\partial\Omega\times\mathbb{R}_+,\\
&(1.4) & f(0,x,y) &= f^{in}(x,y), \quad (x,y)\in\Omega\times\mathbb{R}_+.\\
&(1.5) & &
\end{aligned}$$

Here, Ω is an open bounded subset of $\mathbb{R}^N$, $N \geq 1$, with smooth boundary $\partial\Omega$, $\partial_n f$ denotes the outward normal derivative of f and the coagulation-fragmentation reaction term $Q(f)$ is given by

$$Q(f) = Q_1(f) - Q_2(f) - Q_3(f) + Q_4(f), \tag{1.6}$$

with

$$\begin{aligned}
Q_1(f)(x,y) &= \frac{1}{2}\int_0^y a(y',y-y')\; f(x,y')\; f(x,y-y')\; dy',\\
Q_2(f)(x,y) &= \frac{1}{2}\int_0^y b(y',y-y')\; dy'\; f(x,y),\\
Q_3(f)(x,y) &= L(f)(x,y)\; f(x,y)\\
&\quad \text{with} \quad L(f)(x,y) := \int_0^\infty a(y,y')\; f(x,y')\; dy',\\
Q_4(f)(x,y) &= \int_0^\infty b(y,y')\; f(x,y+y')\; dy'.
\end{aligned}$$

The meaning of the different contributions to the reaction term $Q(f)$ is the following : $Q_1(f)$ accounts for the formation of clusters C_y by coalescence of smaller clusters and $Q_2(f)$ for the breakage of clusters C_y into two smaller pieces. The term $Q_3(f)$ describes the depletion of clusters C_y by coagulation with other clusters, while $Q_4(f)$ represents the gain of clusters C_y as a result of the fragmentation of larger clusters. Observe that there is no source nor sink of clusters in the reactions described above and one thus expects the total volume of the clusters to remain unchanged throughout time evolution. From a mathematical point of view, it would read

$$\int_\Omega\int_0^\infty y\; f(t,x,y)\; dydx = \int_\Omega\int_0^\infty y\; f^{in}(x,y)\; dydx.$$

This equality comes from the following formal identity with $\phi(y) = y$

$$\begin{aligned}
&\int_0^\infty Q(f)\;\phi\; dy \\
&= \frac{1}{2}\int_0^\infty\int_0^\infty \left(a(y,y')\; f\; f' - b(y,y')\; f''\right)\; \left(\phi'' - \phi - \phi'\right)\; dy'dy
\end{aligned} \tag{1.7}$$

for $\phi\; : \mathbb{R}_+ \to \mathbb{R}$ which is obtained by changing variables and applying (without justification) the Fubini theorem to $Q_1(f)$ and $Q_2(f)$. Here and below we put $f = f(t,x,y)$, $f' = f(t,x,y')$ and $f'' = f(t,x,y+y')$ to shorten notation. Similarly $\phi = \phi(y)$, $\phi' = \phi(y')$ and $\phi'' = \phi(y+y')$. It is however well-known by now that, even in the spatially homogeneous case, this property may fail to be true, a phenomenon known as gelation. We will however not consider this issue here and refer to [1, 18, 21] (and the references therein) for results in that direction. In general, it is only possible to prove that the total volume at time t does not exceed the initial total volume, i.e.

$$\int_\Omega\int_0^\infty y\; f(t,x,y)\; dydx \leq \int_\Omega\int_0^\infty y\; f^{in}(x,y)\; dydx, \tag{1.8}$$

provided the latter is finite. Still, the inequality (1.8) provides a natural functional space to work with together with a first *a priori* estimate.

Unfortunately, in the case of general rate coefficients a and b, (1.8) is the only available natural *a priori* estimate and it is not sufficient to yield an existence theory for (1.2)-(**??**) in the spatially inhomogeneous case (in contrast with the spatially homogeneous case for which it is almost enough in many cases, see [21, 22, 37]). The main reason for the discrepancy between these two situations arises from the coagulation terms $Q_1(f)$ and $Q_3(f)$ which may be seen as convolutions with respect to y but pointwise products with respect to x. While an L^1-bound is clearly sufficient to give a meaning to the former, it is certainly not to have suitable functional properties for the latter. Nevertheless, in some situations, additional *a priori* estimates are available which guarantee the weak compactness in L^1 of both f and $Q(f)$. In that case, we are in a situation which looks like the one encountered for the Boltzmann equation [14] (and is even more favourable, thanks to the diffusion operator Δ_x) and some arguments in the spirit of those developed in [14] yield the strong compactness of the volume averages of f. Such a result in turn allows to prove the existence of weak solutions to (1.2)-(**??**).

The purpose of this paper is thus to investigate the existence of weak solutions to (1.2)-(**??**) in two situations where the weak compactness in L^1 of f and $Q(f)$ can be obtained, namely when the coagulation and fragmentation coefficients enjoy the so-called detailed balance condition, and when the coagulation coefficients satisfy a monotonicity property introduced by Galkin & Tupchiev [19]. We briefly describe these two situations now and roughly indicate how the weak compactness in L^1 of f and $Q(f)$ is obtained. Concerning the former, it turns out that, in some physical cases, the coagulation and fragmentation reactions enjoy a reversibility property also called detailed balance condition, and a mathematical formulation of the detailed balance condition reads

$$\left\{\begin{array}{l}\text{there is a positive function } M\in L^1(\mathbb{R}_+,(1+y)\ dy) \text{ such that}\\[2ex] \qquad\qquad a(y,y')\ M(y)\ M(y')=b(y,y')\ M(y+y'),\quad (y,y')\in\mathbb{R}_+^2.\end{array}\right. \tag{1.9}$$

The function M in (1.9) is usually called an *equilibrium* of (1.2) and it readily follows from (1.9) that it is a stationary solution to (1.2)-(**??**). Furthermore, the existence of such an equilibrium guarantees that an additional *a priori* estimate is available which is similar to the celebrated H–theorem for the Boltzmann equation (see [14] and the references therein), and reads

$$\begin{aligned}\frac{d}{dt}H(f|M)&+\int_\Omega\int_0^\infty d(y)\ \frac{|\nabla_x f|^2}{f}\ dydx\\ &+\frac{1}{2}\int_\Omega\int_0^\infty\int_0^\infty e(f)\ dy'dydx=0,\end{aligned} \tag{1.10}$$

where

$$\begin{aligned}H(f|M)&=\int_\Omega\int_0^\infty h(f|M)\ dydx\\ h(f|M)&:=f\ \left(\ln\left(\frac{f}{M(y)}\right)-1\right)+M(y)\geq 0,\end{aligned} \tag{1.11}$$

$$e(f)=j\left(a(y,y')\ f\ f',b(y,y')\ f''\right) \tag{1.12}$$

$$j(r,s)=\left\{\begin{array}{lll}(r-s)\ \ (\ln r-\ln s)\geq 0 & \text{if} & (r,s)\in\mathbb{R}_+^2,\\[2ex] 0 & \text{if} & (r,s)=(0,0),\\[2ex] +\infty & \text{otherwise.} & \end{array}\right. \tag{1.13}$$

Formally (1.10) follows from (1.7) with $\phi = dh(f|M)/df = \ln f/M$. Under suitable assumptions on the initial datum f^{in} the following "natural" bounds may be obtained from (1.8) and (1.10)

$$\sup_{t\in[0,T]} \int_\Omega \int_0^\infty f(t) \left((1+y) + \left| \ln \left(\frac{f(t)}{M(y)} \right) \right| \right) \, dydx < \infty, \tag{1.14}$$

$$\int_0^T \int_\Omega \int_0^\infty d(y)^{1/2} \, |\nabla_x f| \, dydxdt < \infty, \tag{1.15}$$

$$\int_0^T \int_\Omega \int_0^\infty \int_0^\infty e(f) \, dy' dydxdt < \infty. \tag{1.16}$$

It is then possible to deduce from the estimates (1.14)-(1.16) that f and $Q(f)$ lie in a weakly compact subset of L^1 (provided a, b and d fulfil some technical assumptions which will be specified later). We refer to Section 3.1 below for a detailed proof of this fact.

In addition it readily follows from (1.9) that $M_\alpha(y) = M(y)\, e^{\alpha y}$, $y \in \mathbb{R}_+$, satisfies the detailed balance condition and is thus also an equilibrium. Consequently it is a stationary solution to (1.2)-(**??**) for $\alpha \in \mathbb{R}$ and the stabilization of solutions to (1.2)-(**??**) towards the equilibria will also be investigated. Roughly speaking we prove that any non-zero cluster point as $t \to +\infty$ of the solutions to (1.2)-(**??**) coincides with M_α for some $\alpha \in \mathbb{R}$. Convergence towards a single equilibrium is also achieved in some particular cases.

In the other situation to be considered in this paper, a monotonicity condition is required on the coagulation coefficients a [8, 17, 19], namely

$$a(y', y-y') \le a(y', y) \quad \text{for} \quad y \ge y' \ge 0. \tag{1.17}$$

Such a monotonicity property guarantees that, in the absence of fragmentation ($b \equiv 0$), the $L^p(\Omega \times \mathbb{R}_+)$-norm of weak solutions to (1.2)-(**??**) is non-increasing with respect to time for every $p \in [1, \infty]$. In addition, assuming the fragmentation coefficients to be suitably dominated by the coagulation ones, we are able to prove that

$$\sup_{t\in[0,T]} \int_\Omega \int_0^\infty \Phi(f(t)) \, dydx < \infty,$$
$$\int_0^T \int_\Omega \int_0^\infty \int_0^\infty a(y,y') \; f \; f' \; \Phi'(f) \, dy'dydxdt < \infty$$

for any $T \in \mathbb{R}_+$, where Φ is a suitably constructed non-negative convex function such that $\Phi(r)/r \to +\infty$ and $\Phi'(r) \to +\infty$ as r increases to $+\infty$. The properties of Φ then allow us to conclude that f and $Q(f)$ lie in a weakly compact subset of L^1 and we refer to Section 3.2 below for a detailed proof.

Before stating our main results, let us mention some related works. It turns out that the continuous coagulation-fragmentation equations with diffusion have not been much studied and it is rather its discrete version which has been the object of several papers recently. Concerning the continuous model, the only work we are aware of is due to Amann [3]. The approach used by Amann is completely different and consists in viewing (1.2)-(**??**) as a single semilinear evolution equation of the form

$$\frac{du}{dt} + \mathcal{A}(t)u = R(t,u), \quad u(0) = u^0,$$

where u is a Banach-space-valued function of $(t,x) \in (0,+\infty) \times \Omega$. Abstract results from the theory of general linear and quasilinear parabolic problems may then be applied after specifying a suitable functional setting in which the operator $\mathcal{A}$ has the desired properties [3]. Though this approach requires the strong assumption of the boundedness of the kinetic coefficients together with some additional regularity of the initial datum with respect to space, it provides the existence, uniqueness, total volume conservation and continuous dependence with respect to the data of solutions to (1.2)-(**??**), but only locally in time in the general case (global existence is obtained in one-space dimension or in the absence of coagulation ($a \equiv 0$) or if d does not depend on y). Let us also mention here that existence results are also available for the continuous coagulation model with spatial transport where the operator $d(y)\ \Delta_x$ is replaced by a spatial transport operator $\mathrm{div}_x(V(x,y)\ f(t,x,y))$ (see, e.g., [8, 10, 17] and the references therein). Finally, as already mentioned, the discrete version of (1.2)-(**??**) where y ranges in the set of positive integers (and the integrals in $Q(f)$ are replaced by series) has been investigated in several papers since the pioneering work [5] where the pure coagulation equation with diffusion was considered. Existence results have subsequently been obtained in [3, 11, 20, 22, 23, 24, 26, 34, 40, 41] under various assumptions on the kinetic and diffusion coefficients, and large time asymptotics have been considered in [12, 25]. Let us point out that the approach we develop here for the continuous model would yield existence results for the discrete model with similar assumptions. However existence of a solution to the discrete model holds true with the growth assumption (2.2) below but without the structure conditions (1.9) or (1.17) [22].

We now outline the contents of this paper : in the next section, we state precisely our assumptions on the kinetic and diffusion coefficients and on the initial datum f^{in}, together with our main results. Our first result (Theorem 2.2) is a weak stability principle for weak solutions to (1.2)-(**??**) : a similar result is already known for the renormalized solutions to the Boltzmann equation [14, Theorem II.1]. Roughly speaking the stability result states that given a sequence (f_n) of weak solutions to (1.2)-(**??**) such that (f_n) and $(Q(f_n))$ are weakly compact in L^1, there is a subsequence of (f_n) which converges weakly in L^1 to a weak solution to (1.2)-(**??**). This stability property of weak solutions to (1.2)-(**??**) will be the main tool in our existence proof. Weak solutions are constructed in Theorems 2.3 and 2.6 when the kinetic coefficients satisfy the detailed balance condition (1.9) and the monotonicity condition (1.17), respectively. In the former case, additional information on the large time behaviour of the weak solutions constructed in Theorem 2.3 are available and the stabilization towards steady states is described in Theorem 2.4. In Section 3 we perform formal computations to show how the assumptions made on the kinetic coefficients yield the weak compactness in L^1 of both f and $Q(f)$ with the hope of clarifying the role played by these assumptions. Section 4 is devoted to the proof of the weak stability result. The proof of this result actually proceeds along the same lines as that of [14, Theorem II.1]. We then construct in Section 5 a sequence of approximating problems to which we may apply the weak stability result and deduce Theorems 2.3 and 2.6. The weak stability result is used once more in Section 6 to investigate the large time behaviour of the solutions to (1.2)-(**??**) constructed in Theorem 2.3. Some extensions of our approach to other kinetic coefficients are outlined in the last section.

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2 Main results

Throughout the paper we assume a positivity condition on the diffusion coefficient d, namely,

$$
(2.1) \qquad \begin{cases} \text{the functions } d \text{ and } 1/d \text{ both belong to } L^\infty(1/R,R) \text{ for each } R \in (1,+\infty) \\ \text{and we put} \\ \qquad d_R := \operatorname*{ess\,inf}_{(1/R,R)} d > 0 \quad \text{and} \quad D_R := \operatorname*{ess\,sup}_{(1/R,R)} d > 0. \end{cases}
$$

together with the following growth condition on the kinetic coefficients :

$$
(2.2) \qquad \begin{cases} \text{for each } R \in \mathbb{R}_+ \text{ the functions } a \text{ and } b \text{ belong to } L^\infty((0,R)\times(0,R)) \\ \text{with} \\ \qquad M_R := \|a\|_{L^\infty((0,R)\times(0,R))} + \|b\|_{L^\infty((0,R)\times(0,R))}, \\ \text{and there is a function } \omega_R \in L^\infty_0(\mathbb{R}_+) \text{ such that} \\ \qquad a(y,y') + b(y,y') \le \omega_R(y')\ (1+y'), \quad (y,y') \in (0,R)\times\mathbb{R}_+. \end{cases}
$$

Here and below we denote by $L_0^\infty(\mathbb{R}_+)$ the set of functions $\omega \in L^\infty(\mathbb{R}_+)$ such that $\omega(y)$ converges to zero as $y \to +\infty$. Observe that (2.1) ensures that (1.2) is uniformly parabolic in the x-variable on $(0,T) \times \Omega \times (1/R, R)$ for each $R > 1$. Typical examples of diffusion coefficients include $d(y) = D_0\, y^{-\delta}$, $\delta \in (0,1]$ [33], which clearly satisfy (2.1). Also, kinetic coefficients a and b satisfying

$$a(y,y') + b(y,y') \le C_0\ \left(1 + (yy')^\alpha\right)$$

with $\alpha \in [0,1)$ clearly fulfil (2.2).

In view of the expected properties of f and $Q(f)$ mentioned above we are naturally led to the following definition of a weak solution to the initial-boundary value problem (1.2)-(**??**).

Definition 2.1 *Let $T \in (0,+\infty]$ and f^{in} be a non-negative function in $L^1(\Omega \times \mathbb{R}_+; (1+y)dxdy)$. A weak solution to (1.2)-(**??**) on $[0,T)$ is a non-negative function*

$$f \in \mathcal{C}([0,T); L^1(\Omega \times \mathbb{R}_+)) \cap L^\infty(0,T; L^1(\Omega \times \mathbb{R}_+; ydxdy))$$

satisfying $f(0) = f^{in}$ together with

$$f \in L^1((0,T) \times (1/R,R); W^{1,1}(\Omega)), \tag{2.3}$$

$$Q_i(f) \in L^1((0,T) \times \Omega \times (0,R)), \quad i \in \{1,\ldots,4\}, \tag{2.4}$$

*for each $R \in \mathbb{R}_+$ and satisfies (1.2)-(**??**) in the following weak sense*

$$\begin{aligned} & \int_\Omega \int_0^\infty \left(\psi(t)\ f(t) - \psi(0)\ f^{in}\right)\ dydx \\ & + \int_0^t \int_\Omega \int_0^\infty \left(d(y)\ \nabla_x f\ \nabla_x \psi - f\ \partial_t \psi\right)\ dydxds \\ = & \ \frac{1}{2} \int_0^t \int_\Omega \int_0^\infty Q(f)\ \psi\ dydxds \end{aligned} \tag{2.5}$$

for each $t \in (0,T)$ and $\psi \in \mathcal{C}^1([0,T] \times \bar{\Omega} \times \mathbb{R}_+)$ with compact support in $[0,T) \times \bar{\Omega} \times \mathbb{R}_+$.

We now state a weak stability principle for weak solutions to (1.2)-(**??**).

Theorem 2.2 *Assume that the kinetic coefficients a and b and the diffusion coefficient d fulfil the assumptions (2.2) and (2.1), respectively. Let $T \in (0,+\infty)$ and, for each $n \ge 1$, let f_n be a weak solution to (1.2)-(**??**) on $[0,T)$ with initial datum $f_n(0)$. Assume further that there are a weakly compact subset $\mathcal{K}_w$ of $L^1(\Omega \times \mathbb{R}_+)$ and a constant C_T such that*

$$f_n(t) \in \mathcal{K}_w \ \text{ for each } \ t \in [0,T], \tag{2.6}$$

$$\sup_{t \in [0,T)} \int_\Omega \int_0^\infty (1+y)\ f_n(t,x,y)\ dydx \le C_T \tag{2.7}$$

for every $n \ge 1$ and

$$(Q_i(f_n)) \ \text{ is weakly compact in } \ L^1((0,T) \times \Omega \times (0,R)) \tag{2.8}$$

for each $R \ge 0$ and $i \in \{1,\ldots,4\}$. Then there are a subsequence (f_{n_k}) of (f_n) and a function f such that

$$f \ \text{ is a weak solution to (1.2)-(??) on } [0,T), \tag{2.9}$$

$$\left\{ \begin{array}{l} f_{n_k} \longrightarrow f \quad \text{in} \quad \mathcal{C}([0,T); w - L^1(\Omega \times \mathbb{R}_+)), \\ \\ Q_i(f_{n_k}) \rightharpoonup Q_i(f) \quad \text{in} \quad L^1((0,T) \times \Omega \times (0,R)) \end{array} \right. \tag{2.10}$$

for $R \in \mathbb{R}_+$ and $i \in \{1, \dots, 4\}$, and

$$\int_0^\infty \psi(y)\ f_{n_k}\ dy \longrightarrow \int_0^\infty \psi(y)\ f\ dy \quad in \quad L^1((0,T)\times\Omega) \tag{2.11}$$

for $\psi \in \mathcal{D}(\mathbb{R}_+)$. Here $\mathcal{D}(\mathbb{R}_+)$ denotes the space of $\mathcal{C}^\infty$-smooth and compactly supported functions in $\mathbb{R}_+$ and $\mathcal{C}([0,T); w - L^1(\Omega\times\mathbb{R}_+))$ the space of weakly continuous functions from $[0,T)$ in $L^1(\Omega\times\mathbb{R}_+)$.

A more general result is actually available and we may also allow the kinetic and diffusion coefficients to depend on n (see Theorem 4.1 below). In particular, we will be able to consider sequences of solutions to approximating problems for which existence of a solution follows by a classical fixed point argument. We shall then employ the extended version of Theorem 2.2 to obtain a weak solution to (1.2)-(**??**) as a limit of such sequences.

We are thus in a position to state our existence results and first consider the case where the kinetic coefficients enjoy the detailed balance condition (1.9) besides the symmetry condition (1.1) and the growth condition (2.2). We further assume that, for each $R \in \mathbb{R}_+$, the equilibrium M in (1.9) satisfies the positivity condition

$$\operatorname*{ess\,inf}_{y\in(0,R)} M(y) > 0. \tag{2.12}$$

For instance, a possible choice of kinetic coefficients a and b satisfying the detailed balance condition (1.9) is

$$\left\{\begin{array}{lcl} a(y,y') & = & A_0\ (1+y)^\alpha\ (1+y')^\alpha, \\ & & \\ b(y,y') & = & B_0\ a(y,y')\ \dfrac{\exp(\lambda\ (y+y')^p)}{\exp(\lambda\ (y^p+y'^p))}\ \dfrac{(1+y+y')^\tau}{(1+y)^\tau\ (1+y')^\tau}, \end{array}\right. \tag{2.13}$$

where $\alpha \in [0,1]$, $\lambda > 0$, $p \in [0,1)$, $\tau \in [0,+\infty)$, and A_0, B_0 are positive real numbers. In that case, $M(y) = (1+y)^{-\tau}\ \exp(-\lambda\ y^p - y)$, $y \in \mathbb{R}_+$. The case of constant coefficients a and b is included in the above example (with $\alpha = \tau = p = 0$) and the additional requirements (2.12) and (2.2) are clearly fulfilled if $\alpha < 1$.

As for the initial datum f^{in} we assume that

$$\begin{cases} f^{in} \in L^1(\Omega\times\mathbb{R}_+, ydxdy) \text{ is non-negative a.e. and} \\ H(f^{in}|M) < \infty. \end{cases} \tag{2.14}$$

Our first existence result reads as follows (a more precise result is actually available, see Theorem 5.8 below).

Theorem 2.3 *Assume that the kinetic coefficients a and b satisfy (1.1), (1.9), (2.2) and (2.12), the diffusion coefficient d satisfy (2.1) and the initial datum f^{in} satisfy (2.14). Then there is a weak solution f to (1.2)-(**??**) on $[0,+\infty)$ satisfying (1.8).*

Owing to the detailed balance condition (1.9) it is also possible to investigate the large time behaviour of the weak solutions to (1.2)-(**??**) constructed in the previous theorem. Indeed we first observe that, for each $\alpha \in \mathbb{R}$, the function M_α defined by $M_\alpha(y) = M(y)\ \exp(\alpha\ y)$, $y \in \mathbb{R}_+$, also satisfies the detailed balance equation (1.9) and is thus a stationary solution to (1.2)-(**??**). Such properties are also enjoyed by the function $M_{-\infty} := 0$ and the function Φ defined by $\Phi(-\infty) = 0$,

$$\Phi(\alpha) = \int_0^\infty y\ M_\alpha(y)\ dy, \quad \alpha \in \mathbb{R},$$

is an increasing function on $D(\Phi) := \{\alpha \in [-\infty,+\infty)\ ,\ \ \Phi(\alpha) < +\infty\}$. Since we are only interested in equilibria with finite total volume we introduce

$$\alpha_s := \sup\left\{\alpha \in \mathbb{R}\ ,\ M_\alpha \in L^1(\mathbb{R}_+, y\ dy)\right\} \in [0,+\infty], \tag{2.15}$$

so that $[-\infty, \alpha_s) \subset D(\Phi) \subset [-\infty, \alpha_s]$, and

$$\varrho_s := \sup_{\alpha \in (-\infty, \alpha_s)} \int_0^\infty y \ M_\alpha(y) \ dy \in (0, +\infty]. \tag{2.16}$$

Observe that ϱ_s may be finite whatever the value of α_s. In that case there is no equilibrium with total volume larger than ϱ_s. For instance, in the previous example (2.13), $\alpha_s = 1$ but ϱ_s is finite if $p \in (0,1)$ or $\tau > 1$ and infinite if $p = 0$ and $\tau \in [0,1)$. With these notations we have the following result.

Theorem 2.4 *Consider an initial datum f^{in} satisfying (2.14), $f^{in} \not\equiv 0$, and define $\alpha^{in} \in \mathbb{R}$ as the unique real number such that*

$$\Phi\left(\alpha^{in}\right) = M^{in} := \frac{1}{|\Omega|} \int_\Omega \int_0^\infty f^{in}(x,y) \ y \ dydx \tag{2.17}$$

if $M^{in} < \varrho_s$ and $\alpha^{in} = \alpha_s$ if $M^{in} \geq \varrho_s$. Assume also that the assumptions of Theorem 2.3 are fulfilled and that a and b are positive a.e. in $\mathbb{R}_+^2$. For any weak solution f to (1.2)-(??) with initial datum f^{in} given by Theorem 2.3 and any sequence $(t_n)_{n\geq 1}$ of positive real numbers satisfying $t_n \to +\infty$ there are a subsequence (t_{n_k}) and $\alpha \in \left[-\infty, \alpha^{in}\right]$ such that the sequence of functions (f_{n_k}) defined by $f_{n_k}(t,x,y) = f(t + t_{n_k}, x, y)$ satisfies

$$f_{n_k} \longrightarrow M_\alpha \quad in \ \ \mathcal{C}((0,T]; L^1(\Omega \times \mathbb{R}_+)) \ \ for\ every \ \ T > 0. \tag{2.18}$$

Moreover, under the additional assumption $\alpha_s = +\infty$, there holds $\varrho_s = +\infty$ and the equation (2.17) always has a solution α^{in}. In that case $\alpha = \alpha^{in}$ and there holds

$$\int_\Omega \int_0^\infty f(t,x,y) \ y \ dydx = \int_\Omega \int_0^\infty f^{in}(x,y) \ y \ dydx \tag{2.19}$$

for each $t \in \mathbb{R}_+$ together with

$$f(t) \longrightarrow M_{\alpha^{in}} \quad in \ \ L^1(\Omega \times \mathbb{R}_+; (1+y)dxdy) \ \ as \ \ t \to +\infty. \tag{2.20}$$

Let us mention here that the discrepancy between the convergence results when $\alpha_s < +\infty$ and $\alpha_s = +\infty$ stems from the fact that the latter assumption provides a uniform control of $f(t)$ in $L^1(\Omega \times (Y, +\infty), ydxdy)$ for large values of Y which allows us to prove that the total volume of any cluster point of $(f(t))$ as $t \to +\infty$ is equal to the initial total volume.

Remark 2.5 *(i) Let us mention here that there are kinetic coefficients a and b satisfying (1.1), (1.9), (2.2) and (2.12), and $\alpha_s = +\infty$ as well. For instance,*

$$a(y,y') = \exp\left\{-(y^2 + (y')^2)\right\} \quad and \quad b(y,y') = \exp\left\{-(y-y')^2\right\}, \quad (y,y') \in \mathbb{R}_+^2,$$

and (1.9) is fulfilled with $M(y) = \exp\left\{-y^2\right\}$, $y \in \mathbb{R}_+$.
(ii) Let us also point out that Theorem 2.4 seems to be new even in the spatially homogeneous case where only the case of constant coefficients a and b has already been considered in [2, 38].

We next turn to the case where the coagulation coefficients a enjoy the monotonicity property (1.17). Besides (1.1) and (1.17) we also assume that a and b satisfy the growth condition (2.2) and b is suitably dominated by a in the following sense : there are a constant $A > 0$ and a non-negative function

$$B \in L^1(\mathbb{R}_+) \quad \text{with} \quad y \mapsto y \ B(y) \in L^\infty(\mathbb{R}_+) \tag{2.21}$$

such that

$$b(y, y'-y) \leq A \ a(y,y') + B(y'), \quad 0 \leq y \leq y'. \tag{2.22}$$

We finally require the initial datum to fulfil

$$f^{in} \in L^1(\Omega \times \mathbb{R}_+, (1+y)dxdy) \ \ \text{is non-negative a.e.} \tag{2.23}$$

We then have the following result.

Theorem 2.6 *Assume that the kinetic coefficients a and b satisfy (1.1), (1.17), (2.2) and (2.22), the diffusion coefficient d satisfy (2.1) and the initial datum f^{in} satisfy (2.23). Then there is a weak solution f to (1.2)-(**??**) on $[0,+\infty)$ satisfying (1.8).*

As an example of kinetic coefficients enjoying the requirement of Theorem 2.6 we put

$$a(y,y') = \mu_a \ \left(y^\alpha + (y')^\alpha\right), \quad (y,y') \in \mathbb{R}_+^2,$$

where μ_a is a positive real number and $\alpha \in [0,1)$. For the fragmentation coefficients we put

$$b(y,y') = \lambda_b + \mu_b \ \left(y + y'\right)^\beta, \quad (y,y') \in \mathbb{R}_+^2,$$

where λ_b and μ_b are non-negative real numbers and $\beta \in [0,\alpha]$. Such kinetic coefficients clearly satisfy (1.1), (1.17), (2.2) and (2.22) with $A = 2\ (\lambda_b + \mu_b)/\mu_a$ and

$$B(y) = (\lambda_b + \mu_b)\ \mathbf{1}_{[0,1]}(y).$$

Remark 2.7 *If $B \equiv 0$ in (2.22) and $f^{in} \in L^p(\Omega \times \mathbb{R}_+)$ for some $p \in (1,\infty]$ then the solution f to (1.2)-(**??**) constructed in Theorem 2.6 satisfies $f \in L^\infty(0,T; L^p(\Omega \times \mathbb{R}_+))$ for $T \in \mathbb{R}_+$. For $p = \infty$ this fact has been noticed in [17, Section 5.2] in the spatially homogeneous case and in [8] for the spatially inhomogeneous coagulation equation.*

3 *a priori* estimates

As already mentioned in the previous section the weak stability principle stated in Theorem 2.2 is the main tool in the proofs of Theorems 2.3–2.6. Still we first have to check that the requirements needed to apply this result, namely the weak compactness in L^1 of f and $Q(f)$, are fulfilled. The goal of this section is thus to show how the assumptions on the kinetic coefficients lead to the desired compactness properties. Let us point out here that some of the computations of this section are performed at a formal level but may be justified on the approximating solutions we construct in Section 5.

3.1 The detailed balance condition (1.9)

Throughout this section we assume that the kinetic coefficients a and b satisfy (1.1), (1.9), (2.12) and (2.2), the diffusion coefficient d satisfy (2.1) and we consider a solution f to (1.2)-(**??**) with an initial datum f^{in} satisfying (2.14) such that

$$C_0 := \sup_{t \in [0,+\infty)} \int_\Omega \int_0^\infty y\ f(t,x,y)\ dydx < \infty. \tag{3.1}$$

Hereafter we denote by C any positive constant which depends only on Ω, M, C_0 and $H(f^{in}|M)$. The dependence of C upon additional parameters will be indicated explicitly. We also put $\mathcal{U} := \Omega \times \mathbb{R}_+$, $\mathcal{U}_t := (0,t) \times \mathcal{U}$ and $\mathcal{U}_{t,R} := (0,t) \times \Omega \times (0,R)$ for $t \in [0,T]$ and $R \in \mathbb{R}_+$.

As mentioned in the Introduction the detailed balance condition (1.9) ensures that an analogue of the Boltzmann H–theorem holds for the coagulation-fragmentation equations which we formally derive now. We multiply (1.2) by $\ln(f/M)$ and integrate over $\mathcal{U}_t$. Noticing that

$$\int_\Omega \int_0^\infty \partial_t f \ \ln(f/M)\ dydx = \frac{d}{dt} H(f|M)$$

and recalling (1.7) we obtain

$$\begin{aligned} H(f(t)|M) &+ \int_0^t \int_\Omega \int_0^\infty d(y)\ \frac{|\nabla_x f|^2}{f}\ dydxds \\ &+ \frac{1}{2} \int_0^t \int_\Omega \int_0^\infty \int_0^\infty e(f)\ dy'dydxds = H(f^{in}|M), \end{aligned}$$

where $H(f|M)$ and $e(f)$ are defined by (1.11) and (1.12), respectively, and we recall that we use the following notations $f = f(t,x,y)$, $f' = f(t,x,y')$ and $f'' = f(t,x,y+y')$. Since the three terms of the left-hand side of the above inequality are non-negative we conclude that for every $t \in [0,+\infty)$

$$H(f(t)|M) \leq C, \tag{3.2}$$

$$\int_0^t \int_\Omega \int_0^\infty d(y)\ |\nabla_x f|^2\ f^{-1}\ dydxds \leq C, \tag{3.3}$$

$$\int_0^t \int_\Omega \int_0^\infty \int_0^\infty e(f)\ dy'dydxds \leq C. \tag{3.4}$$

We now derive additional estimates on f from (3.1)-(3.4) which eventually imply that f and $Q(f)$ belong to a weakly compact subset of L^1. We first state a preliminary result.

Lemma 3.1 *Let ξ be a measurable function from $\mathbb{R}_+ \times \mathcal{U}$ with discrete values in $\{0,1\}$ and $\alpha \geq e^2$. For $t \in \mathbb{R}_+$ there holds*

$$\begin{aligned} \int_\Omega \int_0^\infty \xi(t)\ f(t)\ dydx \quad &\leq \quad 2\ \left(\alpha + e^{-1}\right)\ \int_\Omega \int_0^\infty \xi(t)\ M\ dydx \\ &+ \quad \frac{2}{\ln\alpha}\ H(f(t)|M). \end{aligned} \tag{3.5}$$

Proof. Let $t \in \mathbb{R}_+$. On the one hand we notice that

$$\begin{aligned} \int_\mathcal{U} \xi(t)\ f(t)\ dydx \quad &\leq \quad \alpha\ \int_\mathcal{U} \xi(t)\ \mathbf{1}_{\{f(t)\leq\alpha M\}}\ M\ dydx \\ &+ \frac{1}{\ln\alpha}\ \int_\mathcal{U} \xi(t)\ \mathbf{1}_{\{f(t)>\alpha M\}}\ f(t)\ \left|\ln\left(\frac{f(t)}{M(y)}\right)\right|\ dydx \\ &\leq \quad \alpha\ \int_\mathcal{U} \xi(t)\ M\ dydx \\ &+ \frac{1}{\ln\alpha}\ \int_\mathcal{U} \xi(t)\ f(t)\ \left|\ln\left(\frac{f(t)}{M(y)}\right)\right|\ dydx. \end{aligned}$$

On the other hand, since $r\ \ln r \geq r\ |\ln r| - 2/e$ for $r > 0$, we have

$$f(t)\ \ln\left(\frac{f(t)}{M(y)}\right) \geq f(t)\ \left|\ln\left(\frac{f(t)}{M(y)}\right)\right| - \frac{2\ M}{e},$$

whence

$$\begin{aligned} \int_\mathcal{U} \xi(t)\ f(t)\ \left|\ln\left(\frac{f(t)}{M(y)}\right)\right|\ dydx \quad &\leq \quad H(f(t)|M) \\ &+ \quad \int_\mathcal{U} \xi(t)\ \left(f(t) + \frac{2\ M}{e}\right)\ dydx. \end{aligned} \tag{3.6}$$

Combining the above two inequalities yields

$$\begin{aligned} \left(1 - \frac{1}{\ln\alpha}\right)\ \int_\mathcal{U} \xi(t)\ f(t)\ dydx \quad &\leq \quad \left(\alpha + \frac{2}{e\ \ln\alpha}\right)\ \int_\mathcal{U} \xi(t)\ M\ dydx \\ &+ \quad \frac{1}{\ln\alpha}\ H(f(t)|M). \end{aligned}$$

Since $\alpha \geq e^2$ (3.5) follows at once from the above inequality. □

After this preparation we have the following result.

Lemma 3.2 *For $t \in \mathbb{R}_+$ there holds*

$$\int_\Omega \int_0^\infty f(t)\ \left(1 + \left|\ln\left(\frac{f(t)}{M(y)}\right)\right|\right)\ dydx \le C. \tag{3.7}$$

Proof. Let $t \in \mathbb{R}_+$ and take $\xi = \mathbf{1}_\mathcal{U}$ and $\alpha = e^2$ in (3.5). We thus obtain, thanks to (3.2),

$$\int_\mathcal{U} f(t)\ dydx \le C\ |\Omega|\ |M|_{L^1} + C \le C.$$

We next write (3.6) with the function $\xi = \mathbf{1}_\mathcal{U}$ and use (3.2) and the above inequality to conclude that (3.7) holds true. $\square$

We next show that (3.7) and (3.3) yield some estimates for the gradient of f. Though these estimates will not be used in the existence proof they ensure that the solution to (1.2)-(**??**) we construct in Theorem 2.3 is such that $d^{1/2}\nabla_x f$ belongs to $L^2(0,+\infty; L^1(\Omega \times \mathbb{R}_+))$. This property plays an important role in the study of the large time behaviour as it guarantees that the cluster points of $\{f(t\}$ as $t \to +\infty$ are spatially homogeneous.

Lemma 3.3 *For each $T \in \mathbb{R}_+$, we have*

$$\int_0^T \left(\int_\Omega \int_0^\infty d(y)^{1/2}\ |\nabla_x f|\ dydx\right)^2\ dt \le C, \tag{3.8}$$

and

$$\int_E d(y)^{1/2}\ |\nabla_x f|\ dydxdt \le C\ \left(\int_E f\ dydxdt\right)^{1/2} \tag{3.9}$$

for every measurable subset E of $\mathbb{R}_+ \times \Omega \times \mathbb{R}_+$.

Proof. Let $T \in \mathbb{R}_+$. By the Hölder inequality we have

$$\begin{aligned} & \int_0^T \left(\int_\mathcal{U} d(y)^{1/2}\ |\nabla_x f|\ dydx\right)^2\ dt \\ \le\ & \int_0^T \left(\int_\mathcal{U} f\ dydx\right)\ \left(\int_\mathcal{U} d(y)\ |\nabla_x f|^2\ f^{-1}\ dydx\right)\ dt, \end{aligned}$$

and (3.8) follows at once from (3.3), (3.7) and the above inequality.

We next consider a measurable subset E of $\mathbb{R}_+ \times \mathcal{U}$. Proceeding as above we infer from (3.3) that

$$\begin{aligned} & \int_E d(y)^{1/2}\ |\nabla_x f|\ dydxdt \\ \le\ & \left(\int_E d(y)\ \frac{|\nabla_x f|^2}{f}\ dydxdt\right)^{1/2}\ \left(\int_E f\ dydxdt\right)^{1/2} \\ \le\ & C\ \left(\int_E f\ dydxdt\right)^{1/2}, \end{aligned}$$

whence (3.9). $\square$

We now show that a sequence (f_n) of non-negative functions satisfying the bounds (3.1) and (3.7) uniformly with respect to $n \ge 1$ enjoys the desired compactness properties.

Lemma 3.4 *Let $T \in \mathbb{R}_+$ and (f_n) be a sequence of non-negative functions such that, for each $n \ge 1$, there holds*

$$\sup_{t\in[0,T]} \int_\Omega \int_0^\infty f_n(t)\ \left(1 + y + \left|\ln\left(\frac{f_n(t)}{M(y)}\right)\right|\right)\ dydx \le K_T, \tag{3.10}$$

$$\int_0^T \int_\Omega \int_0^\infty \int_0^\infty e(f_n)\ dy'dydxdt \le K_T \tag{3.11}$$

for some constant K_T (which does not depend on n). Then the sequence (f_n) is weakly compact in $L^1(\mathcal{U}_T)$ and the sequence $(Q_i(f_n))$ is weakly compact in $L^1(\mathcal{U}_{T,R})$ for each $i \in \{1, \ldots, 4\}$ and $R \in \mathbb{R}_+$. Furthermore there is a weakly compact subset $\mathcal{K}_w$ of $L^1(\mathcal{U})$ such that $f_n(t) \in \mathcal{K}_w$ for each $t \in [0,T]$ and $n \ge 1$.

Proof. We fix $t \in [0,T]$ and consider a measurable subset E of $\mathcal{U}$ with finite measure. It follows from (3.10) and (3.5) with f_n instead of f and $\xi = \mathbf{1}_E$ that, for $\alpha \ge e^2$,

$$\int_E f_n(t)\ dydx \le 4\ \alpha\ \int_E M(y)\ dydx + \frac{2\ K_T}{\ln \alpha}.$$

It also follows from (3.10) that

$$\int_\Omega \int_\alpha^{+\infty} f_n(t)\ dydx \le \frac{K_T}{\alpha}.$$

We now introduce the subset $\mathcal{K}_w$ of $L^1(\mathcal{U})$ defined by : $g \in \mathcal{K}_w$ if $g \in L^1(\mathcal{U})$ and satisfies

$$\begin{aligned} \int_E g\ dydx &\le 4\ \alpha\ \int_E M(y)\ dydx + \frac{2\ K_T}{\ln \alpha}, \\ \int_\alpha^\infty g\ dydx &\le \frac{K_T}{\alpha} \end{aligned}$$

for every measurable subset E of $\mathcal{U}$ with finite measure and $\alpha \ge e^2$.
On the one hand the above analysis ensures that $f_n(t) \in \mathcal{K}_w$ for each $t \in [0,T]$ and $n \ge 1$. On the other hand, since $M \in L^1(\mathbb{R}_+)$ and Ω has finite measure, the Dunford-Pettis theorem entails that $\mathcal{K}_w$ is a weakly compact subset of $L^1(\mathcal{U})$ and the last assertion of Lemma 3.4 is proved. Similar computations yield the weak compactness of (f_n) in $L^1(\mathcal{U}_T)$.

We next fix $R \in \mathbb{R}_+$. By (2.2) we have

$$Q_2(f_n) \le R\ M_R\ f_n,$$

and the weak compactness of (f_n) in $L^1(\mathcal{U}_{T,R})$ entails that of $(Q_2(f_n))$.

We next consider a measurable subset E of $\mathcal{U}_{T,R}$. Recalling the following inequality

$$\eta \le \alpha\ \xi + \frac{1}{\ln \alpha}\ (\eta - \xi)\ (\ln \eta - \ln \xi), \quad (\xi, \eta) \in \mathbb{R}_+^2,\ \alpha > 1, \tag{3.12}$$

we have for $0 \le y' \le y$ and $\alpha > 1$

$$\begin{aligned} & a(y', y-y')\ f_n(t,x,y')\ f_n(t,x,y-y') \\ \le\ & \alpha\ b(y', y-y')\ f_n(t,x,y) + \frac{1}{\ln \alpha}\ e(f_n)(t,x,y',y-y'), \end{aligned}$$

from which we easily deduce that

$$\begin{aligned} \int_E Q_1(f_n)\ dydxdt &\le \alpha\ \int_E Q_2(f_n)\ dydxdt \\ &+ \frac{1}{\ln \alpha}\ \int_E \int_0^y e(f_n)(t,x,y',y-y')\ dy'dydxdt. \end{aligned}$$

The second term of the right-hand side of the above inequality being bounded from above by the L^1-norm of $e(f_n)$ it follows from (3.11) that

$$\int_E Q_1(f_n)\ dydxdt \le \alpha\ \sup_{n \ge 1} \int_E Q_2(f_n)\ dydxdt + \frac{K_T}{\ln \alpha}.$$

Consequently, we infer from the weak compactness of $(Q_2(f_n))$ in $L^1(\mathcal{U}_{T,R})$ that $(Q_1(f_n))$ is bounded in $L^1(\mathcal{U}_{T,R})$ and

$$0 \leq \limsup_{|E|\to 0} \sup_{n\geq 1} \int_E Q_1(f_n)\ dydxdt \leq \frac{K_T}{\ln \alpha}.$$

The above inequality being valid for every $\alpha > 1$, we finally obtain

$$\lim_{|E|\to 0} \sup_{n\geq 1} \int_E Q_1(f_n)\ dydxdt = 0$$

by letting $\alpha \to +\infty$, whence the weak compactness of $(Q_1(f_n))$ in $L^1(\mathcal{U}_{T,R})$.

We now consider $Q_4(f_n)$. By (2.2) and (3.10) we have for $\alpha \geq 2R$

$$\begin{aligned}
\int_E Q_4(f_n)\ dydxdt &\leq \int_E \int_0^\alpha b(y, y'-y)\ f_n'\ dy'dydxdt \\
&+ \int_E \int_\alpha^\infty \omega_R(y'-y)\ (1+y'-y)\ f_n'\ dy'dydxdt \\
&\leq M_\alpha \int_{E\times(0,\alpha)} f_n(t,x,y')\ dy'dydxdt \\
&+ C(T,R)\ |\omega_R|_{L^\infty(\alpha-R,+\infty)}.
\end{aligned}$$

The weak compactness of (f_n) in $L^1(\mathcal{U}_T)$ warrants that the first term of the right-hand side of the above inequality converges to zero uniformly with respect to $n \geq 1$ as $|E| \to 0$. Consequently,

$$0 \leq \limsup_{|E|\to 0} \sup_{n\geq 1} \int_E Q_4(f_n)\ dydxdt \leq C(T,R)\ |\omega_R|_{L^\infty(\alpha-R,+\infty)},$$

and the weak compactness in $L^1(\mathcal{U}_{T,R})$ of $(Q_4(f_n))$ then follows since $\omega_R \in L_0^\infty(\mathbb{R}_+)$.

Finally the weak compactness in $L^1(\mathcal{U}_{T,R})$ of $(Q_3(f_n))$ is a consequence of that of $(Q_4(f_n))$ and (3.11) by the same argument as the one used for $(Q_1(f_n))$. □

3.2 The monotonicity property (1.17)

Throughout this section we assume that the kinetic coefficients a and b satisfy (1.1), (1.17), (2.2) and (2.22), the diffusion coefficient d satisfy (2.1) and we consider a solution f to (1.2)-(**??**) with an initial datum f^{in} satisfying (2.23) such that

$$C_0 := \sup_{t\in[0,+\infty)} \int_\Omega \int_0^\infty y\ f(t,x,y)\ dydx < \infty. \tag{3.13}$$

As we shall see below (Lemma 3.5) the monotonicity condition (1.17) is well-suited to obtain the weak compactness in L^1 of the coagulation terms. This is not the case for the fragmentation terms which have a tendency to concentrate the distribution function f near $y = 0$. The stringent condition (2.22) is then needed to prevent this phenomenon to occur and cannot be relaxed by strengthening the coagulation.

Hereafter we denote by C any positive constant which depends only on Ω, M, C_0, $|f^{in}|_{L^1(\Omega\times(0,+\infty))}$, A, $|B|_{L^1(0,+\infty)}$ and the $L^\infty(\mathbb{R}_+)$-norm of $y \mapsto y\ B(y)$. The dependence of C upon additional parameters will be indicated explicitly. We also put $\mathcal{U} := \Omega \times \mathbb{R}_+$, $\mathcal{U}_t := (0,t) \times \mathcal{U}$ and $\mathcal{U}_{t,R} := (0,t) \times \Omega \times (0,R)$ for $t \in [0,T]$ and $R \in \mathbb{R}_+$.

Lemma 3.5 *Let $\Phi \in W^{1,\infty}_{loc}([0,+\infty))$ be a non-negative and convex function such that*

$$0 \leq \Phi(r) \leq r\,\Phi'(r) \quad \textit{for} \quad r \geq 0. \tag{3.14}$$

Then for any $R \in (0,+\infty]$ one has

$$\begin{aligned}
& \int_\Omega \int_0^R (Q_1(f) - Q_3(f))\, \Phi'(f)\, dydx \\
\leq & -\frac{1}{2} \int_\Omega \int_0^R \int_0^\infty a(y,y')\, f\, \Phi'(f)\, f'\, dy'dydx.
\end{aligned}$$

Proof. We infer from the convexity of Φ that

$$\begin{aligned}
& \int_\Omega \int_0^R (Q_1(f) - Q_3(f))\, \Phi'(f)\, dydx \\
\leq & \frac{1}{2} \int_\Omega \int_0^R \int_0^y a(y', y-y')\, f(y-y')\, (f'-f)\, \Phi'(f)\, dy'dydx \\
+ & \frac{1}{2} \int_\Omega \int_0^R \int_0^y a(y', y-y')\, f(y-y')\, f\, \Phi'(f)\, dy'dydx \\
- & \int_\Omega \int_0^R \int_0^\infty a(y,y')\, f'\, f\, \Phi'(f)\, dy'dydx \\
\leq & \frac{1}{2} \int_\Omega \int_0^R \int_0^y a(y', y-y')\, f(y-y')\, \Phi(f')\, dy'dydx \\
- & \int_\Omega \int_0^R \int_0^\infty a(y,y')\, f'\, f\, \Phi'(f)\, dy'dydx \\
+ & \frac{1}{2} \int_\Omega \int_0^R \int_0^y a(y', y-y')\, f(y-y')\, (f\, \Phi'(f) - \Phi(f))\, dy'dydx.
\end{aligned}$$

Performing the change of variables $(y,y') \to (y', z = y-y')$ in the first term and $(y,y') \to (y, z = y-y')$ in the last term of the right-hand side of the above inequality we obtain

$$\begin{aligned}
& \int_\Omega \int_0^R (Q_1(f) - Q_3(f))\, \Phi'(f)\, dydx \\
\leq & \frac{1}{2} \int_\Omega \int_0^R \int_0^{R-y'} a(y', z)\, f(z)\, \Phi(f')\, dy'dzdx \\
- & \int_\Omega \int_0^R \int_0^\infty a(y,y')\, f'\, f\, \Phi'(f)\, dy'dydx \\
+ & \frac{1}{2} \int_\Omega \int_0^R \int_0^y a(y-z, z)\, f(z)\, (f\, \Phi'(f) - \Phi(f))\, dzdydx \\
\leq & -\frac{1}{2} \int_\Omega \int_0^R \int_0^\infty a(y,y')\, f'\, f\, \Phi'(f)\, dy'dydx \\
- & \frac{1}{2} \int_\Omega \int_0^R \int_0^\infty (a(y,y') - \mathbf{1}_{[0,y]}(y')\ a(y-y', y'))\, f'\, (f\, \Phi'(f) - \Phi(f))\, dy'dydx.
\end{aligned}$$

Now, Lemma 3.5 follows at once from the above inequality by remarking that the last term of the right-hand side of the above inequality is non-negative thanks to (1.17) and (3.14). □

Lemma 3.6 *For each $R \geq 1$ we have*

$$\int_\Omega \int_0^R (Q_2(f) + Q_4(f))\, dydx \leq C(R, M_R, \omega_R) \left(1 + \int_\Omega \int_0^R f\, dydx\right).$$

Proof. On the one hand, using (2.2), (3.13) and performing the change of variables $(y,y') \to (y, z = y+y')$ we obtain

$$\begin{aligned}
\int_\Omega \int_0^R Q_4(f)\, dydx &\le |\omega_R|_{L^\infty} \int_\Omega \int_0^R \int_0^\infty (1+y')\, f(y+y')\, dy'\, dydx \\
&\le R\, |\omega_R|_{L^\infty} \int_\Omega \int_0^\infty f(z)\,(1+z)\; dzdx \\
&\le C(R,\omega_R) \left(\int_\Omega \int_0^R f\; dydx + 2 \int_\Omega \int_0^\infty f\; y\; dydx \right) \\
&\le C(R,\omega_R) \left(1 + \int_\Omega \int_0^R f\; dydx \right).
\end{aligned}$$

On the other hand, it follows from (2.2) that

$$\int_\Omega \int_0^R Q_2(f)\, dydx \le \frac{1}{2} \int_\Omega \int_0^R R\; M_R\, f\, dydx.$$

Combining the above two inequalities yields Lemma 3.6. □

Corollary 3.7 *For $T \in \mathbb{R}_+$ and $R \ge 1$ there holds*

$$\sup_{t\in[0,T]} \int_\Omega \int_0^R f(t)\, dydx \le C(T,R,M_R,\omega_R,),$$

$$\int_0^T \int_\Omega \int_0^R \int_0^\infty a(y,y')\, f\, f'\, dy' dydxdt \le C(T,R,M_R,\omega_R).$$

In particular,

$$\sup_{t\in[0,T]} \int_\Omega \int_0^\infty f(t)\, dydx \le C(T,M_1,\omega_1). \tag{3.15}$$

Proof. Let $t \in [0,T]$. We just integrate (1.2) over $\mathcal{U}_{t,R}$ and infer from Lemma 3.5 (with $\Phi = Id$), Lemma 3.6 and (2.23) that

$$\begin{aligned}
&\int_\Omega \int_0^R f(t)\, dydx + \frac{1}{2} \int_0^t \int_\Omega \int_0^R \int_0^\infty a(y,y')\, f'\, f\, dy' dydxds \\
\le\ &\int_\Omega \int_0^R f^{in}\; dydx + C(T,R,M_R,\omega_R) \left(1 + \int_0^t \int_\Omega \int_0^R f\, dydxds \right).
\end{aligned}$$

Using the Gronwall lemma we conclude that the first assertion of Corollary 3.7 holds true. This assertion with $R = 1$ together with (3.13) yield (3.15). □

Lemma 3.8 *Let $\Phi \in W^{1,\infty}_{loc}([0,+\infty))$ be a non-negative convex and non-decreasing function such that*

$$\begin{cases} \Phi(r) = 0 & \text{for} \quad r \in [0,4A], \\ \Phi^*(\Phi'(r)) \le \kappa\; (\Phi(r) + r) & \text{for} \quad r \ge 0, \end{cases} \tag{3.16}$$

where κ and A are positive constants. Here and below Φ^* denotes the conjugate function of Φ, that is,

$$\Phi^*(r) = \sup_{s\in[0,+\infty)} \left(r\; s - \Phi(s) \right), \quad r \ge 0.$$

Then

$$\begin{aligned}\int_\Omega \int_0^\infty Q_4(f)\ \Phi'(f)\ dydx &\leq \frac{1}{4} \int_\Omega \int_0^\infty \int_0^\infty a(y,y')\ f\ f'\ \Phi'(f)\ dy'dydx \\ &+ C\ (1+\kappa) \int_\Omega \int_0^\infty (\Phi(f)+f)\ dydx.\end{aligned}$$

Proof. By (2.22) we have

$$\begin{aligned}\int_\Omega \int_0^\infty Q_4(f)\ \Phi'(f)\ dydx &= \int_\Omega \int_0^\infty \int_y^\infty b(y,y'-y)\ f'\ \Phi'(f)\ dy'dydx \\ &\leq A \int_\Omega \int_0^\infty \int_y^\infty a(y,y')\ f'\ \Phi'(f)\ dy'dydx \\ &+ \int_\Omega \int_0^\infty \int_y^\infty B(y')\ f'\ \Phi'(f)\ dy'dydx.\end{aligned}$$

On the one hand it follows from (3.16) that $4A\ \Phi'(r) \leq r\ \Phi'(r)$ and thus

$$\begin{aligned}& A \int_\Omega \int_0^\infty \int_y^\infty a(y,y')\ f'\ \Phi'(f)\ dy'dydx \\ \leq\ & \frac{1}{4} \int_\Omega \int_0^\infty \int_0^\infty a(y,y')\ f\ f'\ \Phi'(f)\ dy'dydx.\end{aligned}$$

On the other hand the Young inequality, (3.16) and (2.21) entail

$$\begin{aligned}& \int_\Omega \int_0^\infty \int_y^\infty B(y')\ f'\ \Phi'(f)\ dy'dydx \\ \leq\ & \int_\Omega \int_0^\infty \int_y^\infty B(y')\ \left(\Phi(f')+\Phi^*(\Phi'(f))\right)\ dy'dydx \\ \leq\ & \sup_{y\geq 0}\{y\ B(y)\} \int_\Omega \int_0^\infty \Phi(f)\ dydx \\ +\ & \kappa \int_\Omega \int_0^\infty \int_y^\infty B(y')\ (\Phi(f)+f)\ dy'dydx \\ \leq\ & C \int_\Omega \int_0^\infty \Phi(f)\ dydx \\ +\ & \kappa\ |B|_{L^1(0,+\infty)} \int_\Omega \int_0^\infty (\Phi(f)+f)\ dydx.\end{aligned}$$

Combining the above three inequalities yields Lemma 3.8. □

Corollary 3.9 *There is a non-negative and convex function $\Phi \in W^{1,\infty}_{loc}([0,+\infty))$ depending only on f^{in} and A and satisfying (3.14), (3.16) and*

$$\lim_{r\to+\infty} \Phi'(r) = \lim_{r\to+\infty} \frac{\Phi(r)}{r} = +\infty, \tag{3.17}$$

such that

$$\sup_{t\in[0,T]} \int_\Omega \int_0^\infty \Phi(f(t))\, dydx \leq C(T, M_1, \omega_1, f^{in}), \tag{3.18}$$

$$\int_0^T \int_\Omega \int_0^\infty \int_0^\infty a(y,y')\, f\, f'\, \Phi'(f)\, dy'dydxdt \leq C(T, M_1, \omega_1, f^{in}) \tag{3.19}$$

for every $T \in \mathbb{R}_+$.

Proof of Corollary 3.9. Since f^{in} belongs to $L^1(\mathcal{U})$ we first recall that a refined version of the de la Vallée-Poussin theorem [27, Proposition I.1.1] guarantees that there exists a non-negative and convex function $\Phi_0 \in \mathcal{C}^1([0,+\infty))$ satisfying

$$\begin{cases} \Phi_0(0) = 0, \;\; \Phi_0'(0) \geq 0 \;\text{ and }\; \Phi_0' \;\text{ is concave on }\; [0,+\infty), \\ \\ \displaystyle\lim_{r\to+\infty} \Phi_0'(r) = \lim_{r\to+\infty} \frac{\Phi_0(r)}{r} = +\infty, \end{cases} \tag{3.20}$$

and $\Phi_0(f^{in}) \in L^1(\mathcal{U})$. It then follows from the properties enjoyed by Φ_0 that

$$0 \leq \Phi_0(r) \leq r\ \Phi_0'(r) \leq 2\ \Phi_0(r), \quad r \geq 0,$$

(see Lemma B.1). In addition, denoting by $r_+ := \max\{r, 0\}$ the positive part of the real number r, the function Φ defined by

$$\Phi(r) = \left(\Phi_0(r) - \Phi_0(4A)\right)_+, \quad r \geq 0,$$

is a non-negative and non-decreasing convex function which is bounded from above by Φ_0 and satisfies (3.14) and (3.16) with $\kappa = 1 + (\Phi_0(4A)/2A)$ (see Lemma B.2).

Let $t \in [0,T]$. We now multiply (1.2) by $\Phi'(f)$, integrate over $(0,t)\times\Omega\times\mathbb{R}_+$ and use Lemma 3.5 and Lemma 3.8 to obtain

$$\begin{aligned} &\int_\Omega \int_0^\infty \Phi(f(t))\ dydx + \int_0^t \int_\Omega \int_0^\infty d(y)\ \Phi''(f)\ |\nabla_x f|^2\ dydxds \\ &= \int_\Omega \int_0^\infty \Phi(f^{in})\ dydx + \int_0^t \int_\Omega \int_0^\infty Q(f)\ \Phi'(f)\ dydxds, \end{aligned}$$

$$\begin{aligned} \int_\Omega \int_0^\infty Q(f)\ \Phi'(f)\ dydx \quad &\leq \quad -\frac{1}{2} \int_\Omega \int_0^\infty \int_0^\infty a(y,y')\, f\, \Phi'(f)\, f'\ dy'dydx \\ &+ \quad \frac{1}{4} \int_\Omega \int_0^\infty \int_0^\infty a(y,y')\ f\ f'\ \Phi'(f)\ dy'dydx \\ &+ \quad C\ (1+\kappa) \int_\Omega \int_0^\infty (\Phi(f) + f)\ dydx. \end{aligned}$$

Recall that the term involving $Q_2(f)$ has a non-positive contribution since f is non-negative and Φ is non-decreasing. Thanks to (3.15) and the convexity of Φ we end up with

$$\begin{aligned} \int_\Omega \int_0^\infty \Phi(f(t))\ dydx \quad &\leq \quad \int_\Omega \int_0^\infty \Phi_0(f^{in})\ dydx \\ &- \quad \frac{1}{4} \int_0^t \int_\Omega \int_0^\infty \int_0^\infty a(y,y')\, f\, \Phi'(f)\, f'\ dy'dydxds \\ &+ \quad C(T, M_1, \omega_1, \Phi_0) \left(1 + \int_0^t \int_\Omega \int_0^\infty \Phi(f)\ dydxds\right), \end{aligned}$$

and Corollary 3.9 follows at once from the Gronwall lemma. □

We now show that a sequence (f_n) of non-negative functions satisfying the bounds (3.18) and (3.19) uniformly with respect to $n \geq 1$ enjoys the desired compactness properties.

Lemma 3.10 *Let $T \in \mathbb{R}_+$ and (f_n) be a sequence of non-negative functions such that there are a non-negative and convex function $\Phi \in W^{1,\infty}_{loc}([0,+\infty))$ satisfying (3.14), (3.16) and (3.17) and a constant K_T such that there holds*

$$\sup_{t\in[0,T]} \int_\Omega \int_0^\infty ((1+y)\ f_n(t) + \Phi(f_n(t)))\ dydx \leq K_T, \tag{3.21}$$

$$\int_0^T \int_\Omega \int_0^\infty \int_0^\infty a(y,y')\, f_n\, f_n'\, \Phi'(f_n)\, dy'dydxdt \le K_T \tag{3.22}$$

for any $n \ge 1$. Then the sequence (f_n) is weakly compact in $L^1(\mathcal{U}_T)$ and the sequence $(Q_i(f_n))$ is weakly compact in $L^1(\mathcal{U}_{T,R})$ for each $i \in \{1,\ldots,4\}$ and $R \in \mathbb{R}_+$. Furthermore there is a weakly compact subset $\mathcal{K}_w$ of $L^1(\mathcal{U})$ such that $f_n(t) \in \mathcal{K}_w$ for each $t \in [0,T]$ and $n \ge 1$.

Proof. Observe first that the last assertion of Lemma 3.10 follows from (3.21) and the Dunford-Pettis theorem since Φ satisfies (3.17).

We next fix $R \in \mathbb{R}_+$. By (2.2) we have

$$Q_2(f_n) \le R\ M_R\ f_n,$$

and the weak compactness of (f_n) in $L^1(\mathcal{U}_{T,R})$ entails that of $(Q_2(f_n))$.

We next consider a measurable subset E of $\mathcal{U}_{T,R}$. By (2.2) and the convexity of Φ we have for α large enough

$$\begin{aligned}
& \int_E Q_1(f_n)\ dydxdt \\
\le\ & \int_0^T \int_\Omega \int_0^R \int_0^{R-y'} \mathbf{1}_E''\ a(y,y')\ f_n\ f_n'\ dydy'dxdt \\
\le\ & \alpha^2 \int_0^T \int_\Omega \int_0^R \int_0^R \mathbf{1}_E''\ a(y,y')\ \mathbf{1}_{\{f_n\le\alpha\}}\ \mathbf{1}_{\{f_n\le\alpha\}}'\ dydy'dxdt \\
+\ & \int_0^T \int_\Omega \int_0^R \int_0^R \mathbf{1}_E''\ a(y,y')\ \mathbf{1}_{\{f_n\le\alpha\}}\ \mathbf{1}_{\{f_n\ge\alpha\}}'\ f_n\ f_n'\ dydy'dxdt \\
+\ & \int_0^T \int_\Omega \int_0^R \int_0^R \mathbf{1}_E''\ a(y,y')\ \mathbf{1}_{\{f_n\ge\alpha\}}\ f_n\ f_n'\ dydy'dxdt \\
\le\ & \alpha^2\ R\ M_R\ |E| \\
+\ & \frac{1}{\Phi'(\alpha)} \int_0^T \int_\Omega \int_0^R \int_0^R a(y,y')\ f_n\ f_n'\ \Phi'(f_n)\ dydy'dxdt \\
+\ & \frac{1}{\Phi'(\alpha)} \int_0^T \int_\Omega \int_0^R \int_0^R a(y,y')\ f_n\ \Phi'(f_n)\ f_n'\ dydy'dxdt.
\end{aligned}$$

We now use (3.22) to deduce

$$\int_E Q_1(f_n)\ dydxdt \le \alpha^2\ R\ M_R\ |E| + \frac{2\ K_T}{\Phi'(\alpha)}.$$

Therefore $(Q_1(f_n))$ is bounded in $L^1(\mathcal{U}_{T,R})$ and

$$0 \le \limsup_{|E|\to 0}\ \sup_{n\ge 1} \int_E Q_1(f_n)\ dydxdt \le \frac{2\ K_T}{\Phi'(\alpha)}$$

for any α large enough. Since Φ' fulfils (3.17) we may let $\alpha \to +\infty$ in the above inequality and obtain

$$\lim_{|E|\to 0}\ \sup_{n\ge 1} \int_E Q_1(f_n)\ dydxdt = 0,$$

whence the weak compactness of $(Q_1(f_n))$ in $L^1(\mathcal{U}_{T,R})$.

As for $Q_3(f_n)$, we have for α large enough

$$\begin{aligned}
& \int_E Q_3(f_n)\ dydxdt \\
\le\ & \alpha \int_{E\cap\{f_n\le\alpha\}} \int_0^\infty a(y,y')\ f_n'\ dy'dydxdt \\
+\ & \frac{1}{\Phi'(\alpha)} \int_{E\cap\{f_n\ge\alpha\}} \int_0^\infty a(y,y')\ f_n'\ f_n\ \Phi'(\alpha)\ dy'dydxdt.
\end{aligned}$$

On the one hand we infer from (2.2) and (3.21) that

$$\int_{E\cap\{f_n\le\alpha\}}\int_0^\infty a(y,y')\ f_n'\ dy'dydxdt \le |E|\ |\omega_R|_{L^\infty}\ K_T.$$

On the other hand (3.22) and the convexity of Φ entail

$$\begin{aligned}&\int_{E\cap\{f_n\ge\alpha\}}\int_0^\infty a(y,y')\ f_n'\ f_n\ \Phi'(\alpha)\ dy'dydxdt\\ \le\ &\int_0^T\int_\Omega\int_0^\infty\int_0^\infty a(y,y')\ f_n\ \Phi'(f_n)\ f_n'\ dy'dydxdt \le K_T.\end{aligned}$$

Combining the above inequalities yields

$$\int_E Q_3(f_n)\ dydxdt \le \alpha\ |E|\ |\omega_R|_{L^\infty}\ K_T + \frac{K_T}{\Phi'(\alpha)},$$

and we argue as for $(Q_1(f_n))$ to prove the weak compactness of $(Q_3(f_n))$ in $L^1(\mathcal{U}_{T,R})$.

Finally the weak compactness of $(Q_4(f_n))$ in $L^1(\mathcal{U}_{T,R})$ follows from (2.2), (3.21) and the weak compactness of (f_n) as in the proof of Lemma 3.4. $\square$

4 Weak stability

In this section we study the convergence of sequences (f_n) of solutions to (1.2)-(**??**) such that both (f_n) and $(Q(f_n))$ enjoy weak compactness properties in L^1. In view of applying such a result to the proof of the existence of weak solutions to (1.2)-(**??**) we actually consider a more general framework in which the kinetic and diffusion coefficients also vary. More precisely, we are given sequences of coagulation and fragmentation coefficients $(a_n)_{n\ge1}$ and $(b_n)_{n\ge1}$ (which may now also depend on t and x) and a sequence of diffusion coefficients $(d_n)_{n\ge1}$ satisfying

$$\left\{\begin{array}{l} a_n(t,x,y,y') = a_n(t,x,y',y) \ge 0,\\ \\ b_n(t,x,y,y') = b_n(t,x,y',y) \ge 0,\end{array}\right. \tag{4.1}$$

and

(4.2) $\begin{cases}\text{the kinetic coefficients } a_n \text{ and } b_n \text{ and the diffusion coefficient } d_n \text{ satisfy}\\ \text{(2.2) and (2.1), respectively, uniformly with respect to } n\ge1 \text{ and } (t,x)\in\\ \mathbb{R}_+\times\Omega,\end{cases}$

(4.3) (a_n,b_n,d_n) converges to (a,b,d) a.e. in $\mathbb{R}_+\times\Omega\times\mathbb{R}_+$.

We define $Q_n(f)$, $Q_{i,n}(f)$, $1\le i\le4$, and $L_n(f)$, as $Q(f)$, $Q_i(f)$, $1\le i\le 4$, and $L(f)$, respectively, with a_n, b_n instead of a, b. The weak stability result then reads as follows.

Theorem 4.1 *Let $T\in(0,+\infty)$ and, for each $n\ge1$, let f_n be a weak solution to (1.2)-(**??**) on $[0,T)$ with kinetic coefficients a_n, b_n, diffusion coefficient d_n, and initial datum $f_n(0)$. Assume further that there are a constant C_T and a weakly compact subset $\mathcal{K}_w$ of $L^1(\Omega\times\mathbb{R}_+)$ such that, for each $n\ge1$, there holds*

$$\sup_{t\in[0,T)}\int_\Omega\int_0^\infty f_n(t)\ (1+y)\ dydx \le C_T, \tag{4.4}$$

$$f_n(t)\in\mathcal{K}_w \quad \textit{for each} \quad t\in[0,T), \tag{4.5}$$

and

$$(Q_{i,n}(f_n)) \quad \textit{is weakly compact in} \quad L^1((0,T)\times\Omega\times(0,R)) \tag{4.6}$$

for each $R\ge0$ and $i\in\{1,\dots,4\}$.

Under the assumptions (4.1)-(4.3) there are a subsequence (f_{n_k}) of (f_n) and a function f such that

(4.7)
$$\begin{cases} f_{n_k} \longrightarrow f \quad in \quad \mathcal{C}([0,T]; w - L^1(\Omega \times \mathbb{R}_+)), \\ \\ Q_{i,n_k}(f_{n_k}) \rightharpoonup Q_i(f) \quad weakly\ in \quad L^1((0,T) \times \Omega \times (0,R)) \end{cases}$$

for $R \in \mathbb{R}_+$ and $i \in \{1, \ldots, 4\}$. Consequently

(4.8) $\quad f$ *is a weak solution to (1.2)-(**??**) on $[0,T)$.*

Furthermore there holds

(4.9)
$$\lim_{k \to +\infty} \int_0^T \int_\Omega \left| \int_0^\infty \psi(y)\ (f_{n_k} - f)\ dy \right|\ dxdt = 0$$

for $\psi \in \mathcal{D}(\mathbb{R}_+)$ which implies

(4.10)
$$\begin{cases} L_{n_k}(f_{n_k}) \longrightarrow L(f) \quad strongly\ in \quad L^1((0,T) \times \Omega \times (0,R)), \\ \\ Q_{4,n_k}(f_{n_k}) \longrightarrow Q_4(f) \quad strongly\ in \quad L^1((0,T) \times \Omega \times (0,R)) \end{cases}$$

for every $R \in \mathbb{R}_+$.

Observe that Theorem 4.1 applies in particular to the case where $a_n = a$, $b_n = b$ and $d_n = d$, and thus includes Theorem 2.2 as a particular case.
Proof of Theorem 4.1. We put $\mathcal{U} := \Omega \times \mathbb{R}_+$, $\mathcal{U}_t := (0,t) \times \mathcal{U}$ and $\mathcal{U}_{t,R} := (0,t) \times \Omega \times (0,R)$ for $t \in [0,T]$ and $R \in \mathbb{R}_+$. We first observe that (4.5), (4.6) and (1.2) allow us to improve the compactness properties of (f_n).

Lemma 4.2 *The family $\{f_n\ ,\ n \geq 1\}$ is weakly equicontinuous in $L^1(\mathcal{U})$ at each point $t_0 \in [0,T)$.*

Proof. Consider $t_0 \in [0,T)$ and $\varphi \in \mathcal{C}^2(\bar{\Omega} \times \mathbb{R}_+)$ with compact support in $\bar{\Omega} \times [1/R, R]$ for some $R \geq 1$. It follows from Definition 2.1 that, for $t \in [0,T)$,

$$\begin{aligned} & \left| \int_{\mathcal{U}} (f_n(t) - f_n(t_0))\ \varphi\ dydx \right| \\ \leq\ & \int_{t_0}^t \int_\Omega \int_{1/R}^R f_n\ dydxds\ \ |d_n \Delta_x \varphi|_{L^\infty(\mathcal{U})} \\ & + |\varphi|_{L^\infty(\mathcal{U})} \int_{t_0}^t \int_\Omega \int_{1/R}^R |Q_n(f_n)|\ dydxds. \end{aligned}$$

Since d_n is uniformly bounded on the support of φ by (4.2) we obtain for $\alpha > 1$,

$$\begin{aligned} & \left| \int_{\mathcal{U}} (f_n(t) - f_n(t_0))\ \varphi\ dydx \right| \\ \leq\ & C_T(\varphi)\ \left(\alpha\ |t - t_0| + \sup_{n \geq 1} \int \mathbf{1}_{\{f_n > \alpha\}}\ f_n\ dydxds \right) \\ & + C_T(\varphi)\ \sup_{n \geq 1} \int \mathbf{1}_{\{|Q_n(f_n)| > \alpha\}}\ |Q_n(f_n)|\ dydxds. \end{aligned}$$

We first take the lim sup as $t \to t_0$ in the above inequality and next use the weak compactness in $L^1((0,T) \times \Omega \times (1/R, R))$ of (f_n) and $(Q_n(f_n))$ given by (4.5) and (4.6) to let $\alpha \to +\infty$ and obtain that

(4.11)
$$\lim_{t \to t_0}\ \sup_{n \geq 1} \left| \int_\Omega \int_0^\infty (f_n(t) - f_n(t_0))\ \varphi\ dydx \right| = 0.$$

We next use (4.5) and the fact that any function $\varphi \in L^\infty(\mathcal{U})$ is an almost everywhere limit of smooth functions with compact support to conclude that (4.11) holds true for each $\varphi \in L^\infty(\mathcal{U})$. The proof of Lemma 4.2 is thus complete. $\square$

Owing to (4.5), Lemma 4.2 and the non-negativity of f_n, it follows from a variant of the Arzelà-Ascoli theorem (see, e.g., [39, Theorem 1.3.2]) that there are a subsequence of (f_n) (not relabeled) and a non-negative function $f \in \mathcal{C}([0,T); w - L^1(\mathcal{U}))$ such that

$$f_n \longrightarrow f \quad \text{in } \mathcal{C}([0,T); w - L^1(\mathcal{U})). \tag{4.12}$$

Furthermore, we infer from (4.6) that, extracting a further subsequence if necessary, we may also assume that, for each $i \in \{1, \ldots, 4\}$, there is a function $\overline{Q}_i$ in $L^1_{\text{loc}}(\mathcal{U}_T)$ such that

$$Q_{i,n}(f_n) \rightharpoonup \overline{Q}_i \quad \text{in } L^1(\mathcal{U}_{T,R}) \tag{4.13}$$

for each $R \geq 1$.

The remainder of the proof of Theorem 4.1 is then devoted to the identification of $\overline{Q}_i$ for $i \in \{1, \ldots, 4\}$ in terms of f. As we aim at passing to the limit in the coagulation terms involving quadratic terms, weak convergences are not sufficient for our purpose and we need some strong compactness properties of the sequence (f_n). As the Laplace operator only acts upon the variable x it is expected that its compactness properties will only be effective upon the variables (t, x) : we are thus led to study the averages of f_n with respect to the volume variable y. More precisely we have the following result.

Proposition 4.3 *For each $\psi \in \mathcal{D}(\mathbb{R}_+)$ there holds*

$$\int_0^\infty f_n \ \psi(y) \ dy \longrightarrow \int_0^\infty f \ \psi(y) \ dy \quad \text{in } L^1((0,T) \times \Omega). \tag{4.14}$$

Proof. Owing to (4.12), (4.6) and (4.2) we are in a position to apply Proposition 5.5 below (with $J =$ supp ψ) from which Proposition 4.3 readily follows. $\square$

We gather in the next proposition some useful consequences of (4.12) and Proposition 4.3.

Corollary 4.4 *(i) If ψ is a measurable function in $\mathcal{U}_T$ satisfying*

$$|\psi(t,x,y)| \leq \omega(y) \ (1+y) \quad \text{a.e. in } \mathcal{U}_T \tag{4.15}$$

for some positive function ω in $L^\infty_0(\mathbb{R}_+)$, there holds

$$\lim_{n \to +\infty} \int_0^T \int_\Omega \left| \int_0^\infty (f_n - f) \ \psi \ dy \right| \ dxdt = 0. \tag{4.16}$$

(ii) If $(\psi_n)_{n \geq 1}$ and ψ are measurable functions in $\mathcal{U}_T$ satisfying

$$\lim_{n \to +\infty} \psi_n(t,x,y) = \psi(t,x,y) \quad \text{a.e. in } \mathcal{U}_T, \tag{4.17}$$

$$|\psi_n(t,x,y)| \leq \omega(y) \ (1+y) \quad \text{a.e. in } \mathcal{U}_T \tag{4.18}$$

for each $n \geq 1$ and some positive function ω in $L^\infty_0(\mathbb{R}_+)$, there holds

$$\lim_{n \to +\infty} \int_0^T \int_\Omega \left| \int_0^\infty (\psi_n \ f_n - \psi \ f) \ dy \right| \ dxdt = 0. \tag{4.19}$$

(iii) If $(\psi_n)_{n \geq 1}$ and ψ are measurable functions in $(0,T) \times \Omega \times \mathbb{R}^2_+$ satisfying

$$\lim_{n \to +\infty} \psi_n(t,x,y,y') = \psi(t,x,y,y') \quad \text{a.e. in } (0,T) \times \Omega \times \mathbb{R}^2_+, \tag{4.20}$$

$$|\psi_n(t,x,y,y')| \leq \omega(y') \ (1+y') \quad \text{a.e. in } (0,T) \times \Omega \times \mathbb{R}^2_+ \tag{4.21}$$

for each $n \geq 1$ and some positive function ω in $L_0^\infty(\mathbb{R}_+)$, there holds

$$\lim_{n\to+\infty} \int_0^T \int_\Omega \left| \int_0^\infty \int_0^\infty (\psi_n\ F_n\ f_n' - \psi\ F\ f')\ dy'dy \right|\ dxdt = 0, \tag{4.22}$$

where

$$F_n(t,x,y) = \frac{f_n(t,x,y)}{1+\varrho_n(t,x)} \quad \textit{with} \quad \varrho_n(t,x) = \int_0^\infty f_n(t,x,y)\ dy,$$
$$F(t,x,y) = \frac{f(t,x,y)}{1+\varrho(t,x)} \quad \textit{with} \quad \varrho(t,x) = \int_0^\infty f(t,x,y)\ dy,$$

In order not to delay further the proof of Theorem 4.1 we postpone the proof of Corollary 4.4 to the Appendix. We just mention here that it relies on arguments similar to those employed in [14, Section IV] and go on with the proof of Theorem 4.1.

We first check that (4.10) holds true. Indeed consider $R \in \mathbb{R}_+$ and $y \in (0,R)$. On the one hand it follows from (4.2), (4.4) and (4.12) that

$$\begin{aligned}\int_0^T \int_\Omega \int_0^\infty a_n(t,x,y,y')\ f_n'\ dy'dxdt &\leq |\omega_R|_{L^\infty(0,+\infty)} \int_0^T \int_\Omega \int_0^\infty (1+y')\ f_n'\ dy'dxdt \\ &\leq |\omega_R|_{L^\infty(0,+\infty)}\ C_T,\end{aligned}$$

and the right-hand side of the above inequality clearly belongs to $L^1(0,R)$. On the other hand, using again (4.2), we deduce from Corollary 4.4 (ii) (with $\psi_n = a_n$) that

$$\lim_{n\to+\infty} \int_0^T \int_\Omega \left| \int_0^\infty (a_n(t,x,y,y')\ f_n' - a(y,y')\ f')\ dy' \right|\ dxdt = 0.$$

We may then apply the Lebesgue dominated convergence theorem to conclude that $(L_n(f_n))$ converges strongly towards $L(f)$ in $L^1(\mathcal{U}_{T,R})$ as $n \to +\infty$. Next, since

$$Q_{4,n}(f_n) = \int_0^\infty \mathbf{1}_{[y,+\infty)}(y')\ b_n(t,x,y,y'-y)\ f_n'\ dy',$$

a similar argument yields the strong convergence in $L^1(\mathcal{U}_{T,R})$ of $(Q_{4,n}(f_n))$ towards $Q_4(f)$ as $n \to +\infty$ and the proof of (4.10) is complete. In particular we have proved that $\overline{Q}_4 = Q_4(f)$.

It remains to identify $\overline{Q}_i$ for $i \in \{1,2,3\}$. By Corollary 4.4 (i) we may extract a further subsequence of (f_n) (not relabeled) such that

$$\lim_{n\to+\infty} \varrho_n(t,x) = \varrho(t,x) \ \text{ a.e. in } \ (0,T)\times\Omega, \tag{4.23}$$

where the notation ϱ_n and ϱ have been introduced in Corollary 4.4. We consider $\psi \in L^\infty(\mathcal{U}_T)$ with compact support in $\overline{\mathcal{U}_{T,R}}$ for some $R > 1$. We begin with the fragmentation term $\overline{Q}_2$ and put

$$\psi_n(t,x,y) = \psi(t,x,y) \int_0^y b_n(t,x,y',y-y')\ dy'.$$

Observe that

$$\int_{\mathcal{U}_T} \psi\ Q_{2,n}(f_n)\ dydxdt = \int_{\mathcal{U}_T} \psi_n\ f_n\ dydxdt.$$

Then, on the one hand, (4.13) and the compactness of the support of ψ yield

$$\lim_{n\to+\infty} \int_{\mathcal{U}_T} \psi\ Q_{2,n}(f_n)\ dydxdt = \int_{\mathcal{U}_T} \psi\ \overline{Q}_2\ dydxdt.$$

On the other hand we infer from (4.2) and (4.3) that

$$|\psi_n(t,x,y)| \le R\ M_R\ |\psi|_{L^\infty},$$

and

$$\lim_{n\to+\infty} \psi_n(t,x,y) = \psi(t,x,y)\ \int_0^y b(y',y-y')\ dy' \quad \text{a.e.}$$

We then infer from Corollary 4.4 (ii) that

$$\lim_{n\to+\infty} \int_{\mathcal{U}_T} \psi_n\ f_n\ dydxdt = \int_{\mathcal{U}_T} \psi\ f\ \int_0^y b(y',y-y')\ dy'\ dydxdt,$$

hence

$$\int_{\mathcal{U}_T} \psi\ \overline{Q}_2\ dydxdt = \int_0^T \int_\Omega \int_0^\infty \psi\ Q_2(f)\ dydxdt. \tag{4.24}$$

We next consider the coagulation term $\overline{Q}_3$. We put

$$\psi_n(t,x,y,y') = a_n(t,x,y,y')\ \psi(t,x,y),$$

and notice that

$$\int_{\mathcal{U}_T} \psi\ \frac{Q_{3,n}(f_n)}{1+\varrho_n}\ dydxdt = \int_0^T \int_\Omega \int_0^\infty \int_0^\infty \psi_n\ F_n\ f_n'\ dy'dydxdt,$$

where F_n and ϱ_n are defined in Corollary 4.4. Owing to (4.13), (4.23) and the compactness of the support of ψ we deduce from Lemma A.2 that

$$\lim_{n\to+\infty} \int_{\mathcal{U}_T} \psi\ \frac{Q_{3,n}(f_n)}{1+\varrho_n}\ dydxdt = \int_{\mathcal{U}_T} \psi\ \frac{\overline{Q}_3}{1+\varrho}\ dydxdt.$$

Using again the compactness of the support of ψ together with (4.2) and (4.3) we obtain that

$$\lim_{n\to+\infty} \psi_n(t,x,y,y') = a(y,y')\ \psi(t,x,y) \quad \text{a.e.}$$

$$|\psi_n(t,x,y,y')| \le |\psi|_{L^\infty}\ \omega_R(y')\ (1+y').$$

Therefore, it follows from Corollary 4.4 (iii) that

$$\lim_{n\to+\infty} \int_0^T \int_\Omega \int_0^\infty \int_0^\infty \psi_n\ F_n\ f_n' = \int_0^T \int_\Omega \int_0^\infty \int_0^\infty a(y,y')\ \psi\ F\ f'.$$

The above analysis entails

$$\int_{\mathcal{U}_T} \psi\ \frac{\overline{Q}_3}{1+\varrho}\ dydxdt = \int_{\mathcal{U}_T} \psi\ \frac{Q_3(f)}{1+\varrho}\ dydxdt. \tag{4.25}$$

We finally consider $\overline{Q}_1$. We put

$$\psi_n(t,x,y,y') = a_n(t,x,y,y')\ \psi(t,x,y+y'),$$

so that

$$\int_{\mathcal{U}_T} \psi\ \frac{Q_{1,n}(f_n)}{1+\varrho_n}\ dydxdt = \int_0^T \int_\Omega \int_0^\infty \int_0^\infty \psi_n\ F_n\ f_n'\ dy'dydxdt.$$

We then argue as for $\overline{Q}_3$ to conclude that

$$\int_{\mathcal{U}_T} \psi\ \frac{\overline{Q}_1}{1+\varrho}\ dydxdt = \int_{\mathcal{U}_T} \psi\ \frac{Q_1(f)}{1+\varrho}\ dydxdt. \tag{4.26}$$

Since (4.24)-(4.26) are valid for every ψ in $L^\infty(\mathcal{U}_T)$ with compact support in $\overline{\mathcal{U}_T}$ we end up with

$$\overline{Q}_i = Q_i(f) \quad \text{a.e. for } i \in \{1,\ldots,4\}. \tag{4.27}$$

Now, owing to (4.12), (4.13) and (4.27) it is straightforward to pass to the limit in the equation satisfied by f_n. We thus obtain that f satisfies (2.5) with initial datum $f(0)$ and the proof of Theorem 4.1 is complete. ◻

5 Existence

This section is devoted to the proof of Theorems 2.3 and 2.6 and is divided into three parts. We first study the partially diffusive heat equation $\partial_t u - \nu(y)\ \Delta_x u = g$ with homogeneous Neumann boundary conditions and establish existence and uniqueness of solutions in a L^1-framework, together with the L^1-compactness of volume averages. The second part is devoted to the construction of approximations of (1.2)-(**??**), the solutions of which enjoy the properties required to apply Theorem 4.1. The convergence of these approximating solutions towards a solution of (1.2)-(**??**) is performed in the last part which completes the proofs of Theorems 2.3 and 2.6.

5.1 A partially diffusive heat equation

Let J be a non-empty and bounded interval of $\mathbb{R}_+$ and consider a non-negative function $\nu \in L^\infty(J)$ satisfying

$$0 < m_1 \le \nu(y) \le m_2 \quad \text{a.e. in } J \tag{5.1}$$

for some positive real numbers m_1 and m_2. Here and below we put $U := \Omega \times J$ and $Q_t := (0,t) \times U$ for $t \in \mathbb{R}_+$.

Proposition 5.1 *Let $T \in \mathbb{R}_+$ and consider $u^{in} \in L^1(U)$ and $g \in L^1(Q_T)$. There is a unique function*

$$u \in L^\infty(0,T;L^1(U)) \cap L^1((0,T)\times J; W^{1,1}(\Omega)) \tag{5.2}$$

satisfying $u(0) = u^{in}$ and

$$\begin{aligned} &\int_0^T \int_\Omega \int_J \left(\nu(y)\ \nabla_x u . \nabla_x \varphi - u\ \partial_t \varphi\right)\ dydxdt \\ &= \int_U u^{in}\ \varphi(0)\ dydx + \int_0^T \int_\Omega \int_J g\ \varphi\ dydxdt \end{aligned} \tag{5.3}$$

for each $\varphi \in \mathcal{C}^1([0,T]\times \bar{U})$ with $\varphi(T) = 0$. In addition, $u \in \mathcal{C}([0,T];L^1(U))$ with

$$|u(t)|_{L^1(U)} \le |u^{in}|_{L^1(U)} + |g|_{L^1(Q_t)}, \quad t \in [0,T], \tag{5.4}$$

and there is a constant C_T depending only on Ω, J, T, m_1, $|u^{in}|_{L^1(U)}$ and $|g|_{L^1(Q_T)}$ such that

$$|u|_{L^1((0,T)\times J; W^{1,1}(\Omega))} \le C_T. \tag{5.5}$$

Remark 5.2 *Owing to the regularity (5.2) of u the formula (5.3) is also valid for functions $\varphi \in L^\infty(Q_T)$ with $\nabla_x \varphi \in L^\infty(Q_T)$, $\partial_t \varphi \in L^\infty(Q_T)$ and $\varphi(T) = 0$.*

As a first step towards the proof of Proposition 5.1 we consider the case of L^2-data.

Lemma 5.3 *Consider $u^{in} \in L^2(U)$ and $g \in L^2(Q_T)$. There is a unique function*

$$\left\{ \begin{array}{l} u \in \mathcal{C}([0,T];L^2(U)) \cap L^2((0,T)\times J; H^1(\Omega)), \\ \\ \qquad \partial_t u \in L^2((0,T)\times J; H^1(\Omega)'), \end{array} \right. \tag{5.6}$$

satisfying $u(0) = u^{in}$ and

$$\begin{aligned} &\int_0^t \int_J \langle \partial_t u, \varphi \rangle\ dyds + \int_0^t \int_U \nu(y)\ \nabla_x u . \nabla_x \varphi\ dydxds \\ &= \int_0^t \int_U g\ \varphi\ dydxds \end{aligned} \tag{5.7}$$

for each $t \in (0,T)$ and $\varphi \in L^2((0,T)\times J; H^1(\Omega))$. Here $H^1(\Omega)'$ denotes the dual space of $H^1(\Omega)$ and $\langle .,. \rangle$ the duality pairing between $H^1(\Omega)$ and $H^1(\Omega)'$. Moreover, for $t \in [0,T]$, there holds

$$|u(t)|_{L^2(U)} \leq |u^{in}|_{L^2(U)} + \int_0^t |g(s)|_{L^2(U)}\ ds. \tag{5.8}$$

Proof. Observe first that, if u is a function satisfying (5.6) and (5.7), we may take $\varphi = u$ in (5.7) and obtain

$$\begin{aligned} |u(t)|_{L^2}^2 \quad + \quad & 2 \int_0^t \int_U \nu(y)\ |\nabla_x u|^2\ dydxds \\ = \quad & |u^{in}|_{L^2}^2 + 2 \int_0^t \int_U |g(s)|_{L^2(U)}\ |u(s)|_{L^2(U)}\ ds \end{aligned}$$

for $t \in [0,T]$. As (5.7) is a linear equation the uniqueness part of Lemma 5.3 and (5.8) follow at once from the above inequality and a variant of the Gronwall lemma [7, Lemma A.5].

In order to prove the existence part of Lemma 5.3 we consider the following regularised problem

$$\begin{aligned} \partial_t u^\varepsilon - \nu(y)\ \Delta_x u^\varepsilon - \varepsilon\ \partial_y^2 u^\varepsilon &= g \quad \text{in} \quad (0,T)\times\Omega\times J, \\ \partial_n u^\varepsilon &= 0 \quad \text{on} \quad (0,T)\times\partial\Omega\times J, \\ \partial_y u^\varepsilon &= 0 \quad \text{on} \quad (0,T)\times\Omega\times\partial J, \\ u^\varepsilon(0) &= u^{in} \quad \text{in} \quad \Omega\times J, \end{aligned}$$

for $\varepsilon \in (0,1)$. Classical arguments ensure the existence and uniqueness of a weak solution u^ε to the above initial-boundary value problem satisfying

$$u^\varepsilon \in W^{1,2}(0,T;L^2(U)) \cap L^2(0,T;H^1(U)),$$

together with the estimates

$$\sup_{t\in[0,T]} |u^\varepsilon(t)|_{L^2(U)} + \left|\nu^{1/2}\ \nabla_x u^\varepsilon\right|_{L^2(Q_T)} + \left|\varepsilon^{1/2}\ \partial_y u^\varepsilon\right|_{L^2(Q_T)} \leq K, \tag{5.9}$$

$$|\partial_t u^\varepsilon|_{L^2(0,T;H^1(U)')} \leq K,$$

where K does not depend on $\varepsilon \in (0,1)$. We are thus in a position to apply a classical Aubin lemma (see, e.g., [32, Corollary 4]) to conclude that there are a function $u \in L^2(Q_T)$ and a subsequence of (u^ε) (not relabeled) such that

$$u^\varepsilon \longrightarrow u \quad \text{weakly in} \quad L^2(Q_T) \quad \text{and strongly in} \quad \mathcal{C}([0,T];H^1(U)'). \tag{5.10}$$

We may then pass to the limit as $\varepsilon \to 0$ in the equation satisfied by u^ε and obtain that u satisfies (5.7) for $\varphi \in L^2(0,T;H^1(U))$. A density argument then yields that $\partial_t u$ has the regularity (5.6) and that (5.7) holds true. Finally u belongs to $L^\infty(0,T;L^2(U))$ by (5.9), which, together with (5.10), entail that u belongs to $\mathcal{C}([0,T];w-L^2(U))$. Since $t \mapsto |u(t)|_{L^2}$ is in $\mathcal{C}([0,T])$ as a consequence of (5.7) (with $\varphi = u$) the reflexivity of $L^2(U)$ and (5.9) actually imply that u enjoys the regularity (5.6). ☐

For further use we derive additional regularity properties of the solution to (5.6)-(5.7).

Lemma 5.4 *Under the assumptions and notations of Lemma 5.3 there holds*

$$|u(t)|_{L^1(U)} \leq |u^{in}|_{L^1(U)} + |g|_{L^1(Q_t)} \tag{5.11}$$

for $t \in [0,T]$ and there is a positive constant $C(\Omega,T,m_1)$ such that

$$|\nabla_x u|_{L^1(Q_T)} \leq C(\Omega,T,m_1)\ \left(|u^{in}|_{L^1(U)} + |g|_{L^1(Q_T)}\right). \tag{5.12}$$

Furthermore, if $u^{in} \equiv 0$ and $g \in \mathcal{D}(Q_T)$, we have

$$\left|\partial_t^i \partial_x^j u\right|_{L^\infty(Q_T)} \leq T\ \left|\partial_t^i \partial_x^j g\right|_{L^\infty(Q_T)} \tag{5.13}$$

for $i \in \mathbb{N}$ and $j \in \mathbb{N}$.

Proof. We first observe that the Fubini theorem and (5.1) entail that, for almost every $y \in J$, $\nu(y) \geq m_1$ and $u^{in}(.,y)$ and $g(.,.,y)$ belong to $L^2(\Omega)$ and $L^2((0,T)\times\Omega)$, respectively. For such y we denote by v^y the unique solution in $\mathcal{C}([0,T];L^2(\Omega))$ with $\nabla_x v^y \in L^2((0,T)\times\Omega)$ to

$$\left\{ \begin{array}{rcl} \partial_t v^y - \nu(y)\ \Delta_x v^y = g(.,.,y) & \text{in} & (0,T)\times\Omega, \\ \partial_n v^y = 0 & \text{on} & (0,T)\times\partial\Omega, \\ v^y(0,.) = u^{in}(.,y) & \text{in} & \Omega. \end{array} \right. \tag{5.14}$$

Introducing $V(t,x,y) = v^y(t,x)$ for $(t,x,y) \in Q_T$ the properties of the v^y's yield that V satisfies (5.6) and (5.7) and we infer from the uniqueness statement of Lemma 5.3 that $u(.,.,y) = V(.,.,y)$ for almost every $y \in J$. The assertion (5.11) then readily follows from the $L^1(\Omega)$-contraction property of (5.14) after a further integration with respect to $y \in J$. Next, if $u^{in} \equiv 0$ and $g \in \mathcal{D}(Q_T)$, (5.13) is a straightforward consequence of (5.14) and the maximum principle. Finally, to prove (5.12), we notice that the smoothing properties of (5.14) guarantee that there is a constant $C(\Omega,T,m_1)$ depending only on Ω, T and m_1 such that

$$|\nabla_x v^y|_{L^1((0,T)\times\Omega)} \leq C(\Omega,T,m_1)\ \left(|u^{in}(.,y)|_{L^1(\Omega)} + |g(.,.,y)|_{L^1((0,T)\times\Omega)} \right) \tag{5.15}$$

for almost every $y \in J$. The assertion (5.15) can be proved either by using the integral formulation of (5.14) and decay estimates of the linear heat semigroup or by a duality method [4, Section 3] or with the device introduced in [6, Section IV]. Integrating (5.15) with respect to $y \in J$ yields (5.12). □

Proof of Proposition 5.1. We proceed along the lines of the proof of [4, Lemmes 3.3 & 3.4].
– Existence : let $(u_k^{in})_{k\geq 1}$ and $(g_k)_{k\geq 1}$ be sequences in $L^2(U)$ and $L^2(Q_T)$, respectively, such that

$$\left(u_k^{in}, g_k\right) \longrightarrow \left(u^{in}, g\right) \quad \text{in} \ \ L^1(U)\times L^1(Q_T). \tag{5.16}$$

For $k \geq 1$ we denote by u_k the solution to (5.6)-(5.7) with data $\left(u_k^{in}, g_k\right)$. The linearity of (5.7), (5.11) and (5.12) warrant that (u_k) is a Cauchy sequence in $\mathcal{C}([0,T];L^1(U))$ and $L^1((0,T)\times J; W^{1,1}(\Omega))$ and we denote its limit by u. Clearly u enjoys the regularity properties (5.2) and (5.5). It is then straighforward to pass to the limit as $k \to +\infty$ in (5.7) and check that u satisfies (5.3).
– Uniqueness : Since (5.2)-(5.3) is a linear problem it is sufficient to prove that any function satisfying (5.2)-(5.3) with $u^{in} = g = 0$ vanishes identically. We thus consider a function u satisfying (5.2)-(5.3) with $u^{in} = g = 0$ and proceed along the lines of [4, Lemme 3.4] by a duality method. Given $\vartheta \in \mathcal{D}(Q_T)$ we define $\tilde{\vartheta}(t,x,y) = \vartheta(T-t,x,y)$ for $(t,x,y) \in Q_T$ and denote by $\tilde{\psi}$ the solution to (5.6)-(5.7) with initial datum and right-hand side $(0,\tilde{\vartheta})$. By (5.13) the function ψ defined by $\psi(t,x,y) = \tilde{\psi}(T-t,x,y)$, $(t,x,y) \in Q_T$, enjoys the required properties to be used in (5.3) (see Remark 5.2) and we obtain after some computations

$$\int_{Q_T} u\ \vartheta\ dydxdt = 0.$$

Consequently $u \equiv 0$ and the proof of Proposition 5.1 is complete. □

We next establish some compactness properties of the volume averages of sequences of solutions to (5.2)-(5.3).

Proposition 5.5 *Let $T \in \mathbb{R}_+$ and consider sequences of functions $(u_k^{in})_{k\geq 1}$ in $L^1(U)$, $(g_k)_{k\geq 1}$ in $L^1(Q_T)$ and $(\nu_k)_{k\geq 1}$ in $L^\infty(J)$. Assume that there are positive constants m_1 and m_2 such that*

$$|u_k^{in}|_{L^1(U)} + |g_k|_{L^1(Q_T)} \leq m_2, \tag{5.17}$$

$$0 < m_1 \leq \nu_k(y) \leq m_2 \quad a.e.\ in\ J \tag{5.18}$$

for each $k \geq 1$. For $k \geq 1$ we denote by u_k the solution to (5.2)-(5.3) with diffusion coefficient ν_k instead of ν, initial datum u_k^{in} and right-hand side g_k given by Proposition 5.1. Assuming moreover that there is $u \in L^1(Q_T)$ such that

$$u_k \rightharpoonup u \quad \text{in} \quad L^1(Q_T), \tag{5.19}$$

we have

$$\int_J u_k\ \varphi(y)\ dy \longrightarrow \int_J u\ \varphi(y)\ dy \quad \text{in} \quad L^1((0,T)\times\Omega) \tag{5.20}$$

for any $\varphi \in \mathcal{D}(J)$.

Proof. Owing to (5.17), (5.18), (5.4) and (5.5) there is a constant C depending only m_1, m_2, T, J and Ω such that

$$\sup_{t\in[0,T]} |u_k(t)|_{L^1(U)} + \int_0^T \int_\Omega \int_J |\nabla_x u_k|\ dydxdt \leq C \tag{5.21}$$

for each $k \geq 1$. For $\varphi \in L^\infty(J)$ we put

$$\varrho_k^\varphi(t,x) := \int_J u_k(t,x,y)\ \varphi(y)\ dy$$

and infer from (5.21) and the Fubini theorem that

$$\sup_{t\in[0,T]} |\varrho_k^\varphi(t)|_{L^1(\Omega)} + \int_0^T \int_\Omega |\nabla_x \varrho_k^\varphi|\ dxdt \leq C\ |\varphi|_{L^\infty(J)} \tag{5.22}$$

for each $k \geq 1$. We next infer from (5.3) that, if φ belongs to $\mathcal{D}(J)$, ϱ_k^φ satisfies

$$\partial_t \varrho_k^\varphi = \Delta_x \varrho_k^{\nu_k\varphi} + \int_J g_k(.,.,y)\ \varphi(y)\ dy \quad \text{in} \quad \mathcal{D}'((0,T)\times\Omega).$$

It therefore follows from (5.18) and (5.22) that

$$(\partial_t \varrho_k^\varphi) \quad \text{is bounded in} \quad L^1(0,T;H^m(\Omega)') \tag{5.23}$$

for m large enough. Owing to (5.22), (5.23) and the compactness of the embedding of $W^{1,1}(\Omega)$ into $L^1(\Omega)$ it follows from [32, Corollary 4] that (ϱ_k^φ) is compact in $L^1((0,T)\times\Omega)$. Now it is clear from (5.19) that the only possible cluster point of (ϱ_k^φ) as $k \to +\infty$ is

$$\varrho^\varphi\ : (t,x) \mapsto \int_J u(t,x,y)\ \varphi(y)\ dy,$$

which warrants that (ϱ_k^φ) converges towards ϱ^φ in $L^1((0,T)\times\Omega)$. □

5.2 A regularized problem

Let a, b and d be kinetic and diffusion coefficients satisfying (1.1), (2.2) and (2.1), respectively, and assume further that

$$\operatorname*{ess\,inf}_{[0,m]} d > 0 \quad \text{and} \quad a(y,y') = b(y,y') = 0 \quad \text{whenever} \quad y + y' \geq m \tag{5.24}$$

for some $m \in \mathbb{R}_+$. Observe that (5.24) implies that $\operatorname{supp} Q(f) \subset [0,m]$. For $\delta \in (0,1)$, we further introduce as in [14]

$$\tilde{Q}(f) = \frac{Q(f)}{1+\delta\ \varrho_m(f)} \quad \text{with} \quad \varrho_m(f) = \int_0^m f(y)\ dy,$$

$$\tilde{Q}_i(f) = \frac{Q_i(f)}{1+\delta\ \varrho_m(f)}, \quad i \in \{1,\ldots,4\}.$$

With these notations we are in a position to state the main result of this section.

Proposition 5.6 *Let f^{in} be a non-negative function in $L^\infty(\Omega \times \mathbb{R}_+)$ with support in $\bar{\Omega} \times [0,m]$ and $\delta \in (0,1)$. Under the assumptions (1.1), (2.2), (5.24) and (2.1) on the kinetic and diffusion coefficients there is a unique non-negative function*

$$f \in \mathcal{C}([0,+\infty); L^2(\Omega \times \mathbb{R}_+))$$

with $f(0) = f^{in}$ satisfying supp $f(t) \subset \bar{\Omega} \times [0,m]$ for each $t \in [0,+\infty)$ and f is a solution to (5.6)-(5.7) in $(0,T) \times \Omega \times (0,m)$ with $(\nu, u^{in}, g) = (d, f^{in}, \tilde{Q}(f))$ for each $T \in \mathbb{R}_+$. Also, for $t \in \mathbb{R}_+$ there holds

$$\int_\Omega \int_0^m y\ f(t,x,y)dydx = \int_\Omega \int_0^m y\ f^{in}(x,y)dydx. \tag{5.25}$$

Assume further that there is $\gamma_0 > 0$ such that $f^{in} \geq \gamma_0$ a.e. in $\Omega \times (0,m)$. Then, for each $t \in \mathbb{R}_+$ there is a positive constant $\gamma(t)$ depending on a, b, m and t such that

$$f(s,x,y) \geq \gamma(t) > 0 \quad \text{a.e. in} \quad (0,t) \times \Omega \times (0,m). \tag{5.26}$$

The proof of Proposition 5.6 will be performed by a classical fixed point argument. For that purpose we first need to check that the reaction terms have Lipschitz continuity properties. We summarize the required properties in the next lemma, the proof being left to the reader.

Lemma 5.7 *Let a and b be kinetic coefficients satisfying (1.1), (2.2) and (5.24) and put*

$$K_{a,b} := \sup_{(y,y')\in\mathbb{R}_+^2} \left\{ a(y,y') + (1+y+y')^2\ b(y,y') \right\} < +\infty.$$

If $\delta \in (0,1)$ and $f \in L^2(\Omega \times (0,m))$ then $\tilde{Q}_i(f)$ also belongs to $L^2(\Omega \times (0,m))$ for $i \in \{1,\ldots,4\}$. In addition, putting $\Lambda = 4\ K_{a,b}\ \delta^{-1}$ there holds

$$\left|\tilde{Q}_i(f)\right|_{L^2(\Omega\times(0,m))} \leq \Lambda\ |f|_{L^2(\Omega\times(0,m))}$$
$$\left|\tilde{Q}_i(f) - \tilde{Q}_i(\hat{f})\right|_{L^1(\Omega\times(0,m))} \leq \Lambda\ \left|f - \hat{f}\right|_{L^1(\Omega\times(0,m))}$$

for every $(f,\hat{f}) \in L^2(\Omega \times (0,m); \mathbb{R}^2)$.

In other words each $\tilde{Q}_i$ maps bounded subsets of $L^2(\Omega \times (0,m))$ into bounded subsets of $L^2(\Omega\times(0,m))$ and is Lipschitz continuous in $L^2(\Omega\times(0,m))$ for the norm of $L^1(\Omega\times(0,m))$. Observe that the constant Λ in Lemma 5.7 only depends on a and b through $K_{a,b}$ and, in particular, does not depend explicitly on m.

Proof of Proposition 5.6. For $u \in L^2(\Omega \times (0,m))$ we put

$$G(u) = \tilde{Q}_1(u)_+ - \tilde{Q}_2(u) - \tilde{Q}_3(u) + \tilde{Q}_4(u)_+,$$

where $r_+ := \max(r,0)$ denotes the positive part of $r \in \mathbb{R}$. It readily follows from Lemma 5.7 that

$$|G(u)|_{L^2(\Omega\times(0,m))} \leq 4\ \Lambda\ |u|_{L^2(\Omega\times(0,m))}$$

$$|G(u) - G(\hat{u})|_{L^1(\Omega\times(0,m))} \leq 4\ \Lambda\ |u - \hat{u}|_{L^1(\Omega\times(0,m))}$$

for $(u,\hat{u}) \in L^2(\Omega \times (0,m); \mathbb{R}^2)$. For $R \geq 1$ and $T \in \mathbb{R}_+$ we introduce the complete metric space $(X_{R,T},\ \text{dist}\ )$ defined by

$$X_{R,T} := \left\{ u \in \mathcal{C}([0,T]; L^1(\Omega \times (0,m)),\ \sup_{t\in[0,T]} |u(t)|_{L^2(\Omega\times(0,m))} \leq R \right\},$$
$$\text{dist}\ (u,\hat{u}) = \sup_{t\in[0,T]} |(u-\hat{u})(t)|_{L^1(\Omega\times(0,m))}.$$

For $u \in X_{R,T}$ $\tilde{Q}(u)$ belongs to $L^\infty(0,T;L^2(\Omega\times(0,m))$ and we denote by $\mathcal{G}(u)$ the unique solution to (5.6)-(5.7) given by Lemma 5.3 with $(\nu,u^{in},g)=(d,f^{in},G(u))$. Thanks to (5.8) and (5.11) we have

$$\sup_{t\in[0,T]} |\mathcal{G}(u)(t)|_{L^2(\Omega\times(0,m))} \le |f^{in}|_{L^2(\Omega\times(0,m))} + 4\ T\ \Lambda \sup_{t\in[0,T]} |u(t)|_{L^2(\Omega\times(0,m))},$$

$$\text{dist}\ (\mathcal{G}(u),\mathcal{G}(\hat{u})) \le 4\ T\ \Lambda\ \ \text{dist}\ (u,\hat{u}).$$

Choosing R large enough and T small enough it is easily seen that $\mathcal{G}$ is a strict contraction from $X_{R,T}$ in $X_{R,T}$, whence there are $T>0$ and $f\in X_{R,T}$ such that f is the unique solution to (5.6)-(5.7) given by Lemma 5.3 with $(\nu,u^{in},g)=(d,f^{in},G(f))$. By classical arguments f might be extended to a unique maximal solution to (5.6)-(5.7) with $(\nu,u^{in},g)=(d,f^{in},G(f))$ defined on $[0,T_\star)$ for some $T_\star\in(0,+\infty]$ with the following alternative : either $T_\star=+\infty$ or $T_\star<+\infty$ and $|f(t)|_{L^2(\Omega\times(0,m))}$ blows up as $t\to T_\star$. However (5.8), Lemma 5.7 and the Gronwall lemma exclude the latter. Consequently, $T_\star=+\infty$.

Next, by (2.2) we have

$$-G(f)(-f)_+ \le \frac{M_m}{\delta}\ (-f)_+^2,$$

and it follows from (5.7) (after multiplication by $-(-u)_+$ and integration), the Gronwall lemma and the non-negativity of f^{in} that $(-f)_+=0$ almost everywhere. Consequently f is non-negative and $G(f)=\tilde{Q}(f)$. We may then take y as a test function in the equation satisfied by f and conclude that (5.25) hold true.

We finally check the last assertion of Proposition 5.6 under the additional assumption that $f^{in}\ge\gamma_0>0$ a.e. in $\Omega\times(0,m)$. Introducing

$$\gamma(t)=\gamma_0\ \exp\{-((1+m)\ M_m\ t)/\delta\},$$

we infer from (2.2) and (5.7) that

$$\begin{aligned}\partial_t(\gamma-f) \quad &- \quad d\ \Delta_x(\gamma-f)+\frac{(1+m)\ M_m}{\delta}\ (\gamma-f)\\ &\le \quad \tilde{Q}_2(f)+\tilde{Q}_3(f)-\frac{(1+m)\ M_m}{\delta}\ f\le 0\end{aligned}$$

in $L^2((0,T)\times(0,m);H^1(\Omega)')$ with $(\gamma-f)(0)\le 0$. Consequently $(\gamma-f)_+=0$, whence (5.26). □

5.3 Proof of Theorem 2.3

Let a, b, and d be kinetic and diffusion coefficients satisfying (1.1), (1.9), (2.12), (2.2), and (2.1), respectively, and consider an initial datum f^{in} satisfying (2.14). The aim of this section is to show that the above assumptions on the data guarantee the existence of a weak solution to (1.2)-(**??**) as stated in Theorem 2.3. We actually have a more precise result.

Theorem 5.8 *Under the assumptions of Theorem 2.3 there is at least one weak solution f to (1.2)-(**??**) on $[0,+\infty)$ satisfying (1.8) and there is a constant κ_0 depending only on Ω, M and f^{in} such that*

(5.27)
$$\int_\Omega\int_0^\infty f(t)\ (1+y)\ dydx + H(f(t)|M)\le\kappa_0,$$

(5.28)
$$\int_0^t\left(\int_\Omega\int_0^\infty d(y)^{1/2}\ |\nabla_x f|\ dydx\right)^2\ ds\le\kappa_0,$$

(5.29)
$$\int_0^t\int_\Omega\int_0^\infty\int_0^\infty e(f)\ dy'dydxds\le\kappa_0$$

for every $t \in \mathbb{R}_+$. There also holds

$$\int_E d(y)^{1/2} \ |\nabla_x f| \ dydxdt \le \kappa_0 \ \left(\int_E f \ dydxdt \right)^{1/2} \tag{5.30}$$

for every measurable subset E of $\mathbb{R}_+ \times \Omega \times \mathbb{R}_+$.

Furthermore, if $\alpha_s = +\infty$ (recall that α_s is defined in (2.15)) there holds

$$\int_\Omega \int_0^\infty y \ f(t) \ dydx = \int_\Omega \int_0^\infty y \ f^{in} \ dydx, \quad t \ge 0. \tag{5.31}$$

Notice that we do not obtain an L^2-estimate on $d^{1/2} \ \nabla_x f \ f^{-1/2}$ in Theorem 5.8 as it could be expected from the formal computation (1.10) and as one gets in the case of the Boltzmann-Fokker-Planck equation, see [13]. This is due to the fact that, though (1.10) is satisfied by the solutions to the approximating problems considered below, the compactness properties of these solutions are not sufficient to recover (1.10) in the limit. Instead, we obtain the weaker estimates (5.28) which is nevertheless useful for the study of the large time behaviour in Section 6.

Proof of Theorem 5.8. Let $m \ge 1$ be an integer and put

$$a_m(y,y') = \mathbf{1}_{[0,m]}(y+y') \ a(y,y'), \quad b_m(y,y') = \mathbf{1}_{[0,m]}(y+y') \ b(y,y'),$$

$$d_m(y) = \min\{m, \max\{1/m, d(y)\}\},$$

$$f_m^{in}(x,y) = \min\left\{ \max\left\{ f^{in}(x,y), \frac{M(y)}{m} \right\}, m \right\} \ \mathbf{1}_{[0,m]}(y)$$

for $(x,y,y') \in \Omega \times \mathbb{R}_+^2$. We first point out worthy properties enjoyed by the above approximations of the data. First (a_m, b_m) still satisfies the detailed balance condition (1.9) with the same function M, that is,

$$a_m(y,y') \ M(y) \ M(y') = b_m(y,y') \ M(y+y'), \ \ (y,y') \in \mathbb{R}_+^2.$$

Also, (2.12) ensures that there is $\gamma_{0,m} > 0$ such that $f_m^{in} \ge \gamma_{0,m}$ a.e. in $\Omega \times (0,m)$. We finally infer from (2.14) that there is a constant κ_0 depending only on Ω, M and f^{in} such that

$$H(f_m^{in}|M) + \int_\Omega \int_0^\infty y \ f_m^{in} \ dydx \le \kappa_0. \tag{5.32}$$

We next define $Q_{i,m}$ as Q_i with (a_m, b_m) instead of (a,b) for $i \in \{1,\ldots,4\}$,

$$\tilde{Q}_{i,m}(f) = \frac{Q_{i,m}(f)}{1 + \varrho_m(f)/m} \ \text{ with } \ \varrho_m(f) = \int_0^m f(y) \ dy,$$

and $\tilde{Q}_m(f) = \tilde{Q}_{1,m}(f) - \tilde{Q}_{2,m}(f) - \tilde{Q}_{3,m}(f) + \tilde{Q}_{4,m}(f)$. Owing to the properties enjoyed by a_m, b_m, d_m and f_m^{in} we are in a position to apply Proposition 5.6 and obtain a non-negative solution f_m to (5.6)-(5.7) on $(0,+\infty) \times \Omega \times (0,m)$ with $(\nu, u^{in}, g) = (d_m, f_m^{in}, \tilde{Q}_m(f_m))$ which satisfies $\text{supp } f_m(t) \subset \bar{\Omega} \times [0,m]$,

$$\int_\Omega \int_0^\infty y \ f_m(t) \ dydx = \int_\Omega \int_0^\infty y \ f_m^{in} \ dydx, \tag{5.33}$$

$$f_m(s,x,y) \ge \gamma_m(t) > 0 \quad \text{a.e. in } \ (0,t) \times \Omega \times (0,m)$$

for some positive constant $\gamma_m(t)$ (depending on t and m) and for each $t \in \mathbb{R}_+$. Observe next that this positivity property allows us to take $\ln(f_m/M)$ as a test function in the equation satisfied by f_m ; since (a_m, b_m) satisfies the detailed balance condition (1.9) we obtain

$$\begin{aligned} & H(f_m(t)|M) + \int_0^t \int_\Omega \int_0^m d_m(y) \ \frac{|\nabla_x f_m|^2}{f_m} \ dydxds \\ & + \frac{1}{2} \ \int_0^t \int_\Omega \int_0^m \int_0^m j\left(\tilde{a}_m \ f_m \ f_m', \tilde{b}_m \ f_m''\right) \ dy'dydxds \le \ H(f_m^{in}|M), \end{aligned} \tag{5.34}$$

with

$$\tilde{a}_m(s,x,y,y') = \frac{a_m(y,y')}{1+\varrho_m(f_m(s,x))/m}, \quad \tilde{b}_m(s,x,y,y') = \frac{b_m(y,y')}{1+\varrho_m(f_m(s,x))/m}.$$

Now, on the one hand, it follows from (5.32) that the right-hand sides of (5.33) and (5.34) are bounded from above by a constant κ_0 which depends neither on $m \ge 1$ nor on $t \in \mathbb{R}_+$. On the other hand, we remark that f_m is a weak solution to (1.2)-(**??**) on $[0,+\infty)$ with kinetic coefficients $(\tilde{a}_m, \tilde{b}_m)$, diffusion coefficient d_m and initial datum f_m^{in} which obviously fulfil (1.1), (1.9), (2.12), (2.2), and (2.1), uniformly with respect to $m \ge 1$. We may then proceed along the lines of Section 3.1 to show that

$$\int_\Omega \int_0^\infty (1+y)\ f_m(t)\ dydx + H(f_m(t)|M) \le \kappa_0, \tag{5.35}$$

$$\int_0^t \left(\int_\Omega \int_0^\infty d_m(y)^{1/2}\ |\nabla_x f_m|\ dydx \right)^2\ ds \le \kappa_0, \tag{5.36}$$

$$\int_0^t \int_\Omega \int_0^\infty \int_0^\infty j\left(\tilde{a}_m\ f_m\ f_m',\tilde{b}_m\ f_m''\right)\ dy'dydxds \le \kappa_0 \tag{5.37}$$

for each $t \in \mathbb{R}_+$ (with some possibly larger κ_0) while for each $T \in \mathbb{R}_+$ there is a weakly compact subset $\mathcal{K}_w$ of $L^1(\Omega \times \mathbb{R}_+)$ such that

$$f_m(t) \in \mathcal{K}_w \ \text{ for each } \ t \in [0,T], \tag{5.38}$$

In addition,

$$\left(\tilde{Q}_{i,m}(f_m)\right) \ \text{ is weakly compact in } \ L^1((0,T)\times\Omega\times(0,R)) \tag{5.39}$$

for each $R \in \mathbb{R}_+$, $T \in \mathbb{R}_+$ and $i \in \{1,\ldots,4\}$. Also,

$$\int_E d_m(y)^{1/2}\ |\nabla_x f_m|\ dydxdt \le \kappa_0\ \left(\int_E f_m(y)\ dydxdt \right)^{1/2} \tag{5.40}$$

for every measurable subset E of $\mathbb{R}_+ \times \Omega \times \mathbb{R}_+$.

Consequently, on the one hand, $\tilde{a}_m$, $\tilde{b}_m$, and d_m clearly satisfy (4.1) and (4.2) and

$$(a_m, b_m, d_m) \longrightarrow (a,b,d) \ \text{ a.e. in } \ \mathbb{R}_+ \times \Omega \times \mathbb{R}_+.$$

On the other hand, owing to (5.35), (5.38) and (5.39) the sequence (f_m) enjoys the properties required by Theorem 4.1. We are thus almost in a position to apply Theorem 4.1 but we do not know yet whether the condition (4.3) is fulfilled. Nevertheless it is clear from the proof of Theorem 4.1 that (4.3) is only needed to identify the weak limits of the reaction terms and that the strong compactness property (4.9) holds true without (4.3). But (4.9) is exactly what is needed to guarantee that $(\tilde{a}_m, \tilde{b}_m)$ converges towards (a,b) a.e. in $\mathbb{R}_+ \times \Omega \times \mathbb{R}_+$. This last remark justifies that Theorem 4.1 can be used to conclude that there are a subsequence of (f_m) (not relabeled) and a weak solution f to (1.2)-(**??**) on $[0,+\infty)$ with kinetic coefficients a, b and diffusion coefficient d such that

$$f_m \longrightarrow f \ \text{ in } \ \mathcal{C}([0,T]; w - L^1(\Omega \times \mathbb{R}_+)) \tag{5.41}$$

for each $T \in \mathbb{R}_+$. In particular, it readily follows from the choice of f_m^{in} and (5.41) that $f(0) = f^{in}$ while (5.35), (5.36), (5.41) and the convexity of $H(.|M)$ ensure that (5.27) holds true. We next infer from (5.40), (5.41) and (2.1) that $(d_m^{1/2}\nabla_x f_m)$ and $(\nabla_x f_m)$ are weakly compact in $L^1((0,T)\times\Omega\times(0,R))$ and $L^1((0,T)\times\Omega\times(1/R,R))$, respectively, for each $R \ge 1$ and $T \in \mathbb{R}_+$. Therefore,

$$d_m^{1/2}\nabla_x f_m \rightharpoonup d^{1/2}\nabla_x f \ \text{ in } \ L^1((0,T)\times\Omega\times(0,R)) \tag{5.42}$$

for each $R \ge 1$ and $T \in \mathbb{R}_+$. The assertions (5.28) and (5.30) then follow at once from (5.41) and (5.42) by letting $m \to +\infty$ in (5.36) and (5.40), respectively.

It remains to check (5.29). For that purpose we argue as in [15] and claim that

$$
\left\{
\begin{array}{lll}
a_m\, f_m\, f'_m \rightharpoonup a\, f\, f' & \text{in} & L^1((0,T)\times\Omega\times(0,R)^2), \\
b_m\, f''_m \rightharpoonup b\, f'' & \text{in} & L^1((0,T)\times\Omega\times(0,R)^2),
\end{array}
\right. \tag{5.43}
$$

for every $R \geq 1$ and $T \in \mathbb{R}_+$. Indeed the weak compactness (5.41) of (f_m) guarantees the claimed weak compactness of $(b_m f''_m)$ which in turn, together with (5.37) and (3.12), entails that of $(a_m f_m f'_m)$. Next, if $\varphi \in L^\infty((0,T)\times\Omega\times(0,R)^2)$, we proceed as in the proof of (4.25) (with $\psi_m = a_m\, \varphi$) to conclude that (5.43) holds true. Since j is a lower semicontinuous convex function in $\mathbb{R}^2$ the functional

$$
(A,B) \in L^1((0,T)\times\Omega\times(0,R)^2;\mathbb{R}^2) \mapsto \int_0^T \int_\Omega \int_0^R \int_0^R j(A,B)\; dy' dy dx dt
$$

is a lower semicontinuous and convex functional and the weak convergences (5.43) and (5.37) entail that

$$
\int_0^T \int_\Omega \int_0^R \int_0^R e(f) \leq \liminf_{n\to+\infty} \int_0^T \int_\Omega \int_0^R \int_0^R e_n(f_n) \leq \kappa_0 .
$$

Letting $R \to +\infty$ yields (5.29) and the proof of Theorem 2.3 is complete in the general case.

It remains to check that (5.41) can be improved to yield (5.31) when $\alpha_s = +\infty$. Consider $Y \geq 1$, $t \in \mathbb{R}_+$, $m \geq 1$ and $\alpha > 1$. Since $\alpha_s = +\infty$ the function M_α defined by $M_\alpha(y) = M(y)\, e^{\alpha y}$, $y \in \mathbb{R}_+$, belongs to $L^1(\mathbb{R}_+, ydy)$. Consequently,

$$
\begin{aligned}
\int_\Omega \int_Y^{+\infty} y\, f_m(t)\; dy dx \quad &\leq \quad \int_\Omega \int_Y^{+\infty} y\, M_\alpha(y)\; \mathbf{1}_{\{f_m(t)\leq M_\alpha\}}(x,y)\; dy dx \\
&\quad + \int_\Omega \int_Y^{+\infty} y\, f_m(t)\; \mathbf{1}_{\{f_m(t)> M_\alpha\}}(x,y)\; dy dx \\
&\leq \quad |\Omega| \int_Y^{+\infty} y\, M_\alpha(y)\; dy \\
&\quad + \frac{1}{\alpha} \int_\Omega \int_Y^{+\infty} f_m(t)\; \ln\left(\frac{f_m(t)}{M}\right)\; dy dx \\
&\leq \quad |\Omega| \int_Y^{+\infty} y\, M_\alpha(y)\; dy + \frac{\kappa_0}{\alpha},
\end{aligned}
$$

where we have used (5.35) and Lemma 3.4 to bound the last term of the right-hand side of the above inequality. Therefore,

$$
\limsup_{Y\to+\infty} \sup_{m\geq 1} \int_\Omega \int_Y^{+\infty} f_m(t)\; y\; dy dx \leq \frac{\kappa_0}{\alpha}
$$

for each $\alpha > 1$, whence

$$
\lim_{Y\to+\infty} \sup_{m\geq 1} \sup_{t\in\mathbb{R}_+} \int_\Omega \int_Y^{+\infty} f_m(t)\; y\; dy dx = 0. \tag{5.44}
$$

Recalling (5.41) we readily conclude that

$$
f_m \longrightarrow f \;\text{ in }\; \mathcal{C}([0,T]; w - L^1(\Omega\times\mathbb{R}_+, (1+y) dx dy)),
$$

from which (5.31) follows, together with

$$
\lim_{Y\to+\infty} \sup_{t\in\mathbb{R}_+} \int_\Omega \int_Y^{+\infty} f_m(t)\; y\; dy dx = 0. \tag{5.45}
$$

The proof of Theorem 2.3 is thus complete. □

5.4 Proof of Theorem 2.6

Let a, b, and d be kinetic and diffusion coefficients satisfying (1.1), (1.17), (2.2), (2.22) and (2.1), respectively, and consider an initial datum f^{in} satisfying (2.23). As in the proof of Corollary 3.9 we use again a refined version of the de la Vallée-Poussin theorem [27, Proposition I.1.1] to deduce that there exists a non-negative and convex function $\Phi_0 \in \mathcal{C}^1([0,+\infty))$ satisfying (3.20) and $\Phi_0(f^{in}) \in L^1(\Omega \times \mathbb{R}_+)$.

We now introduce approximations of a, b, d and f^{in}. Let us first remark that the monotonicity property (1.17) is not likely to be fulfilled by a compactly supported approximation of a as required by Proposition 5.6 ; the proof of Theorem 2.6 will thus proceed in two steps : for $k \ge 1$, $m \ge k$ and $(x,y,y') \in \Omega \times \mathbb{R}_+^2$ we put

$$a^k(y,y') = \min\{a(y,y'),k\}, \quad b^k(y,y') = \min\{b(y,y'), A\ k\}\ \mathbf{1}_{[0,k]}(y+y'),$$

$$d^k(y) = \min\{\max\{1/k, d(y)\}, k\}, \quad f^{in,k}(x,y) = \min\{f^{in}(x,y),k\}\ \mathbf{1}_{[0,k]}(y),$$

and

$$a^k_m(y,y') = a^k(y,y')\ \mathbf{1}_{[0,m]}(y+y').$$

Observe that $a^k_m \le a^k \le a$, $b^k \le b$,

$$\left\{ \begin{array}{ll} a^k(y', y-y') \le a^k(y,y'), & 0 \le y' \le y, \\ & \\ b^k(y, y'-y) \le A\ a^k(y,y') + B(y'), & 0 \le y \le y', \end{array} \right. \tag{5.46}$$

and, since a and b are locally bounded in $\mathbb{R}_+^2$ by (2.2),

$$(a^k, b^k) \longrightarrow (a,b) \ \text{ a.e. in } \ \mathbb{R}_+^2. \tag{5.47}$$

Also, since Φ_0 is a non-decreasing function we have

$$\int_\Omega \int_0^\infty \left((1+y)\ f^{in,k} + \Phi_0(f^{in,k}) \right)\ dydx \le \kappa_0 \tag{5.48}$$

for each $k \ge 1$, where κ_0 is a constant depending only on f^{in}. Note that (5.48) will enable us to employ the estimates derived in Section 3.2 to perform the passage to the limit as $k \to +\infty$. On the other hand a^k_m clearly does not satisfy (1.17) but

$$\sup_{(y,y') \in \mathbb{R}_+^2} \left(a^k_m(y,y') + (1+y+y')^2\ b^k(y,y') \right) \le 2\ A\ (1+k)^3 \tag{5.49}$$

for $m \ge k$ and $k \ge 1$. We now define $Q^k_{i,m}$ and Q^k_i with (a^k_m, b^k) and (a^k, b^k), respectively, instead of (a,b) for $i \in \{1,\ldots,4\}$,

$$\begin{array}{rcl} \tilde{Q}^k_{i,m}(f) & = & \dfrac{Q^k_{i,m}(f)}{1+\varrho_m(f)/k} \ \text{ with } \ \varrho_m(f) = \displaystyle\int_0^m f(y)\ dy, \\ \tilde{Q}^k_i(f) & = & \dfrac{Q^k_i(f)}{1+\varrho(f)/k} \ \text{ with } \ \varrho(f) = \displaystyle\int_0^\infty f(y)\ dy, \end{array}$$

and

$$\left\{ \begin{array}{l} \tilde{Q}^k_m(f) = \tilde{Q}^k_{1,m}(f) - \tilde{Q}^k_{2,m}(f) - \tilde{Q}^k_{3,m}(f) + \tilde{Q}^k_{4,m}(f), \\ \\ \tilde{Q}^k(f) = \tilde{Q}^k_1(f) - \tilde{Q}^k_2(f) - \tilde{Q}^k_3(f) + \tilde{Q}^k_4(f). \end{array} \right.$$

We now fix $k \ge 1$ and consider $m \ge k$. Owing to the properties enjoyed by a^k_m, b^k, d^k and $f^{in,k}$ we infer from Proposition 5.6 that there is a non-negative solution f^k_m to (5.6)-(5.7) on $\mathbb{R}_+ \times \Omega \times (0,m)$ with $(\nu, u^{in}, g) = (d^k, f^{in,k}, \tilde{Q}^k_m(f^k_m))$ which satisfies $\text{supp}\ f^k_m(t) \subset \bar{\Omega} \times [0,m]$ and

$$\int_\Omega \int_0^\infty y\ f^k_m(t)\ dydx = \int_\Omega \int_0^\infty y\ f^{in,k}\ dydx \tag{5.50}$$

for $t \in \mathbb{R}_+$. We next infer from (5.8), (5.49) and Lemma 5.7 that

$$|f_m^k(t)|_{L^2(\Omega\times\mathbb{R}_+)} \leq |f^{in,k}|_{L^2(\Omega\times\mathbb{R}_+)} + 32\ A\ (1+k)^4 \int_0^t |f_m^k(s)|_{L^2(\Omega\times\mathbb{R}_+)}\ ds$$

and the Gronwall lemma entails that

$$(f_m^k) \text{ is bounded in } L^\infty(0,T;L^2(\Omega\times\mathbb{R}_+)) \tag{5.51}$$

for each $T \in \mathbb{R}_+$. Using again (5.49) and Lemma 5.7 we further obtain that

$$(\tilde{Q}_{i,m}^k(f_m^k)) \text{ is bounded in } L^\infty(0,T;L^2(\Omega\times\mathbb{R}_+)) \tag{5.52}$$

for $i \in \{1,\ldots,4\}$ and $T \in \mathbb{R}_+$.
Consider next $n \geq m \geq k$. For $i \in \{1,\ldots,4\}$ we have

$$\tilde{Q}_{i,m}^k(f_m^k) - \tilde{Q}_{i,n}^k(f_n^k) = \tilde{Q}_{i,m}^k(f_m^k) - \tilde{Q}_{i,n}^k(f_m^k) + \tilde{Q}_{i,n}^k(f_m^k) - \tilde{Q}_{i,n}^k(f_n^k).$$

On the one hand, using once more (5.49) and Lemma 5.7 we deduce that

$$\left|\tilde{Q}_{i,n}^k(f_m^k) - \tilde{Q}_{i,n}^k(f_n^k)\right|_{L^1(\Omega\times\mathbb{R}_+)} \leq 8\ A\ (1+k)^3\ \left|f_m^k - f_n^k\right|_{L^1(\Omega\times\mathbb{R}_+)}.$$

On the other hand,

$$\tilde{Q}_{i,m}^k(f_m^k) - \tilde{Q}_{i,n}^k(f_m^k) = 0$$

if $i \in \{2,4\}$, while (5.49) and (5.50) ensure that

$$\left|\tilde{Q}_{i,m}^k(f_m^k) - \tilde{Q}_{i,n}^k(f_m^k)\right|_{L^1(\Omega\times\mathbb{R}_+)} \leq \frac{2\ k^2}{m} \int_\Omega\int_0^\infty y\ f^{in}\ dydx.$$

Consequently, for $n \geq m \geq k$ we have

$$\begin{aligned} &\sup_{1\leq i\leq 4}\left|\tilde{Q}_{i,m}^k(f_m^k) - \tilde{Q}_{i,n}^k(f_n^k)\right|_{L^1(\Omega\times\mathbb{R}_+)} \\ \leq\ & C(k,A,f^{in})\ \left(\frac{1}{m} + \left|f_m^k - f_n^k\right|_{L^1(\Omega\times\mathbb{R}_+)}\right). \end{aligned} \tag{5.53}$$

Since $(f_n^k - f_m^k)$ is the solution to (5.6)-(5.7) with $(\nu,u^{in},g) = (d^k,0,\tilde{Q}_m^k(f_m^k) - \tilde{Q}_n^k(f_n^k))$ it follows from (5.4), (5.53) and the Gronwall lemma that $(f_m^k)_{m\geq k}$ is a Cauchy sequence in $\mathcal{C}([0,T];L^1(\Omega\times\mathbb{R}_+))$ for each $T \in \mathbb{R}_+$ and we denote by f^k its limit. This last result and (5.53) entail that $\left(\tilde{Q}_{i,m}^k(f_m^k)\right)_{m\geq k}$ is also a Cauchy sequence in $\mathcal{C}([0,T];L^1(\Omega\times\mathbb{R}_+))$ for $i \in \{1,\ldots,4\}$ and $T \in \mathbb{R}_+$ and we denote by $\bar{Q}_i^k$ its limit. We may now argue as in the proof of Theorem 4.1 and conclude that $\bar{Q}_i^k = Q_i^k(f^k)$ with the help of Corollary 4.4. Recalling (5.51) and (5.52) it is straightforward to pass to the limit as $m \to +\infty$ and deduce that f^k is the solution to (5.6)-(5.7) on $(0,+\infty)\times\Omega\times\mathbb{R}_+$ with $(\nu,u^{in},g) = (d^k,f^{in,k},\tilde{Q}^k(f^k))$.

We now pass to the limit as $k \to +\infty$: we first notice that, for $k \geq 1$, f^k is a weak solution to (1.2)-(**??**) on $[0,+\infty)$ with kinetic coefficients $\tilde{a}^k$ and $\tilde{b}^k$ defined by

$$\tilde{a}^k(t,x,y,y') = \frac{a^k(y,y')}{1+\varrho(f^k)/k},\quad \tilde{b}^k(t,x,y,y') = \frac{b^k(y,y')}{1+\varrho(f^k)/k},$$

diffusion coefficient d^k and initial datum $f^{in,k}$. Moreover, we infer from (5.46) that $(\tilde{a}^k,\tilde{b}^k)$ satisfies (5.46) as well, together with (4.1) and (4.2). Also, (5.46), (5.48) and the regularity of $(f^k,\tilde{Q}^k(f^k))$ allow us to justify the computations performed in Section 3.2 and show that the sequence (f^k) enjoys the bounds of Corollaries 3.7 and 3.9 uniformly with respect to $k \geq 1$. We may therefore employ Lemma 3.10 to deduce that (f^k) satisfies the compactness properties required to apply Theorem 4.1. The remainder of the proof of Theorem 2.6 is then similar to that of Theorem 2.3 in the previous section to which we refer. ◻

6 Large time behaviour

In this section we prove the stabilization towards an equilibrium in the long time when the kinetic coefficients a and b satisfy the detailed balance condition (1.9). More precisely we assume that the kinetic and diffusion coefficients fulfil (1.1), (2.2), (2.12) and (2.1), respectively, and that a and b are positive a.e. in $\mathbb{R}_+^2$. We are also given an initial datum f^{in} satisfying (2.14) and denote by f the weak solution to (1.2)-(**??**) constructed in Theorem 5.8. It first follows from (5.27), (5.28) and (5.29) that

$$\text{(6.1)} \qquad \begin{cases} f \in L^\infty(0,+\infty; L^1(\Omega\times\mathbb{R}_+, (1+y)\ dxdy), & H(f|M) \in L^\infty(0,+\infty), \\ d^{1/2}\ \nabla_x f \in L^2(0,+\infty; L^1(\Omega\times\mathbb{R}_+)), & e(f) \in L^1((0,+\infty)\times\Omega\times\mathbb{R}_+^2). \end{cases}$$

Let (t_n) be a sequence of positive real numbers such that $t_n \to +\infty$ and put $f_n(t) = f(t_n+t)$ for $n \geq 1$ and $t \in \mathbb{R}_+$. Owing to Theorem 5.8 it is easily seen that f_n is a weak solution to (1.2)-(**??**) on $[0,+\infty)$ with initial datum $f(t_n)$. We fix $T \in \mathbb{R}_+$ and infer from (6.1) that

$$\text{(6.2)} \qquad \sup_{t\in[0,T]} \int_\Omega \int_0^\infty (1+y)\ f_n(t)\ dydx + H(f_n(t)|M) \leq \kappa_0,$$

$$\text{(6.3)} \qquad \begin{aligned} &\int_0^T \left(\int_\Omega \int_0^\infty d(y)^{1/2}\ |\nabla_x f_n|\ dydx \right)^2\ dt \\ &\leq \int_{t_n}^{t_n+T} \left(\int_\Omega \int_0^\infty d(y)^{1/2}\ |\nabla_x f|\ dydx \right)^2\ dt \longrightarrow 0, \end{aligned}$$

$$\text{(6.4)} \qquad \int_0^T \int_\Omega \int_0^\infty \int_0^\infty e(f_n)\ dy'dydxdt \leq \int_{t_n}^{t_n+T} \int_\Omega \int_0^\infty \int_0^\infty e(f)\ dy'dydxdt \longrightarrow 0.$$

Thanks to Lemma 3.4 we are in a position to apply Theorem 4.1 to deduce that there are a subsequence of (f_n) (not relabeled) and a weak solution f to (1.2)-(**??**) such that

$$\text{(6.5)} \qquad f_n \longrightarrow \bar{f} \ \text{ in } \ \mathcal{C}([0,T]; w-L^1(\Omega\times\mathbb{R}_+)),$$

$$\text{(6.6)} \qquad \begin{cases} Q_i(f_n) \rightharpoonup Q_i(\bar{f}) & \text{in} \quad L^1((0,T)\times\Omega\times(0,R)), \\ L(f_n) \longrightarrow L(\bar{f}) & \text{in} \quad L^1((0,T)\times\Omega\times(0,R)), \\ Q_4(f_n) \longrightarrow Q_4(\bar{f}) & \text{in} \quad L^1((0,T)\times\Omega\times(0,R)) \end{cases}$$

for every $R \in \mathbb{R}_+$. Moreover, (5.30), (2.1) and (6.5) warrant the weak compactness of $(d^{1/2}\ \nabla_x f_n)$ and $(\nabla_x f_n)$ in $L^1((0,T)\times\Omega\times(0,R))$ and $L^1((0,T)\times\Omega\times(1/R,R))$, respectively, for every $R \geq 1$, whence

$$\text{(6.7)} \qquad d^{1/2}\ \nabla_x f_n \rightharpoonup d^{1/2}\ \nabla_x \bar{f} \quad \text{in} \quad L^1((0,T)\times\Omega\times(0,R)).$$

On the one hand we argue as in the proof of (5.29) and use (6.4) to conclude that

$$\text{(6.8)} \qquad e(\bar{f}) = 0 \ \text{ a.e. in } \ (0,T)\times\Omega\times\mathbb{R}_+^2.$$

Consequently,

$$\text{(6.9)} \qquad a(y,y')\ \bar{f}(t,x,y)\ \bar{f}(t,x,y') = b(y,y')\ \bar{f}(t,x,y+y') \ \text{ a.e. in } \ (0,T)\times\Omega\times\mathbb{R}_+^2.$$

In particular (6.9) implies that $Q(\bar{f}) = 0$. On the other hand, (6.3), (6.7) and (2.1) entail that

$$\text{(6.10)} \qquad \nabla_x \bar{f} = 0 \ \text{ a.e. in } \ (0,T)\times\Omega\times\mathbb{R}_+.$$

Therefore $\bar{f}$ satisfies $\partial_t \bar{f} = 0$ in $\mathcal{D}'((0,T)\times\Omega\times\mathbb{R}_+)$ and we end up with $\bar{f}(t,x,y) = \bar{f}(y)$ almost everywhere in $(0,T)\times\Omega\times\mathbb{R}_+$. Recalling (6.9) we conclude from Lemma C.1 that there is $\alpha \in [-\infty,+\infty)$ such that

$$\bar{f} = M_\alpha \ \text{ a.e. in } \ (0,T)\times\Omega\times\mathbb{R}_+.$$

Owing to (1.8) and (6.5) we clearly have $\alpha \in [-\infty, \alpha^{in}]$ and we have thus proved that

$$f_n \longrightarrow M_\alpha \ \text{ in } \ \mathcal{C}([0,T]; w - L^1(\Omega\times\mathbb{R}_+)). \tag{6.11}$$

In order to improve the convergence (6.11) we adapt an argument of P.-L. Lions [28] and first establish that

$$f_n \longrightarrow M_\alpha \ \text{ in } \ L^1((0,T)\times\Omega\times\mathbb{R}_+). \tag{6.12}$$

Indeed if $\alpha = -\infty$ (i.e. $M_\alpha \equiv 0$) the assertion (6.12) follows at once from (6.11) and the non-negativity of f_n for every $n \ge 1$. If $\alpha \in \mathbb{R}$ we have

$$|\eta - \xi| \le (\lambda - 1)\ \xi + \frac{1}{\ln\lambda}\ (\eta - \xi)\ (\ln\eta - \ln\xi), \quad (\xi,\eta) \in \mathbb{R}_+^2, \quad \lambda > 1$$

by (3.12). Consequently, for $R \ge 1$ and $\lambda > 1$,

$$\begin{aligned}\int_0^T \int_\Omega \int_0^R |Q_3(f_n) - Q_4(f_n)|\ dydxdt \quad &\le \quad (\lambda - 1)\ \int_0^T \int_\Omega \int_0^R Q_4(f_n)\ dydxdt \\ &+ \frac{1}{\ln\lambda}\ \int_0^T \int_\Omega \int_0^\infty \int_0^\infty e(f_n)\ dy'dydxdt.\end{aligned}$$

Thanks to (6.4) and (6.6) we may pass to the limit as $n \to +\infty$ in the above inequality and, recalling that $\bar{f} = M_\alpha$, we obtain

$$\limsup_{n\to+\infty} \int_0^T \int_\Omega \int_0^R |Q_3(f_n) - Q_4(f_n)|\ dydxdt \le (\lambda - 1)\ \int_0^T \int_\Omega \int_0^R Q_4(M_\alpha)\ dydxdt.$$

We then let $\lambda \to 1$ and use (6.6) once more to conclude that

$$Q_3(f_n) \longrightarrow Q_4(M_\alpha) \ \text{ in } \ L^1((0,T)\times\Omega\times(0,R))$$

for each $R \in \mathbb{R}_+$. Since $Q_3(M_\alpha) = Q_4(M_\alpha)$ by (1.9) we finally obtain, after extracting a further subsequence if necessary, that

$$\left\{ \begin{array}{lll} Q_3(f_n) \longrightarrow Q_3(M_\alpha) & \text{in} & L^1((0,T)\times\Omega\times(0,R)), \\ & & \\ f_n\ L(f_n) \longrightarrow M_\alpha\ L(M_\alpha) & \text{a.e. in} & (0,T)\times\Omega\times(0,R) \end{array} \right. \tag{6.13}$$

for every $R \in \mathbb{R}_+$. Since $L(f_n)$ converges towards $L(M_\alpha)$ in $L^1((0,T)\times\Omega\times(0,R))$ by (6.6) the positivity (2.12) of M ensures that (f_n) converges almost everywhere towards M_α, whence the claim (6.12) by (6.11) and the Vitali theorem.

It then readily follows from (6.12) that

$$Q_2(f_n) \longrightarrow Q_2(M_\alpha) \ \text{ in } \ L^1((0,T)\times\Omega\times(0,R)) \tag{6.14}$$

for each $R \in \mathbb{R}_+$ and we next argue as in the proof of (6.13) to obtain that

$$Q_1(f_n) \longrightarrow Q_1(M_\alpha) \ \text{ in } \ L^1((0,T)\times\Omega\times(0,R)). \tag{6.15}$$

Combining (6.6), (6.13), (6.14) and (6.15) yields

$$Q(f_n) \longrightarrow Q(M_\alpha) = 0 \ \text{ in } \ L^1((0,T)\times\Omega\times(0,R)). \tag{6.16}$$

We now consider $t \in (0,T]$, $s \in (0,t)$ and infer from (5.4) that

$$\begin{aligned}\int_\Omega \int_0^R |f_n(t) - M_\alpha| \ dydx \quad & \le \quad \int_\Omega \int_0^R |f_n(s) - M_\alpha| \ dydx \\ & + \int_s^t \int_\Omega \int_0^R |Q(f_n)| \ dydxd\sigma.\end{aligned}$$

After integration with respect to s we obtain

$$\begin{aligned}t \int_\Omega \int_0^R |f_n(t) - M_\alpha| \ dydx \quad & \le \quad \int_0^t \int_\Omega \int_0^R |f_n(s) - M_\alpha| \ dydxds \\ & + t \int_0^t \int_\Omega \int_0^R |Q(f_n)| \ dydxds.\end{aligned}$$

Thanks to (6.12) and (6.16) we may pass to the limit as $n \to +\infty$ and end up with

$$\lim_{n\to+\infty} \sup_{t\in(0,T]} t \int_\Omega \int_0^R |f_n(t) - M_\alpha| \ dydx = 0.$$

Combining (6.11) with the above assertion yields (2.18).

Finally, if $\alpha_s = +\infty$, (5.31) and the estimate (5.45) imply that $M_\alpha = M_{\alpha^{in}}$, whence (2.20).

7 Some extensions

The purpose of this section is to outline how some slight modifications of our analysis allow us to prove the existence of weak solutions to (1.2)-(**??**) for kinetic coefficients a and b satisfying the symmetry condition (1.1), the monotonicity condition (1.17) but not (2.2) or (2.22).

We first consider kinetic coefficients a and b satisfying (1.1), (1.17) and (2.2) and assume that there are positive real numbers A and B such that

$$\left\{\begin{array}{ll} 0 \le b(y,y') \le B, & (y,y') \in \mathbb{R}_+^2, \\ & \\ A \le a(y,y'), & (y,y') \in [1,+\infty)^2. \end{array}\right. \tag{7.1}$$

As for the diffusion coefficient we still assume that (2.1) holds while a stronger assumption is required on the initial datum, namely,

$$f^{in} \in L^1(\Omega\times\mathbb{R}_+,(1+y)dxdy) \cap L^2(\Omega\times\mathbb{R}_+) \ \text{ is non-negative a.e.} \tag{7.2}$$

We then have the following result.

Theorem 7.1 *Assume that the kinetic coefficients a and b satisfy (1.1), (1.17), (2.2) and (7.1), the diffusion coefficient d satisfy (2.1) and the initial datum f^{in} satisfy (7.2). Then there is a weak solution f to (1.2)-(**??**) on $[0,+\infty)$ satisfying (1.8) and f belongs to $L^\infty(0,T;L^2(\Omega\times\mathbb{R}_+))$ for each $T \in \mathbb{R}_+$.*

Theorem 7.1 applies in particular when $a(y,y') = (y\ y')^\alpha$ for $\alpha \in [0,1)$ and $b \equiv 1$ which cannot be handled by Theorem 2.6 (as (2.21)-(2.22) are not fulfilled).

The proof of Theorem 7.1 is similar to that of Theorem 2.6 except that the function Φ in Corollary 3.9 and Lemma 3.10 is now $\Phi(r) = r^2$, $r \in \mathbb{R}_+$, and Lemma 3.8 is to be replaced by the following result.

Lemma 7.2 *Under the assumptions of Theorem 7.1 there holds*

$$\int_\Omega \int_0^\infty Q_4(f)\ f\ dydx \le 2\ B \int_\Omega \int_0^\infty f^2\ dydx + \frac{2\ B}{A} \int_\Omega \int_0^\infty Q_3(f)\ f\ dydx.$$

Proof. Thanks to (7.1) we have

$$\begin{aligned}\int_\Omega \int_0^\infty Q_4(f)\ f\ dydx &\le B\ \int_\Omega \int_0^\infty \int_0^\infty f\ f'\ dy'dydx \\ &\le B\ \int_\Omega \left(\int_0^1 f\ dy + \int_1^\infty f\ dy \right)^2 \\ &\le 2\ B\ \int_\Omega \left(\int_0^1 f\ dy \right)^2 + 2\ B\ \int_\Omega \int_1^\infty \int_1^\infty f\ f'\ dy'dydx \\ &\le 2\ B\ \int_\Omega \int_0^1 f^2\ dy + \frac{2\ B}{A}\ \int_\Omega \int_1^\infty \int_1^\infty a(y,y')\ f\ f'\ dy'dydx,\end{aligned}$$

and the proof of Lemma 7.2 is complete. □

Another possible extension of our analysis involves the particular class of coagulation coefficients of multiplicative type and allows to relax the growth condition (2.2). More precisely, we assume that the coagulation coefficient a satisfies (1.1), (1.17) and there are a non-negative measurable function $r : \mathbb{R}_+ \to \mathbb{R}_+$ and $A \ge 1$ such that

$$r(y)\ r(y') \le a(y,y') \le A\ r(y)\ r(y'), \quad (y,y') \in \mathbb{R}_+^2. \tag{7.3}$$

For the sake of simplicity we assume that $b \equiv 0$ (though it is possible to consider fragmentation coefficients suitably dominated by r as in [21]). In that case we have the following result.

Theorem 7.3 *Under the assumptions (1.1), (1.17), (7.3), (2.1) and (2.23) on the coagulation coefficients a, the diffusion coefficient d and the initial datum f^{in}, and if $b \equiv 0$, there is a weak solution f to (1.2)-(**??**) on $[0,+\infty)$ satisfying (1.8).*

Here again, the proof of Theorem 7.3 is similar to that of Theorem 2.6 but we need an additional estimate to control the behaviour of f for large values of y (in order to be able to pass to the limit in $Q_3(f)$ and $Q_4(f)$). Such an estimate is supplied by (7.3) which guarantees that

$$\int_0^T \int_\Omega \left(\int_Y^\infty r(y)\ f(t,x,y)\ dy \right)^2\ dxdt \le \frac{2}{Y}\ \int_\Omega \int_0^\infty (1+y)\ f^{in}\ dydx$$

holds true for every $Y \in \mathbb{R}_+$ (see [21, Lemma 2.4 & 2.5]).

Notice that Theorem 7.3 allows us to obtain an existence result for

$$a(y,y') = y\ y' + A_1\ (y+y') + A_0,$$

where A_0, A_1 are non-negative real numbers, which clearly does not satisfy the growth condition (2.2). Though Theorem 7.3 extends to the coagulation-fragmentation equation it requires a sufficiently weak fragmentation and in particular, the case $a(y,y') = y\ y'$ and $b(y,y') = 1$ cannot be handled by this device.

Remark 7.4 *Let us finally point out that our analysis of the coagulation-fragmentation equation with diffusion for coagulation coefficients satisfying the monotonicity condition (1.17) is likely to extend to the multiple fragmentation case (see, e.g., [21] and the references therein).*

A Proof of Corollary 4.4

Lemma A.1 *Let U be an open bounded subset of $\mathbb{R}^m$, $m \ge 1$, and consider two sequences (v_n) in $L^1(U)$ and (w_k) in $L^\infty(U)$ and a function $w \in L^\infty(U)$ such that*

$$(v_n) \quad \textit{is weakly relatively compact in} \quad L^1(U), \tag{A.1}$$

$$\text{(A.2)} \qquad |w_k(x)| \le C \quad \textit{and} \quad \lim_{k\to+\infty} w_k(x) = w(x) \quad \textit{a.e.}$$

for some $C > 0$. Then

$$\text{(A.3)} \qquad \lim_{k\to+\infty} \sup_{n\ge1} \int_U |v_n|\ |w_k - w|\ dx = 0.$$

The proof of Lemma A.1 relies on the Egorov and Dunford-Pettis theorems and is implicitly contained in [14]. Similarly one can prove the following result, the proof of which is given below for the sake of completeness.

Lemma A.2 *Let U be an open bounded subset of $\mathbb{R}^m$, $m \ge 1$, and consider two sequences (v_n) in $L^1(U)$ and (w_n) in $L^\infty(U)$ and functions $v \in L^1(U)$ and $w \in L^\infty(U)$ such that*

$$\text{(A.4)} \qquad v_n \rightharpoonup v \quad \textit{in} \quad L^1(U),$$

$$\text{(A.5)} \qquad |w_n(x)| \le C \quad \textit{and} \quad \lim_{n\to+\infty} w_n(x) = w(x) \quad \textit{a.e.}$$

for some $C > 0$. Then

$$\text{(A.6)} \qquad \lim_{n\to+\infty} \int_U |v_n|\ |w_n - w|\ dx = 0 \quad \textit{and} \quad v_n\ w_n \rightharpoonup v\ w \quad \textit{in} \quad L^1(U).$$

Proof of Lemma A.2. Let $\varepsilon \in (0,1)$. The Dunford-Pettis theorem and (A.4) entail that there exists $\eta > 0$ such that

$$\sup_{n\ge1} \int_E |v_n(x)|\ dx \le \varepsilon/(2C)$$

for any measurable subset E of U with $|E| \le \eta$. It next follows from (A.5) and the Egorov theorem that there is a measurable subset E_η with $|E_\eta| \le \eta$ such that (w_n) converges towards w uniformly on $U \setminus E_\eta$. Then

$$\text{(A.7)} \qquad \int_U |v_n|\ |w_n - w|\ dx \le |v_n|_{L^1(U)}\ |w_n - w|_{L^\infty(U\setminus E_\eta)} + 2C \int_{E_\eta} |v_n|\ dx.$$

Since (v_n) is bounded in $L^1(U)$ by (A.4) and the last term of the right-hand side of (A.7) is bounded from above by ε, the left-hand side of (A.7) is bounded from above by 2ε for n large enough, whence the first assertion of (A.6). The second assertion of (A.6) then readily follows from the first one and (A.4). □

Proof of Corollary 4.4. We still use the notations $\mathcal{U} = \Omega \times \mathbb{R}_+$ and $\mathcal{U}_T = (0,T) \times \mathcal{U}$.
We first prove the assertion (i) and consider the case where $\psi(t,x,y) = \varphi(t,x)\ \vartheta(y)$ where

$$\varphi \in L^\infty((0,T)\times\Omega) \quad \text{and} \quad \vartheta(y) \le \omega(y)\ (1+y) \quad \text{a.e. in} \quad \mathbb{R}_+$$

for some positive function $\omega \in L_0^\infty(\mathbb{R}_+)$. There is a sequence (ϑ_m) in $\mathcal{D}(\mathbb{R}_+)$ such that

$$\lim_{m\to+\infty} \vartheta_m(y) = \vartheta(y) \quad \text{and} \quad \vartheta_m(y) \le 2\ \omega(y)\ (1+y) \quad \text{a.e. in} \quad \mathbb{R}_+.$$

On the one hand, putting

$$\hat{f}_n(t,x,y) = \omega(y)\ (1+y)\ f_n(t,x,y), \quad \hat{f}(t,x,y) = \omega(y)\ (1+y)\ f(t,x,y),$$

we infer from (4.4) and (4.12) that

$$\hat{f}_n \rightharpoonup \hat{f} \quad \text{in} \quad L^1(\mathcal{U}_T).$$

On the other hand, introducing

$$\tilde{\psi}_m(t,x,y) = \varphi(t,x)\ \frac{\vartheta_m(y)}{\omega(y)\ (1+y)}, \quad \tilde{\psi}(t,x,y) = \varphi(t,x)\ \frac{\vartheta(y)}{\omega(y)\ (1+y)},$$

we see that $(\tilde{\psi}_m)$ is bounded in $L^\infty(\mathcal{U}_T)$ and converges almost everywhere towards $\tilde{\psi}$. We may thus apply Lemma A.1 to conclude that

$$\lim_{m\to+\infty} \sup_{n\geq 1} \int_{\mathcal{U}_T} \left|\hat{f}_n - \hat{f}\right| \left|\tilde{\psi}_m - \tilde{\psi}\right| \ dydxdt = 0.$$

Next, noticing that $f_n \ \varphi \ \vartheta_m = \hat{f}_n \ \tilde{\psi}_m$, we obtain

$$\begin{aligned} \int_0^T \int_\Omega \left| \int_0^\infty (f_n - f) \ \psi \ dy \right| dxdt \quad & \leq \quad |\varphi|_{L^\infty} \int_0^T \int_\Omega \left| \int_0^\infty (\hat{f}_n - \hat{f}) \ \vartheta_m \ dy \right| dxdt \\ & \quad + \sup_{n\geq 1} \int_{\mathcal{U}_T} \left|\hat{f}_n - \hat{f}\right| \left|\tilde{\psi}_m - \tilde{\psi}\right| \ dydxdt. \end{aligned}$$

We first let $n \to +\infty$ in the above inequality and infer from Proposition 4.3 that the first term of the right-hand side converges to zero. We then pass to the limit as $m \to +\infty$ and conclude that (4.16) holds true for functions ψ with separated variables. We next argue as in the proof of [14, Corollary IV.2] to obtain the assertion (i) for arbitrary ψ satisfying (4.15).

As for the assertion (ii), it follows at once from (i) and Lemma A.2. Finally similar arguments lead to (iii). □

B Auxiliary results on convex functions

Let $\Phi \in \mathcal{C}^1([0,+\infty))$ be a non-negative and convex function satisfying (3.20), that is,

$$\begin{cases} \Phi(0) = 0, \ \ \Phi'(0) \geq 0 \ \text{ and } \ \Phi' \ \text{ is concave on } \ [0,+\infty), \\ \\ \displaystyle\lim_{r\to+\infty} \Phi'(r) = \lim_{r\to+\infty} \frac{\Phi(r)}{r} = +\infty, \end{cases}$$

and put

$$\Phi^*(r) = \sup_{s\in\mathbb{R}_+} (r \ s - \Phi(s)), \quad r \in \mathbb{R}_+.$$

Lemma B.1 *For $r \in [0,+\infty)$ there holds*

$$\Phi(r) \leq r \ \Phi'(r) \leq 2 \ \Phi(r), \tag{B.1}$$

$$\Phi^* (\Phi'(r)) = r \ \Phi'(r) - \Phi(r) \leq \Phi(r). \tag{B.2}$$

Proof. Let $r \in (0,+\infty)$. The convexity of Φ first entails that

$$\Phi(0) - \Phi(r) \geq (0 - r) \ \Phi'(r),$$

hence the first inequality in (B.1). Next, Φ' being concave with $\Phi'(0) \geq 0$ we have

$$-\Phi'(s) \leq \Phi'(0) - \Phi'(s) \leq -s \ \Phi''(s), \quad s \in (0,r),$$

hence

$$s \ \Phi''(s) + \Phi'(s) \leq 2 \ \Phi'(s), \quad s \in (0,r).$$

Integrating the above inequality with respect to s over $(0,r)$ yields the second inequality in (B.1). Finally, let $r \in [0,+\infty)$. Using again the convexity of Φ yields that

$$r \ \Phi'(r) - \Phi(r) \geq s \ \Phi'(r) - \Phi(s)$$

for $s \in \mathbb{R}_+$ and the first equality in (B.2) follows. The second inequality in (B.2) is then a consequence of (B.1). □

Lemma B.2 *For $r_0 \in \mathbb{R}_+$ the function Ψ defined by $\Psi(r) = (\Phi(r) - \Phi(r_0))_+$, $r \in \mathbb{R}_+$, is a non-negative and convex function satisfying*

$$\Psi(r) \leq r\ \Psi'(r), \tag{B.3}$$

$$\Psi^*\left(\Psi'(r)\right) \leq \left(1 + \frac{\Phi(2\ r_0)}{r_0}\right)\ (r + \Psi(r))\,, \tag{B.4}$$

for $r \in [0, +\infty)$.

Proof. The convexity and non-negativity of Ψ readily follow from the properties of Φ and of the positive part, while (B.3) is proved as the first inequality in (B.1). We next notice that, since Φ is non-decreasing, we have $\Psi'(r) = \Phi'(r)\ \mathrm{sign}_+(r - r_0)$. Consequently,

$$\Psi^*\left(\Psi'(r)\right) = \Psi^*(0) = 0 \quad \text{for} \quad r \in [0, r_0).$$

For $r \geq r_0$ and $s \in \mathbb{R}_+$ we have

$$\Phi'(r)\ s - \Psi(r) \leq r\ \Phi'(r) + \Phi(s) - \Phi(r) - \Psi(r)$$

by the convexity of Φ and we further infer from (B.1) that

$$\begin{aligned} \Phi'(r)\ s - \Psi(r) &\leq r\ \Phi'(r) - \Phi(r) + \Phi(s) - \Phi(r_0) - (\Phi(r) - \Phi(r_0))_+ + \Phi(r_0) \\ &\leq \Phi(r) + \Phi(r_0) \end{aligned}$$

for $r \geq r_0$. Therefore

$$\begin{aligned} \Psi^*\left(\Psi'(r)\right) &\leq (\Phi(r) + \Phi(r_0))\ \mathrm{sign}_+(r - r_0) \\ &\leq 2\ \Phi(r_0)\ \mathrm{sign}_+(r - r_0) + \Psi(r) \\ &\leq \frac{2\ \Phi(r_0)}{r_0}\ r + \Psi(r) \end{aligned}$$

for $r \geq 0$ and the proof of Lemma B.2 is complete. $\square$

C Equilibria are M_α

We assume here that the kinetic coefficients a and b enjoy the detailed balance condition (1.9) together with the positivity assumption (2.12) and are positive a.e. in $\mathbb{R}_+^2$. The aim of this section is to prove that any function satisfying (1.9) coincides with M_α for some $\alpha \in [-\infty, +\infty)$. More precisely we have the following result.

Lemma C.1 *Consider a non-negative function $f \in L^1(\mathbb{R}_+)$ satisfying*

$$a(y, y')\ f(y)\ f(y') = b(y, y')\ f(y + y') \quad a.e.\ in\ \ \mathbb{R}_+^2. \tag{C.1}$$

Then there is $\alpha \in [-\infty, +\infty)$ such that $f(y) = M(y)\ e^{\alpha y}$ a.e. in $\mathbb{R}_+$.

Proof. It readily follows from (C.1), (1.9) and the positivity of a, b and M that the function g defined by $g(y) = f(y)/M(y)$, $y \in \mathbb{R}_+$, is a non-negative measurable function satisfying

$$g(y)\ g(y') = g(y + y') \quad \text{a.e. in} \quad \mathbb{R}_+^2. \tag{C.2}$$

In addition we infer from (2.12) that $g \in L^1(0, R)$ for every $R \in \mathbb{R}_+$. We may therefore introduce the function $G \in \mathcal{C}([0, +\infty))$ defined by

$$G(y) = \int_0^y g(y')\ dy', \quad y \in [0, +\infty).$$

Integrating (C.2) over $(0,1)$ with respect to y' yields

$$G(1)\ g(y) = G(y+1) - G(y) \quad \text{a.e. in } \mathbb{R}_+. \tag{C.3}$$

If $G(1) = 0$ it follows from (C.3) and the continuity of G that $G(y+1) = G(y)$ for each $y \in \mathbb{R}_+$, whence $G(k) = 0$ for each integer $k \geq 1$. Consequently, $g = 0$ a.e. in $\mathbb{R}_+$ and we conclude that $f = M_{-\infty}$ a.e. in $\mathbb{R}_+$.

Assume now that $G(1) > 0$ and put

$$\tilde{g}(y) = \frac{G(y+1) - G(y)}{G(1)}, \quad y \in [0, +\infty).$$

Then $\tilde{g} \in \mathcal{C}([0,+\infty))$ and coincides with g a.e. in $\mathbb{R}_+$. Therefore $\tilde{g}$ also satisfies (C.2) and the continuity of $\tilde{g}$ entails that (C.2) holds true for every $(y, y') \in \mathbb{R}_+^2$. Classical arguments then ensure that there is $\alpha \in \mathbb{R}$ such that $\tilde{g}(y) = \exp\{\alpha y\}$ for $y \in \mathbb{R}_+$, whence $f = M_\alpha$ a.e. in $\mathbb{R}_+$. □

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